The Strong Convergence for Solutions of Pseudomonotone Variational Inequality Problem in Banach Spaces

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Abstract

In this paper, based on the line-search technique, an iteration method to solve pseudomonotone variational inequality problem in 2-uniformly convex Banach spaces is introduced. The iteration scheme presented in this paper is proved to converge strongly to a solution of the pseudomonotone variational inequality problem. Our result extends the main result in [4].

Keywords: pseudomontone variational inequality, iteration algorithm, Banach space, strong convergence

1 Introduction

Let C be a nonempty closed convex subset of real Banach space E with norm ∥⋅∥, we denote by E∗ the dual space of E and ⟨f, x⟩ the value of f ∈ E∗ at x ∈ E, and let F : E → E∗ be a nonlinear operator. The variational inequality problem (for short VIP) is to find x∗ ∈ C such that

⟨Fx∗, y − x∗⟩ ≥ 0, ∀y ∈ C. (1.1)

The set of solution of VI(C, F) is denoted by Γ.
It is well known that a variational inequality problem may be converted into a fixed point problem of a nonlinear operator. So many fixed point methods are widely used to solve the solution of variational inequality problem. For example, Lecutin E.S [3] proposed the following projection algorithm to solve the strongly monotone variational inequality problem,

\[ x_{k+1} = P_C(x_k - \tau F(x_k)), \quad (1.2) \]

where, \( P_C \) is the projection operator on \( C \), \( F \) is strongly monotone, \( L \)-Lipschitzian and \( \tau \) is a sufficiently small positive number. Further, to weaken the constraint on \( F \), Korpelevich and Antipin [1] proposed the following extragradient method (EGM) to solve monotone variational inequality problem:

\[
\begin{aligned}
& x_0 \in C, \\
& y_n = P_C(x_n - \tau Fx_n), \\
& x_{n+1} = P_C(x_n - \tau Fy_n),
\end{aligned}
\quad (1.3)\]

where, \( F : H \to H \) is monotone and \( L \)-Lipschitzian, \( \tau \in (0, \frac{1}{L}) \). What’s more, many authors have extended the algorithm (1.3) from Hilbert spaces to Banach spaces (see, for instance [2]). Because it is difficult to compute two projections at each iteration step in algorithm (1.3), so Tseng proposed Tseng’s extragradient method (TEGM) [9] that involves only one projection as follows to solve monotone variational inequality problem.

\[
\begin{aligned}
& x_0 \in H, \\
& y_n = P_C(x_n - \tau Fx_n), \\
& x_{n+1} = y_n - \tau (Fy_n - Fx_n),
\end{aligned}
\quad (1.4)\]

where, \( F : H \to H \) is monotone and \( L \)-Lipschitzian on \( H \), \( \tau \in (0, \frac{1}{L}) \).

In 2020, Shehu [17] improved the algorithm (1.4) to investigate the convergence of variational inequality in \( 2 \)-uniformly convex Banach space when \( F \) is \( L \)-Lipschitzian and monotone. The iteration is as follows:

\[
\begin{aligned}
& x_1 \in E, \\
& y_n = \Pi_CJ^{-1}(Jy_n - \tau Fx_n), \\
& x_{n+1} = J^{-1}[Jy_n - \tau (Fy_n - Fx_n)],
\end{aligned}
\quad (1.5)\]

where, \( F : E \to E^* \) is monotone and \( L \)-Lipschitzian on \( E \), \( J : E \to E^* \) is the normalized dual mapping, \( 0 < \tau < \frac{1}{\sqrt{2uKL}} \), and \( \Pi_C \) is the generalized projection on \( C \).

Since pseudomonotone mapping is more general than monotone mapping, so many scholars have paid attentions to study pseudomonotone variational inequality problem. For example, in 2012, Yao and Postolache [16] proposed an iterative algorithm to find the common element of the set of solution of

In 2022, Hu [4] modified Tseng’s extragradient algorithm to study the strong convergence of solution of pseudomonotone variational inequality problem in Hilbert spaces. The iterative algorithm is as follows:

\[
\begin{cases}
  x_1 \in E, \\
y_n = P_C(x_n - \frac{k}{\lambda_n}Ax_n), \\
x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)z_n, \\
\lambda_{n+1} = \max\left\{\frac{\|Ax_n - Ay_n\|}{\mu \|x_n - y_n\|}, \lambda_n\right\}, \text{if } x_n - y_n \neq 0, \\
\lambda_n, \text{ otherwise,} \\
z_n = y_n + \frac{k}{\lambda_n}(Ax_n - Ay_n),
\end{cases}
\]

where, \(\mu \in (0, 1), k > 0,\) and \(F : H \to H\) is a pseudomonotone operator.

