On the Robustness of Informational Cascades with Imprecise Binary Signals

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Abstract
The paper addresses uncertainty analysis in decision theory by applying imprecise probabilities to a herding behavior model, which describes imitative behavior and explains the informational cascade phenomenon. In the literature, the application of the principle of rationality in herding behavior generates informational cascades, i.e., sequences of actions in which each agent makes their choice by observing the decisions taken by those who acted before them, regardless of the private signal they own. Since the probability distribution of the signal may be hard to identify in some cases, this paper studies the herding behavior model by considering imprecise the signal probability. In the simplest case of a binary signal model, the agent’s private information is described by using a set of probability measures and assuming that the signal probability ranges in a probability interval. The paper aims to test the herding behavior model robustness when some assumptions no longer hold due to imprecise probabilities and prove that an informational cascade may occur even with a further noisy signal.

Mathematics Subject Classification: 91B06
Keywords: Imprecise Probabilities, Uncertainty, Herding Behavior, Informational Cascades

1 Introduction

A considerable literature suggesting that economic agents tend to make investment decisions based on information provided by the other market participants’ trades has recently developed. Individuals are influenced by other individuals in many activities, such as investment and financial transactions, political choices, and consumer preferences. In the literature, this type of phenomenon is known as herding behavior and describes the tendency of individuals to follow other individuals and imitate their behavior. In some circumstances, the principle of rationality in herding behavior may generate informational cascades. Informational cascades arise when an individual follows the behavior of predecessors regardless of the private information received. In this case, an individual’s decision becomes uninformative to all subsequent individuals. Therefore, private information vanishes.

One of the first models dealing with informational cascades is proposed in [3]. In a setting with sequential choices, at some stage, an individual would disregard their private information and make a decision only based on the information conveyed from previous actions, and an informational cascade would occur.

This paper provides a robustness analysis of the model described in [3], introducing uncertainty in the agents’ private information. More specifically, each agent has to take an investment decision, and they are endowed with a private signal on the true investment value. Differently from the literature, the probability that the private signal shows the real value of the investment is assumed to lie in an interval. Therefore, our model aims to study the effects of imprecise probabilities on informational cascades. To the best of our knowledge, herding behavior and informational cascades have never been approached in a context of uncertainty on the probability distribution of private information. The motivation behind our model is that even if each agent possesses private information about the object of their decision, in most cases, they are not able to link this information to a precise probability measure. Our model’s primary goal is to evaluate the behavior of a sequence of agents when their private information is given by a binary signal, whose quality is not a single value but lies in a set of values.

Herding behavior is a well-known concept in the economic literature. The causes leading individuals to follow the crowd are deepened in [10]: individuals imitate the crowd behavior since, in their opinion, other group members are better informed. In this perspective, herding can be considered a response to uncertainty and individuals’ perception of their ignorance. In [2], it is shown
that the herding process could lead to optimal choices in a sequential decision model, although the resulting equilibrium is inefficient. As observed in [15], a cascade behavior might arise in the initial public offering (IPO) market when IPO shares are sold sequentially.

A hierarchy of four categories of informational sources able to generate behavior similarity or dissimilarity is provided in [9]. From the broadest set to the narrowest one, they are herding/dispersing, observational influence, rational observational learning, and informational cascade.

An extensive survey on herding behavior is given in [13], which proposed an application of informational cascades to capital markets, in which publicly available information is reflected in prices, and investment decisions are continuous. In [12], herding and contrarian behavior in the financial market is analyzed. More in detail, it is argued that both types of behavior result in more volatile prices and lower liquidity, seeing that social learning generates price paths that are very sensitive to changes in some key parameters. Then, there will be large movements in prices compared to frameworks where these types of social learning are absent.

The Bayesian paradigm is the foundation of the cited literature and it was also the primary approach used in the 20th century to deal with uncertainty in economic models. According to the Bayesian paradigm, uncertainty has to be treated through probabilities, and information has to be updated through the Bayes formula. This approach implies that if an agent has no objective probabilities about the matter on which they have to make a decision, then the agent is able to set subjective probabilities. Roughly speaking, ignorance is not allowed.

