A Characterization of Unicyclic Graphs with the Same Independent Domination Number

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Abstract

A set $D$ of vertices of $G$ is an independent dominating set if no two vertices of $D$ are adjacent and every vertex not in $D$ is adjacent to at lest one vertex in $D$. The independent domination number of a graph $G$, denoted by $i(G)$, is the minimum cardinality of an independent dominating set in $G$. A unicyclic graph is a connected graph containing exactly one cycle. For $k \geq 1$, let $\mathcal{H}(k)$ be the set of unicyclic graphs $H$ satisfying $i(H) = k$. In this paper, we provide a constructive characterization of $\mathcal{H}(k)$ for all $k \geq 1$.

Mathematics Subject Classification: 05C05, 05C69

Keywords: unicyclic graph, independent dominating set, independent domination number

1 Introduction

One of the famous concepts in graph theory is Domination in graphs. The domination problem is NP-complete for an arbitrary graph [3]. Domination in graphs is now well studied in graph theory. A set $D$ of vertices of $G$ is an independent dominating set (IDS) if no two vertices of $D$ are adjacent and every vertex not in $D$ is adjacent to at lest one vertex in $D$. The independent domination number of a graph $G$, denoted by $i(G)$, is the minimum cardinality of an independent dominating set in $G$. If $D$ is an IDS of $G$ with cardinality...
$i(G)$, then we call $D$ an $i$-set of $G$. The independent domination number and the notation $i(G)$ were introduced by Cockayne and Hedetniemi in [2]. Recently, it was then extensively studied for various classes of graphs in the literature (see [4],[5],[6],[7]).

For $k \geq 1$, let $\mathcal{X}(k)$ be the set of unicyclic graphs $H$ satisfying $i(H) = k$. In this paper, we provide a constructive characterization of $\mathcal{X}(k)$ for all $k \geq 1$.

2 Notations and preliminary results

All graphs considered in this paper are finite, loopless, and without multiple edges. For a graph $G$, $V(G)$ and $E(G)$ denote the vertex set and the edge set of $G$, respectively. The (open) neighborhood $N_G(v)$ of a vertex $v$ is the set of vertices adjacent to $v$ in $G$, and the closed neighborhood $N_G[v]$ is $N_G(v) \cup \{v\}$. For any subset $A \subseteq V(G)$, denote $N_G(A) = \bigcup_{v \in A} N_G(v)$ and $N_G[A] = \bigcup_{v \in A} N_G[v]$. The degree of $v$ is the cardinality of $N_G(v)$, denoted by $\deg_G(v)$. A vertex $x$ is said to be a leaf of $G$ if $\deg_G(x) = 1$. A vertex of $G$ is a support vertex if it is adjacent to a leaf in $G$. We denote by $L(G)$, and $U(G)$ the collections of the leaves and support vertices of $G$, respectively. For two sets $A$ and $B$, the difference of $A$ and $B$, denoted by $A - B$, is the set of all the elements of $A$ that are not elements of $B$. For a subset $A \subseteq V(G)$, the deletion of $A$ from $G$ is the graph $G - A$ obtained by removing all vertices in $A$ and all edges incident to these vertices. A $u$-$v$ path $P : u = v_1, v_2, \ldots, v_k = v$ of $G$ is a sequence of $k$ vertices in $G$ such that $v_iv_{i+1} \in E(G)$ for $i = 1, 2, \ldots, k - 1$. For any two vertices $u$ and $v$ in $G$, the distance between $u$ and $v$, denoted by $\text{dist}_G(u, v)$, is the minimum length of the $u$-$v$ paths in $G$. Denote by $P_n$ a $n$-path with $n$ vertices. The length of $P_n$ is $n$. 1. For other undefined notions, the reader is referred to [1] for graph theory.

The following lemmas are useful.

Lemma 2.1. For $n \geq 1$, $i(P_n) = \left\lceil \frac{n}{3} \right\rceil$.