Motivated and inspired by the work of [17] and [4], we extend the main result in [8] from Hilbert spaces to 2-uniformly convex Banach spaces. And the strong convergence of the solution of the pseudomonotone variational inequality problem was obtained when the cost function \(F\) is pseudomonotone and \(L-\)Lipschitzian.

2 Preliminaries

Let \(E^*\) be the dual space of a real Banach space \(E\), and \(J\) be the normalized dual mapping of \(E\) into \(E^*\) (see [14]), which is defined as follows:

\[Jx = \{x^* \in E^* \mid \langle x^*, x \rangle = \|x^*\|^2 = \|x\|^2\}, \quad \forall x \in E.\]

In particular, if \(E\) is smooth, then \(J\) is single-valued; if \(E\) is uniformly smooth, then \(J\) is norm-norm uniformly continuous.

Let \(S_E\) and \(B_E\) be the unit sphere and the closed unit sphere of Banach space \(E\), respectively, \(E\) is said to be smooth if for all \(x, y \in S_E\)

\[
\lim_{n \to \infty} \frac{\|x + ty\| - \|x\|}{t}
\]

exists. The space \(E\) is said to be uniformly smooth if (2.1) converges uniformly in \(x, y \in S_E\). Moreover, \(E\) is said to be strictly convex if \(\|(x + y)/2\| < 1\) whenever \(x, y \in S_E\) and \(x \neq y\); \(E\) is said to be uniformly convex if \(\delta_E(\varepsilon) > 0, \ \forall \varepsilon \in (0, 2]\), where \(\delta_E\) is the modulus of convexity of \(E\) defined by \(\delta_E(\varepsilon) = \inf\{1 - \|\frac{x+y}{2}\| \mid x, y \in B_E, \ \|x - y\| > \varepsilon\} \) for all \(\varepsilon \in [0, 2]\); And the space \(E\) is
said to be 2-uniformly convex if there exists $c > 0$ such that $\delta_E(\varepsilon) \geq c\varepsilon^2$ for all $\varepsilon \in [0, 2]$. It is well known that $E^*$ is 2-uniformly smooth if and only if $E$ is 2-uniformly convex. It is obvious that every 2-uniformly convex Banach space is uniformly convex.

Let $C$ be a nonempty closed and convex subset of $E$. The following functional $\phi : E \times E \to \mathbb{R}$ was introduced in [12, 13, 19] when $E$ is a smooth Banach space

$$\phi(x, y) = \|x\|^2 - 2 \langle x, Jy \rangle + \|y\|^2, \quad (2.2)$$

clearly, $\phi(x, y) \geq (\|x\| - \|y\|)^2 \geq 0$.

The operator $\Pi_C : E \to C$ is called generalized projection if for each $x \in E$, there corresponds a unique element $x_0 \in C$ (denoted by $\Pi_C(x)$) such that $\phi(x_0, x) = \min_{y \in C} \phi(y, x)$. And for more details on the existence of $\Pi_C$, see [19].

**Lemma 2.1** [7] Let $\frac{1}{p} + \frac{1}{q} = 1$, $p, q > 1$. The space $E$ is $q$-uniformly smooth if and only if its dual $E^*$ is $p$-uniformly convex.

**Lemma 2.2** [5] Let $E$ be a real Banach space. The following two statements are equivalent:

1. $E$ is 2-uniformly smooth;
2. There exists constant $k > 0$ such that for any $x, y \in E$,

$$\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, J(x) \rangle + 2k^2\|y\|^2, \quad (2.3)$$

where $k$ is the 2-uniform smoothness constant. In Hilbert space $k = \frac{1}{\sqrt{2}}$.