Different approaches to deal with uncertainty have been developed during the last century. Among these alternatives, one of the most successful approach is given by the max-min expected utility (MMEU). The idea is that when an agent, who shows risk aversion, deals with uncertainty on the probability measure, then the best they can do is to maximize the worst-case scenario [14]. When there is uncertainty about the probability measure, i.e., when a single probability distribution may eventually be hard to identify [1], a powerful mathematical tool allowing to evaluate and quantify uncertainty in the absence of classical probability distributions is represented by imprecise probabilities. Imprecise probabilities arise from the requirement to provide a solution to Ellsberg’s empirical anomaly, in which individuals are not able to assign a subjective probability to an event unequivocally. The most natural approach to solve the paradox is to represent uncertainty using not one probability measure but a set of them. In [8], a set of probability measures and the MMEU approach to solve Ellsberg’s paradox are used. Besides the solution of Ellsberg’s paradox, several models in the literature deal with different types of uncertainty and

\footnote{See [7] for an extensive survey.}
explore several criteria to deal with it. As an example, the MMEU approach in an asset pricing model is used in [5]. In this model, the world is assumed to be binary; an event’s probability is not given by a single value but takes values in a set, and agents act according to the max-min expected utility criterion. In their model, there exists a range of prices in which agents do not want to sell or buy the asset. The MMEU approach is also applied to a labor market model [11]. More specifically, it is studied how unemployed agents react to the increased uncertainty in the labor market.

To test the robustness of the model described in [3], the max-min expected utility is used as a criterion to choose whether to invest or not. The model shows that an informational cascade can start even with a further noisy signal, as long as certain assumptions hold.

The structure of the paper is as follows. In Section 2, the binary signal model with imprecise probabilities notation is introduced. In Section 3, the Bayesian updating process is illustrated, with a focus on the description of an UP cascade and a DOWN cascade. Finally, Section 4 concludes.

2 The Binary Signal Model

In the model, a finite sequence of $n$ economic agents is considered. Each agent privately observes a conditionally independent signal $X_i \in \{X_H, X_L\}$ on the true investment value $V$, $\forall i = 1, \ldots, n$. The signal can be high or low according to $X = X_H$ or $X = X_L$, respectively. The investment value is assumed to be binary, as well: $V = V_h$ when it is high, while $V = V_l$ when it is low, with ex-ante equal probability, namely,

$$\Pr(V = V_h) = \Pr(V = V_l) = \frac{1}{2}.$$

The above probabilities represent the prior probabilities (or priors) in the Bayesian updating.

Individuals act sequentially, one by one, and each of them can only observe the actions of the predecessors but not the private information. Then, knowing the history and private information, they have to decide if accept or reject the investment. Moreover, an investment cost $K$ is supported only by the agents who invest.

Table 1 shows the conditional probabilities of the signal. In the model considered in [3], such a signal probability is common knowledge: it is assumed to be constant and equal to $p > 1/2$ for all agents. In our model, the signal probability is still assumed to be constant and equal to $p$ for all agents. Nevertheless, to generalize the framework, the agents do not know the precise value

\textsuperscript{2}For further details, see [3], p. 999.
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Table 1: Signal Probabilities

<table>
<thead>
<tr>
<th>Investment Value</th>
<th>Pr($X_H \mid V$)</th>
<th>Pr($X_L \mid V$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V = V_h$</td>
<td>$p_i$</td>
<td>$1 - p_i$</td>
</tr>
<tr>
<td>$V = V_l$</td>
<td>$1 - p_i$</td>
<td>$p_i$</td>
</tr>
</tbody>
</table>

of $p$. The only information they have is that the signal probability can take any value in a probability interval $I_p = [\alpha, \beta]$, with $\alpha > 1/2$ and $\beta < 1$.