Proof. It’s true for $n = 1, 2$ and $3$. For $n \geq 4$, let $k = \left\lceil \frac{n}{3} \right\rceil$ and $P_n : v_1, v_2, \ldots, v_n$. Suppose $D = \{v_2, \ldots, v_{3k-1}, \ldots, v_{3k-4}, v_m\}$, where $m = 3k - 2$ or $3k - 1$ is an IDS of $P_n$, then $i(P_n) \leq |D| = (k - 1) + 1 = k$.

Suppose, by contradiction, $i(P_n) = s \leq k - 1$ and $D' = \{v_{i_1}, \ldots, v_{i_s}\}$ is an $i$-set of $P_n$, where $i_1 < i_2 < \ldots, i_s$. We can see that $\text{dist}(v_j, v_{j+1}) \leq 3$ for $j = 1, \ldots, s - 1$. Then $n = |P_n| = |D'| + |P_n - D'| \leq s + [1 + 2(s - 1) + 1] = 3s \leq 3(k - 1) < n$. This is a contradiction, so $i(P_n) = k = \left\lceil \frac{n}{3} \right\rceil$. 

Lemma 2.2. For $n \geq 3$, $i(C_n) = \left\lceil \frac{n}{3} \right\rceil$.

Proof. It’s true for $n = 3$. For $n \geq 4$, let $k = \left\lceil \frac{n}{3} \right\rceil$ and $C_n : v_1, v_2, \ldots, v_n, v_1$. Assume $D$ is an $i$-set of $C_n$ and $v_1 \in D$. Then $v_2 \notin D$ and $v_n \notin D$. Let
Proof. We can see that \( P' = C_n - \{v_1, v_2, v_n\} \) and \( D' = D - \{v_1\} \). Then \( D' \) is an \( i \)-set of \( P' \), where \(|P'| = n - 3\). By Lemma 2.1, \(|D'| = i(P') = \left\lceil \frac{n-3}{3} \right\rceil = \left\lceil \frac{n}{3} \right\rceil - 1\). Thus \( i(C_n) = |D| = |D'| + 1 = (\left\lceil \frac{n}{3} \right\rceil - 1) + 1 = \left\lceil \frac{n}{3} \right\rceil \). \( \square \)

**Lemma 2.3.** Suppose \( H \) is obtained from \( H' \in \mathcal{H}(k) \) by adding one vertex \( v \) and the edge \( uv \), where \( u \in V(H') \), then \( k \leq i(H) \leq k + 1 \). Moreover, the followings hold.

(i) The graph \( H \in \mathcal{H}(k+1) \) if and only if \( w \notin D' \) for every \( i \)-set \( D' \) of \( H' \).

(ii) The graph \( H \in \mathcal{H}(k) \) if and only if \( w \in D' \) for some \( i \)-set \( D' \) of \( H' \).

**Proof.** We can see that \( H \) is unicyclic. If \( D' \) is an \( i \)-set of \( H' \), then \( D' \) or \( D' \cup \{v\} \) is an IDS of \( H \). So \( i(H) \leq |D'| + 1 = k + 1 \). The equalities hold if and if \( D' \cup \{v\} \) is an \( i \)-set of \( H \). Thus we got (i).

If \( D \) is an \( i \)-set of \( H \), then \( D, D - \{v\} \) or \((D - \{v\}) \cup \{w\}\) is an IDS of \( H' \). So \( i(H) = |D| \geq i(H') = k \). The equalities hold if and if \( D_1 \) is an \( i \)-set of \( H' \), where \( D_1 = D \) or \( D_1 = (D - \{v\}) \cup \{w\} \). Note that \( w \in D_1 \). Thus we got (ii). \( \square \)

**Lemma 2.4.** Suppose \( H \) is obtained from \( H' \in \mathcal{H}(k) \) by adding a \( P_2 : v, v' \) and the edge \( uv \), where \( w \in V(H') \), then \( H \in \mathcal{H}(k+1) \).