**Lemma 2.3** [18] Let $E$ be a real uniformly convex and smooth Banach space, then the following identities hold:

(a) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2 \langle x - z, Jz - Jy \rangle, \forall x, y, z \in E$;
(b) $\phi(x, y) + \phi(y, x) = 2 \langle x - y, Jx - Jy \rangle, \forall x, y \in E$.

In addition, the functional $V(x, y) : E \times E^*$ is defined by

$$V(x, y) = \|x\|^2_E - 2 \langle x, y \rangle + \|y\|^2_{E^*}. \quad (2.4)$$

Then, it is easy to see that

$$V(x, y) = \phi(x, J^{-1}y), \forall x \in E, y \in E^*. \quad (2.5)$$

**Definition 2.4** Let $C \subseteq E$ be a nonempty subset of a real Banach space $E$. Then the mapping $F : X \to E^*$ is called

(a) monotone on $C$ if $\langle Fx - Fy, x - y \rangle \geq 0, \forall x, y \in C$;
(b) pseudomonotone on $C$ if $\langle Fx, y - x \rangle \geq 0 \Rightarrow \langle Fy, y - x \rangle \geq 0$;
(c) $L$-Lipschitzian on $C$ if there exists constant $L > 0$ such that for any $x, y \in C$ $\|Fx - Fy\| \leq L\|x - y\|$.
Lemma 2.5 [18] Let $C \subseteq E$ be a nonempty subset of a smooth and strictly convex Banach space $E$. The following results hold:

(a) $\tilde{z} \in C$ is the generalized projection of $z$ on $C$ if and only if the following inequality holds

\[ \langle w - \tilde{z}, J\tilde{z} - Jz \rangle \geq 0, \forall w \in C; \]

(b) $\Pi_C z = \tilde{z}$ if and only if $\phi(w, \tilde{z}) \leq \phi(w, z) - \phi(\tilde{z}, z)$ for any $w \in E$;

(c) $V(x, x^*) + 2 \langle J^{-1}x^* - x, y^* \rangle \leq V(x, x^* + y^*)$ for any $x \in E$, $x^*, y^* \in E$.

Lemma 2.6 [8] If $E$ is a 2-uniformly convex Banach space, then there exists constant $\mu \geq 1$ such that

\[ \frac{1}{\mu} ||x - y||^2 \leq \phi(x, y), \forall x, y \in E. \]

Lemma 2.7 [11] Let $C$ be a nonempty closed and convex subset of space $E$. Let $F : C \to E^*$ be a continuous, pseudomonotone mapping, $\Gamma$ be the set of solution of $VI(C, F)$ and $z \in C$, then $z \in \Gamma \iff \langle Fx, x - z \rangle \geq 0$ for any $x \in C$.

Lemma 2.8 [6] Let $\{a_n\}$ be a nonnegative real sequence and there exists positive integer $N$ such that $a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_nb_n$ as $n \geq N$, if $\{\alpha_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{b_n\}$ is a sequence satisfying $\lim_{n \to \infty} b_n \leq 0$, then, $\lim_{n \to \infty} a_n = 0$.

Lemma 2.9 [15] Let $\{a_n\}$ be a sequence of real numbers and there exist subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all positive integers $i$. Then there exists a nondecreasing sequence $\{m_k\} \subset N$ such that $m_k \to \infty$ and the following properties are satisfied for positive integers $k$ (sufficiently large)

\[ a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}. \]

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

3 Main Results

In this section, we make the following assumptions.

Assumption 3.1 $C$ is a nonempty closed and convex subset of real $2$–uniformly convex and uniformly smooth Banach space $E$ with $2$-uniform smoothness constant $k$, and $\mu$ is the constant appeared in Lemma 2.6;

Assumption 3.2 $F : E \to E^*$ is pseudomonotone and $L$-Lipschitzian;

Assumption 3.3 The solution set $\Gamma$ of $VI(C, F)$ is nonempty.

Now, we introduce our algorithm as follows.

Algorithm 3.4 Let Assumption 3.1–3.3 hold, $\{\alpha_n\}$ be a real sequence in $(0, 1)$. Taking $m > 0$, $\theta \in (0, 1)$, and $x_1 \in E$ is arbitrarily chosen.