The conditional probabilities of the high investment value $V = V_h$ and the low investment value $V = V_l$, given the same signal, are

$$\Pi_{X_i}(V_h) = \Pr(V_h \mid X_i) \quad \text{and} \quad \Pi_{X_i}(V_l) = \Pr(V_l \mid X_i),$$

with $X_i \in \{X_H, X_L\}$, for all $i = 1, \ldots, n$. These probabilities represent the posteriors in the Bayesian updating and they are involved in the agent’s decision problem. The two actions are denoted by $a$ in the case of an accept and by $r$ in the case of a reject.

Therefore, action $a$ pays 1 if the true investment value is $V = V_h$, 0 otherwise; on the other hand, action $r$ pays $q$, with $q \in [0, 1]$, if the true investment value is $V = V_l$, 0 otherwise. The expected value of an accept is defined as $E_{X_i}[a] = \Pi_{X_i}(V_h) - K$, to take into account the cost supported by the agents; conversely, the expected value of a reject is given by $E_{X_i}[r] = q\Pi_{X_i}(V_l)$. By using the expected utility criterion, an individual accepts if the difference between the two expected values is non-negative, that is,

$$(\Pi_{X_i}(V_h) - K) - q\Pi_{X_i}(V_l) > 0. \quad (1)$$

Since the signal probability is imprecise, we can choose a value among different probability values for each posterior. In order to choose a suitable probability value, we use the max-min expected utility criterion, following the approach suggested in [8]. Applying the max-min expected utility criterion to the condition $[\pi]$ to tackle decisions with imprecise probabilities consists of a double strategy. Firstly, assuming that agents are uncertainty averse, we minimize both the expected values of an accept, $E_{X_i}[a]$, and a reject, $E_{X_i}[b]$, as follows,

$$\min_{p \in I_p} E_{X_i}[a] = \min_{p \in I_p} [\Pi_{X_i}(V_h) - K] = \min_{p \in I_p} [\Pi_{X_i}(V_h)] - K,$$

and

$$\min_{p \in I_p} E_{X_i}[b] = \min_{p \in I_p} [q \cdot \Pi_{X_i}(V_l)] = q \cdot \min_{p \in I_p} [\Pi_{X_i}(V_l)].$$

Secondly, we compare these two minimized quantities, as follows,
\[
\min_{p \in I_p} [\Pi X_i(V_h)] - K - q \cdot \min_{p \in I_p} [\Pi X_i(V_l)] .
\]

An agent will accept to invest if the above difference is non-negative, that is,
\[
\min_{p \in I_p} [\Pi X_i(V_h)] - q \min_{p \in I_p} [\Pi X_i(V_l)] > K .
\]  

Let \( \argmin_{p \in I_p} \Pi X_i(V_h) = p' \) and \( \argmin_{p \in I_p} \Pi X_i(V_l) = p'' \) be defined. At every stage, we determine the couple \((p', p'')\) that minimizes the two posteriors and evaluate whether the condition (2) holds or not, namely if the individual accepts or rejects as the signal changes. If the action is the same as in the previous stage, an informational cascade occurs.

### 3 Bayesian Updating in an Informational Cascade

In the literature, herding models are well-founded on Bayesian hypotheses \[6\]. An individual’s decision represents the information another individual later uses to compute their beliefs and expectations. In sequential decision-making, individuals update their beliefs using the Bayes rule. The outcome will be positive or negative according to the history of their predecessors’ actions is correct or incorrect. Such a Bayesian learning process may generate imitative behavior and informational cascades \[2, 15\].

Herding behavior and informational cascades were often confused with each other, but they are conceptually different. In a herding model, agents act similarly, but they could have made different decisions if they had observed different private signals. On the other hand, in an informational cascade, it is optimal for an agent to behave like their predecessors regardless of the private signal observed. In this case, the agent’s belief is too informative that no signal can overcome it \[4\]. Therefore, an informational cascade is also a herding mechanism, but a herding phenomenon does not necessarily lead to an informational cascade. While herding is a mechanism based on rational observational learning, informational cascades represent a sub-rational mechanism based on imitation, leading the agents to be affected by others’ behavior and behave similarly. Despite not being fully rational, this social interaction has a significant advantage: an individual can exploit information conveyed by the other participants about the environment, avoiding adverse conditions, as evidenced by observing behavior in many animals.