**Proof.** We can see that \( H \) is unicyclic. Since \( v' \notin N_H[V(H')] \), this means that \( i(H) \geq i(H') + 1 = k + 1 \). Let \( D' \) be an \( i \)-set of \( H' \). Then \( D = D' \cup \{v\} \) is an ISD of \( H \). So \( k + 1 \leq i(H) \leq |D| = |D'| + 1 = k + 1 \), thus \( H \in \mathcal{H}(k+1) \). \( \square \)

## 3 Characterization

In this section, we characterize the set \( \mathcal{H}(k) \) for all \( k \geq 1 \). Suppose \( H' \) is a unicyclic graph and \( H \) is obtained from \( H' \) by one of the following Operations.

**Operation O1.** Add a new vertex \( v \) and the edge \( uv \), where \( w \in V(H') \) and \( w \notin D' \) for every \( i \)-set \( D' \) of \( H' \).

**Operation O2.** Add a new path \( P_2 \) and the edge \( uv \), where \( w \in V(H') \) and \( v \in V(P_2) \).

**Operation O3.** Add a new vertex \( v \) and the edge \( uv \), where \( w \in V(H') \) and \( w \in D' \) for some \( i \)-set \( D' \) of \( H' \).

**Lemma 3.1.** Let \( H' \in \mathcal{H}(k-1) \). Suppose \( H \) is obtained from \( H' \) by one of the Operation O1 or Operation O2, then \( H \in \mathcal{H}(k) \).
Proof. Suppose $H$ is obtained from some $H'$ by the Operation $O_i$, where $i = 1, 2$. Then $H$ is a unicyclic graphs.

Case 1. $i = 1$. By Lemma 2.3 (i), then $i(H) = i(H') + 1 = k$ and $H \in \mathcal{H}(k)$.

Case 2. $i = 2$. By Lemma 2.4, $i(H) = i(H') + 1 = k$. Therefore, $H \in \mathcal{H}(k)$.

By Case 1 and Case 2, $H \in \mathcal{H}(k)$.

Lemma 3.2. Let $H' \in \mathcal{H}(k)$. Suppose that $H$ is obtained from $H'$ by the Operation $O_3$, then $H \in \mathcal{H}(k)$.

Proof. We can see that $H$ is unicyclic. By Lemma 2.3(ii), $k \leq i(H) \leq |D'| = i(H') = k$, thus $H \in \mathcal{H}(k)$.

Let $\mathcal{C}(1) = \{C_3\}$ and $\mathcal{A}(1) = \{C_3\} \cup \mathcal{A}'(1)$, where $\mathcal{A}'(1)$ is the collection of graphs in Figure 1.

![Figure 1: The collection $\mathcal{A}'(1)$ of graphs](image)

For $k \geq 2$, we define the following collections.

(i) $\mathcal{C}(k) = \{C_{3k-2}, C_{3k-1}, C_{3k}\}$.

(ii) $\mathcal{B}(k)$ is the collection of the unicyclic graphs $H$ which is obtained from some $H' \in \mathcal{A}(k-1)$ by one of the Operation $O_1$ or Operation $O_2$.

(iii) $\mathcal{A}'(k)$ is the collection of the unicyclic graphs $H$ which is obtained from a sequence $H_1$, where $H_1 \in \mathcal{C}(k)$ or $H \in \mathcal{B}(k)$, $H_2, \ldots, H_m = H$ and, if $j = 1, 2, \ldots, m-1$, $H_{j+1}$ is obtained from $H_j$ by the Operation $O_3$.

(iv) $\mathcal{A}(k) = \mathcal{C}(k) \cup \mathcal{B}(k) \cup \mathcal{A}'(k)$

By Lemma 2.2, we have the following lemma.