Step 1 Compute:

\[ y_n = \Pi_C J^{-1}(Jx_n - \frac{m}{\lambda_n}Fx_n), \quad (3.1) \]
stop if \( y_n = x_n \). Otherwise, go to Step 2.

**Step 2** Compute:

\[
  z_n = J^{-1}(Jy_n + \frac{m}{\lambda_n}(Fx_n - Fy_n)), \\
  \text{(3.2)}
\]

where

\[
  \lambda_{n+1} = \begin{cases} 
  \max\{\frac{2\mu k\|Fx_n - Fy_n\|}{\theta\|x_n - y_n\|}, \lambda_n\}, & \text{if } x_n - y_n \neq 0, \\
  \lambda_n, & \text{otherwise.}
\end{cases} \\
  \text{(3.3)}
\]

**Step 3** Compute:

\[
  x_{n+1} = J^{-1}(\alpha_n Jx_1 + (1 - \alpha_n)Jz_n). \\
  \text{(3.4)}
\]

Set \( n := n + 1 \), and go to Step 1.

**Lemma 3.5** If Assumption 3.1–3.3 hold, then sequence \( \{\lambda_n\} \) generated by Algorithm 3.4 is nondecreasing, and

\[
  0 < \lim_{n \to \infty} \lambda_n = \lambda \leq \max\{\frac{2\mu kL}{\theta}, \lambda_1\}.
\]

**Proof:** It is obviously seen that \( \{\lambda_n\} \) is nondecreasing from (3.3). And for some \( n \in N \), if \( x_n = y_n \), then from (3.1) we have \( x_n = \Pi_C J^{-1}(Jx_n - \frac{m}{\lambda_n}Fx_n) \) and by Lemma 2.5(a) we obtain that

\[
  \langle w - x_n, Jx_n - Jx + \frac{m}{\lambda_n}Fx_n \rangle = \frac{m}{\lambda_n} \langle w - x_n, Fx_n \rangle \geq 0, \forall w \in C.
\]

Due to \( \lambda_n > 0 \), we have \( x_n \in \Gamma \). On the other hand, when \( x_n \neq y_n \), since \( F \) is \( L \)-Lipschitzian, so we can get

\[
  \frac{2\mu k\|Fx_n - Fy_n\|}{\theta\|x_n - y_n\|} \leq \frac{2\mu kL\|x_n - y_n\|}{\theta\|x_n - y_n\|} = \frac{2\mu kL}{\theta}.
\]

From (3.3), we have

\[
  \lambda_{n+1} = \max\{\frac{2\mu k\|Fx_n - Fy_n\|}{\theta\|x_n - y_n\|}, \lambda_n\} \leq \max\{\frac{2\mu kL}{\theta}, \lambda_n\}.
\]

By induction, we get that \( \lambda_{n+1} \leq \max\{\frac{2\mu kL}{\theta}, \lambda_1\} \). Therefore, \( \{\lambda_n\} \) has upper bound and there exists \( \lambda > 0 \), such that \( \lim_{n \to \infty} \lambda_n = \lambda \leq \max\{\frac{2\mu kL}{\theta}, \lambda_1\} \).

**Lemma 3.6** If Assumption 3.1–3.3 hold, the sequence \( \{x_n\} \) is generated by Algorithm 3.4, and \( m \in (0, \frac{\sqrt{2\mu}}{2\theta}) \), then \( \{x_n\} \) is bounded.

**Proof** Let \( x^* \in \Gamma \), it follows from the definition of \( \phi \) and (2.5) that

\[
  \phi(x^*, z_n) = V(x^*, Jy_n + \frac{m}{\lambda_n}(Fx_n - Fy_n)) \\
  = \|x^*\|^2 - 2\left\langle x^*, Jy_n + \frac{m}{\lambda_n}(Fx_n - Fy_n) \right\rangle + \|Jy_n + \frac{m}{\lambda_n}(Fx_n - Fy_n)\|^2.
\]

(3.5)
Pseudomonotone variational inequality problem

From Lemma 2.1 we know that $E^*$ is 2-uniformly smooth. So by Lemma 2.2, we get

$$
\|Jy_n + \frac{m}{\lambda_n} (Fx_n - Fy_n)\|^2 \leq \|Jy_n\|^2 + 2 \frac{m}{\lambda_n} \langle Jy_n, Fx_n - Fy_n \rangle + \frac{2k^2m^2}{\lambda_n^2} \|Fx_n - Fy_n\|^2.
$$  (3.6)