When an informational cascade starts, the individual’s choice becomes uninformative to all subsequent participants. As a result, the following individuals will draw the same inference from the history of past decisions. Since followers’
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signals will be drawn independently from the same distribution as their predecessors, they will ignore their private information and imitate the predecessors. In the absence of external disturbances, every successor will act similarly, and this behavior can be viewed as a chain reaction. A description of an UP cascade and a DOWN cascade is illustrated below.

3.1 UP Cascade

A binary signal model of herding behavior with a cost \( K = 1/2 \) is considered. An UP cascade can arise only when the first individual observes a high signal, that is \( X_H \).

Suppose that the first agent is endowed with the private signal \( X_H \). The two posteriors, computed by the Bayes rule\(^3\), are \( \Pi_{X_1=x_H}(V_h) = p \) and \( \Pi_{X_1=x_H}(V_l) = 1 - p \). The first one, \( p \), is a strictly increasing linear function since

\[
\frac{\partial \Pi_{X_1=x_H}(V_h)}{\partial p} = 1, \quad \forall p \in [\alpha, \beta].
\]

On the other hand, the second posterior, \( 1 - p \) is a strictly decreasing linear function as

\[
\frac{\partial \Pi_{X_1=x_H}(V_l)}{\partial p} = -1, \quad \forall p \in [\alpha, \beta].
\]

Due to the linearity of posteriors, the extreme value \( \alpha \) minimizes the posterior \( \Pi_{X_1=x_H}(V_h) \), while the extreme value \( \beta \) minimizes the posterior \( \Pi_{X_1=x_H}(V_l) \). By applying the max-min expected utility criterion in (2), the first agent will accept if

\[
q < \frac{\alpha - K}{1 - \beta},
\]

with \( q \in ]0, 1[ \). If \( q = \frac{\alpha - K}{1 - \beta} \), the same individual accepts and rejects with equal probability; otherwise, if \( q > \frac{\alpha - K}{1 - \beta} \), the first individual rejects.

Condition (3) plays a crucial role in the model. If this condition holds, the acceptance conditions related to the successor agents are also satisfied (see Lemma 3.1 later introduced). In Figure 1, the yellow area highlights the combinations of parameters \((\alpha, \beta)\) such that \( \alpha < \beta \) and the condition in (3) is satisfied, when \( q = 1 \) and \( K = 1/2 \). The dashed line represents how the level curve of condition (3) moves as the value of \( q \) changes. More specifically, as

\[^3\text{According to the Bayes rule, the two posteriors are computed as}

\[
Pr(V \mid X_1) = \frac{Pr(V)Pr(X_1 \mid V)}{Pr(V = V_h)Pr(X_1 \mid V = V_h) + Pr(V = V_l)Pr(X_1 \mid V = V_l)},
\]

with \( V \in \{V_h, V_l\} \) and \( X_1 \in \{X_H, X_L\} \).
Figure 1: Region of parameters \((\alpha, \beta)\) that satisfies condition (3), with \(K = 1/2\).

If \(q\) decreases, the level curve shifts down, and the region of parameters \((\alpha, \beta)\) satisfying condition (3) expands. A lower level of \(q\) means that the reward for rejecting an unprofitable investment reduces (potentially until zero), making the idea of investing more attractive. Therefore, the agent is willing to accept more uncertainty about the signal probability.

On the other hand, if the first individual is endowed with \(X_L\), the private beliefs are \(\Pi_{X_1=X_L}(V_h) = 1 - p\) and \(\Pi_{X_1=X_L}(V_l) = p\). By applying the same procedure previously described, the decision criterion leading the first agent to accept is expressed as follows,

\[
\frac{q\alpha + \beta}{2} < \frac{1 - K}{2}.
\] (4)

However, such a condition never holds due to \(\alpha, \beta > 1/2\). As a consequence, the first individual rejects.