Lemma 3.3. For $k \geq 1$, $\mathcal{C}(k) \subset \mathcal{H}(k)$.

We first prove the following lemma.

Lemma 3.4. For $k \geq 1$, $\mathcal{A}(k) \subseteq \mathcal{H}(k)$.
Proof. We prove it by induction on $k$. It’s true for $k = 1$. Assume that it’s true for $k - 1$, where $k \geq 2$, and $H \in \mathcal{S}(k)$. Then $H$ is unicyclic. We consider three cases.

Case 1. $H \in \mathcal{C}(k)$. By Lemma 3.3, then $H \in \mathcal{H}(k)$.

Case 2. $H \in \mathcal{B}(k)$. Then $H$ is obtained from some $H' \in \mathcal{S}(k - 1)$ by one of the Operation O1 or Operation O2. By the hypothesis, $H' \in \mathcal{H}(k - 1)$. By Lemma 3.1, $H \in \mathcal{H}(k)$.

Case 3. $H \in \mathcal{S}(k)$. Then $H$ is obtained from a sequence $H_1$, where $H_1 \in \mathcal{C}(k)$ or $H \in \mathcal{B}(k)$, $H_2, \ldots, H_m = H$ and, if $j = 1, 2, \ldots, m - 1$, $H_{j+1}$ is obtained from $H_j$ by the Operation O3. By Case 1 and Case 2, we have that $H_1 \in \mathcal{H}(k)$. By Lemma 3.2, $i(H) = i(H_m) = i(H_{m-1}) = \cdots = i(H_1) = k$. Thus $H \in \mathcal{H}(k)$.

By Case 1, Case 2 and Case 3, we have that $H \in \mathcal{H}(k)$. □

Theorem 3.5 is the main theorem.

Theorem 3.5. For $k \geq 1$, $\mathcal{S}(n) = \mathcal{H}(n)$.

Proof. By Lemma 3.4, we need only prove that $\mathcal{H}(k) \subseteq \mathcal{S}(k)$ for all $k \geq 1$ and it is proved by contradiction. Suppose $H \in \mathcal{H}(k)$ and $H \notin \mathcal{S}(k)$ such that $|H|$ is as small as possible. Let $C$ be the cycle of $H$. By Lemma 3.3, then $H \neq C$ and $L(H) \neq \emptyset$. Let $x$ be a leaf of $H$ and $w$ be the neighbor of $x$. Then $H' = H - \{x\}$ is unicyclic. By Lemma 2.3, $k - 1 \leq i(H') \leq k$.

Case 1. $i(H') = k$.

Then $H' \in \mathcal{H}(k)$. Since $|H'| < |H|$, by the hypothesis, $H' \in \mathcal{S}(k)$. Since $i(H) = i(H')$, by Lemma 2.3 (ii), $w \in D'$ for some $i$-set $D'$ of $H'$. Thus $H$ is obtained from $H' \in \mathcal{S}(k)$ by the Operation O3, it means that $H \in \mathcal{S}(k)$. This is a contradiction.

Case 2. $i(H') = k - 1$.

Then $H' \in \mathcal{H}(k - 1)$. Since $|H'| < |H|$, by the hypothesis, $H' \in \mathcal{S}(k - 1)$. Since $i(H) = i(H') + 1$, by Lemma 2.3 (i), $w \notin D'$ for every $i$-set $D'$ of $H'$. Thus $H$ is obtained from $H' \in \mathcal{S}(k - 1)$ by the Operation O1, it means that $H \in \mathcal{B}(k)$. So $H \in \mathcal{S}(k)$, this is a contradiction.

By Case 1 and Case 2, $\mathcal{H}(k) \subseteq \mathcal{S}(k)$ for all $k \geq 1$. We complete the proof. □

Hence we provide a constructive characterization $\mathcal{S}(k)$ of $\mathcal{H}(k)$ for all $k \geq 1$.

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Received: March 15, 2023; Published: April 6, 2023