Suppose that the first individual has accepted at the end of the first stage. The second agent infers its predecessor’s signal: \(X_H\). The two prior probabilities involved in the updating process are now given by the transition priors between the previous and current stages. Transition priors updating depends on the agent’s ability to infer the signal observed by its predecessor. In such a case, the second agent can deduce the signal observed by the first agent exactly. Therefore, transition priors coincide with the two posteriors obtained at the first step when the first individual observes \(X_H\), that is \(\Pr(V_h \mid a) = \Pr(V_h \mid X_H) = p\) and \(\Pr(V_l \mid a) = \Pr(V_l \mid X_H) = 1 - p\).
Consider the case in which the second individual observes $X_H$. The two private beliefs are computed by the repeated application of the Bayes rule\footnote{At the second stage, Bayesian updating returns}:

\[ \Pi_{X_2 = X_H}(V_h) = \frac{p^2}{p^2 + (1-p)^2} \quad \text{and} \quad \Pi_{X_2 = X_H}(V_l) = \frac{(1-p)^2}{p^2 + (1-p)^2}. \]

By adopting the max-min expected utility criterion, the second individual accepts if

\[ \frac{\alpha^2}{\alpha^2 + (1-\alpha)^2} - q \frac{(1-\beta)^2}{\beta^2 + (1-\beta)^2} > K. \quad (5) \]

If the condition (3) holds, even the condition (5) holds due to the following Lemma 3.1.

**Lemma 3.1** If $\frac{\alpha^n}{\alpha^n + (1-\alpha)^n} - q \frac{(1-\beta)^n}{\beta^n + (1-\beta)^n} > K$, then the following inequality holds too, that is,

\[ \frac{\alpha^{(n+1)}}{\alpha^{(n+1)} + (1-\alpha)^{(n+1)}} - q \frac{(1-\beta)^{(n+1)}}{\beta^{(n+1)} + (1-\beta)^{(n+1)}} > K, \]

where $1/2 \leq \alpha < \beta \leq 1$ and $0 < q \leq 1$.

**Proof 3.1** A self-contained proof is given in the appendix\footnote{A self-contained proof is given in the appendix}.

In other words, if the first agent accepts and the second one is endowed with a high signal, $X_H$, then the second agent will also accept.

On the other hand, if the second individual observes $X_L$, the posteriors $\Pi_{X_2 = X_L}(V_h)$ and $\Pi_{X_2 = X_L}(V_l)$ are both equal to $1/2$. In this case, the second individual would accept if

\[ \frac{1}{2} - q \frac{1}{2} > K. \quad (6) \]

However, such a condition never holds for $q \in [0,1]$; as a result, the second individual rejects.

At the third stage, by assuming that the first agent has accepted, the third individual can face two different scenarios. (i) Both previous agents have accepted: in this case, the third individual can infer their predecessors’ signals, which are both high. (ii) The first agent has accepted, and the second one
has rejected: in this case, the transition priors of the third individual are
\( \Pr(V_h \mid a, r) = \Pr(V_l \mid a, r) = 1/2 \), then they are in the same position as the
first agent. We focus on the first case and show that an UP cascade will start
at the third stage.

In the case in which the third agent observes \( X_H \), the two posteriors are given
by \( \Pi_{X_3 \mid X_H} (V_h) = \frac{p^3}{p^3 + (1-p)^3} \) and \( \Pi_{X_3 \mid X_H} (V_l) = \frac{(1-p)^3}{p^3 + (1-p)^3} \). Hence, the individual
accepts if

\[
\frac{\alpha^3}{\alpha^3 + (1-\alpha)^3} - q \frac{(1-\beta)^3}{\beta^3 + (1-\beta)^3} > K. \tag{7}
\]

As in the previous step, if condition (3) holds, the above condition holds too
due to the Lemma 3.1. Consequently, if the first agent accepts and the two
later agents are both endowed with a high signal, \( X_H \), then even the two
successors will accept.

On the other hand, if the third agent observes \( X_L \), their two private beliefs
are given by \( \Pi_{X_3 \mid X_L} (V_h) = p \) and \( \Pi_{X_3 \mid X_L} (V_l) = 1 - p \), and the decision rule
leading the agent to accept is the same condition in the (3), which will be
necessarily satisfied. Therefore, the third individual accepts even when they
observe a low signal. As a result, an UP cascade arises since the individual
accepts regardless of the private information observed.

### 3.2 DOWN Cascade

Given a cost \( K = 1/2 \), a path leading to a DOWN cascade occurrence is
described in this subsection. Such a cascade can arise only when the first
individual rejects.

Two possible paths lead to a DOWN cascade (see Figure 2). Given the model’s
rules, three different scenarios can be achieved if the first agent rejects the
investment.

**S1** \( q < \frac{\alpha-K}{1-\beta} \). The second agent infers their predecessor’s signal, which is \( X_L \).
At the second stage, transition priors are given by the two posteriors
computed at the first stage in the case of a low signal, and a DOWN
cascade occurs since the agent will reject regardless of their private signal.

**S2** \( q > \frac{\alpha-K}{1-\beta} \). The first agent rejects with both signals, \( X_H \) and \( X_L \). It is
the only case in which a DOWN cascade starts at the first stage.

**S3** \( q = \frac{\alpha-K}{1-\beta} \). The second agent cannot infer the previous agent’s signal,
which could be either \( X_H \) or \( X_L \).
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Figure 2: Agent choice Tree with Imprecise Probabilities when $K = 1/2$.

In this last scenario, a DOWN cascade begins at the second stage. In such a case, due to the agent's ignorance about the signal observed by their predecessor, transition priors jointly take into account both the high signal and the low one.

They are $\Pr(V_h | r) = \frac{2-p}{3}$ and $\Pr(V_l | r) = \frac{1+p}{3}$.

If the individual observes a high signal, the two posteriors, $\Pi_{X_2 = X_H}(V_h) = \frac{p(2-p)}{1+2p-2p^2}$ and $\Pi_{X_2 = X_H}(V_l) = \frac{(1-p)(1+p)}{1+2p-2p^2}$, imply a condition leading to accept, that is,

$$\frac{\alpha(2-\alpha)}{1+2\alpha-2\alpha^2} - q \frac{(1-\beta)(1+\beta)}{1+2\beta-2\beta^2} > K,$$

which is never satisfied for $q = \frac{a-K}{1-\beta}$.

A similar procedure is involved when the second individual observes a low signal. The two posteriors are $\Pi_{X_2 = X_L}(V_h) = \frac{(1-p)(2-p)}{2(1-p+p^2)}$ and $\Pi_{X_2 = X_L}(V_l) = \frac{(1-p)(2-p)}{2(1-p+p^2)}$.

Transition priors are computed as follows,

$$\Pr(V | r) = \frac{\Pr(V)(\frac{1}{2}\Pr(X_H | V) + \Pr(X_L | V))}{\Pr(V_h)(\frac{1}{2}\Pr(X_H | V_h) + \Pr(X_L | V_h)) + \Pr(V_l)(\frac{1}{2}\Pr(X_H | V_l) + \Pr(X_L | V_l))},$$

with $V \in \{V_h, V_l\}$. 
\[ \frac{p(1+p)}{2(1-p+p^2)}, \text{ while the investment condition becomes} \]
\[ \frac{(1-\beta)(2-\beta)}{2(1-\beta + \beta^2)} - q \frac{\alpha(1+\alpha)}{2(1-\alpha + \alpha^2)} > K. \] (9)

In this last case, the above condition does not hold for \( q = \frac{\alpha-K}{1-\beta} \), as well. Therefore, a DOWN cascade occurs.

4 Conclusion

Informational cascades represent a particular case of herding behavior, characterized by a complete information blockage because, after some periods, individuals’ actions become uninformative to all other market participants. This phenomenon can explain how some social conventions and norms arise, are maintained, and evolve. Nevertheless, the loss of various information sources represents the main social disadvantage of informational cascades.

In this work, the effect of imprecise probabilities on a herding behavior model is studied to test its robustness and prove that an informational cascade may eventually occur even with imprecise probabilities. At every stage, the agent’s private beliefs are minimized by associating with a reject action a payoff lower, at most equal, than one associated with an accept action. The analysis shows that the individual’s choice depends on the length of the interval \( I_p = [\alpha, \beta] \) in which the signal probability \( p \) ranges. If \( q = 1 \), the agent’s choice depends on the probability interval \( I_p \) midpoint value, as well.

Respect to the framework in [3], in our analysis, with a similar setting and the same values of the exiting parameters, an UP cascade still begins at the third stage, while in contrast, a DOWN cascade starts earlier, probably due to the higher level of uncertainty.

A Appendix

Proof of Lemma 3.1

Lemma 3.1 If \( \frac{\alpha^n}{\alpha^n+(1-\alpha)^n} - q \frac{(1-\beta)^n}{\beta^n+(1-\beta)^n} > K \), then the following inequality holds too, that is,
\[ \frac{\alpha^{(n+1)}}{\alpha^{(n+1)} + (1-\alpha)^{(n+1)}} - q \frac{(1-\beta)^{(n+1)}}{\beta^{(n+1)} + (1-\beta)^{(n+1)}} > K, \] (A.1)

where \( 1/2 \leq \alpha < \beta \leq 1 \) and \( 0 < q \leq 1 \).
Proof A.1 By comparing the two inequalities and proving that

\[
\frac{\alpha^n}{\alpha^n + (1 - \alpha)^n} - q \frac{(1 - \beta)^n}{\beta^n + (1 - \beta)^n} < \frac{\alpha^{(n+1)}}{\alpha^{(n+1)} + (1 - \alpha)^{(n+1)}} - q \frac{(1 - \beta)^{(n+1)}}{(1 - \beta)^{(n+1)} + (1 - \beta)^{(n+1)}};
\]

statement [A.1] holds for transitivity. The above inequality can be expressed in the following way,

\[
\alpha^n \left( \frac{\alpha}{\alpha^{n+1} + (1 - \alpha)^{n+1}} - \frac{1}{\alpha^n + (1 - \alpha)^n} \right) > q(1 - \beta)^n \left( \frac{1 - \beta}{\beta^{n+1} + (1 - \beta)^{n+1}} - \frac{1}{\beta^n + (1 - \beta)^n} \right).
\]

Since the quantities \( \alpha^n, \) \( \alpha^n + (1 - \alpha)^n, \) \( q \) and \( (1 - \beta)^n \) are positive, while the term \( \left( \frac{1 - \beta}{\beta^{n+1} + (1 - \beta)^{n+1}} - \frac{1}{\beta^n + (1 - \beta)^n} \right) \) is negative, the inequality is verified. In fact, \( \alpha^n, \) \( (1 - \beta)^n \) and \( q \) are positive by assumption. Moreover, it can be proven that

\[
\frac{\alpha}{\alpha^{n+1} + (1 - \alpha)^{n+1}} - \frac{1}{\alpha^n + (1 - \alpha)^n} > 0,
\]

seeing that

\[
\frac{\alpha (\alpha^n + (1 - \alpha)^n) - (\alpha^{n+1} + (1 - \alpha)^{n+1})}{(\alpha^{n+1} + (1 - \alpha)^{n+1})(\alpha^n + (1 - \alpha)^n)} > 0,
\]

or, equivalently,

\[
(1 - \alpha)^n (2\alpha - 1) > 0,
\]

which holds for \( 1/2 \leq \alpha < 1. \) Regarding the last term,

\[
\frac{1 - \beta}{\beta^{n+1} + (1 - \beta)^{n+1}} - \frac{1}{\beta^n + (1 - \beta)^n} < 0,
\]

since

\[
\frac{(1 - \beta)(\beta^n + (1 - \beta)^n) - (\beta^{n+1} + (1 - \beta)^{n+1})}{(\beta^{n+1} + (1 - \beta)^{n+1})(\beta^n + (1 - \beta)^n)} < 0,
\]
or, equivalently,

\[ \beta^n(2\beta - 1) > 0, \]

which is true for \( 1/2 < \beta \leq 1 \).

References


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