Substituting (3.6) into (3.5), we have

$$
\phi(x^*, z_n) \leq \|x^*\|^2 - 2 \langle x^*, Jy_n \rangle - 2 \frac{m}{\lambda_n} \langle x^*, Fx_n - Fy_n \rangle
+ \|Jy_n\|^2 + 2 \frac{m}{\lambda_n} \langle Jy_n, Fx_n - Fy_n \rangle + \frac{2k^2m^2}{\lambda_n^2} \|Fx_n - Fy_n\|^2
\leq \phi(x^*, y_n) + 2 \frac{m}{\lambda_n} \langle Jy_n - x^*, Fx_n - Fy_n \rangle + \frac{2k^2m^2}{\lambda_n^2} \|Fx_n - Fy_n\|^2.
$$  (3.7)

In addition, from Lemma 2.3(a), we get

$$
\phi(x^*, y_n) = \phi(x^*, x_n) + \phi(x_n, y_n) + 2 \langle x^* - x_n, Jx_n - Jy_n \rangle.
$$  (3.8)

Substituting (3.8) into (3.7), we obtain that

$$
\phi(x^*, z_n) = \phi(x^*, x_n) + \phi(x_n, y_n) + 2 \langle x^* - x_n, Jx_n - Jy_n \rangle
+ 2 \frac{m}{\lambda_n} \langle Jy_n - x^*, Fx_n - Fy_n \rangle + \frac{2k^2m^2}{\lambda_n^2} \|Fx_n - Fy_n\|^2
\leq \phi(x^*, x_n) + \phi(x_n, y_n) + 2 \langle x^* - y_n, Jx_n - Jy_n \rangle + 2 \langle y_n - x_n, Jx_n - Jy_n \rangle
+ 2 \frac{m}{\lambda_n} \langle y_n - x^*, Fx_n - Fy_n \rangle + \frac{2k^2m^2}{\lambda_n^2} \|Fx_n - Fy_n\|^2.
$$  (3.9)

It follows from Lemma 2.5(a) and (3.1) that

$$
0 \leq \left< x^* - y_n, Jy_n - Jx_n + \frac{m}{\lambda_n} Fx_n \right> = \left< x^* - y_n, Jy_n - Jx_n \right> + \left< x^* - y_n, \frac{m}{\lambda_n} Fx_n \right>,
$$
so

$$
\left< x^* - y_n, Jx_n - Jy_n \right> \leq - \frac{m}{\lambda_n} \langle y_n - x^*, Fx_n \rangle.
$$  (3.10)

Applying Lemma 2.3(b), we have

$$
\phi(x_n, y_n) = -\phi(y_n, x_n) + 2 \langle x_n - y_n, Jx_n - Jy_n \rangle.
$$  (3.11)
Combining (3.9) with (3.11) and (3.10), we get
\[
\phi(x^*, z_n) \leq \phi(x^*, x_n) - \phi(y_n, x_n) + 2 \langle x_n - y_n, Jx_n - Jy_n \rangle + 2 \langle x^* - y_n, Jx_n - Jy_n \rangle \\
+ 2 \langle y_n - x_n, Jx_n - Jy_n \rangle + \frac{2m}{\lambda_n} \langle y_n - x^*, Fx_n - Fy_n \rangle \\
+ \frac{2k^2m^2}{\lambda^2_n} \|Fx_n - Fy_n\|^2 \\
\leq \phi(x^*, x_n) - \phi(y_n, x_n) - \frac{2m}{\lambda_n} \langle y_n - x^*, Fx_n \rangle + \frac{2m}{\lambda_n} \langle y_n - x^*, Fx_n - Fy_n \rangle \\
+ \frac{2k^2m^2}{\lambda^2_n} \|Fx_n - Fy_n\|^2 \\
= \phi(x^*, x_n) - \phi(y_n, x_n) - \frac{2m}{\lambda_n} \langle y_n - x^*, Fy_n \rangle + \frac{2k^2m^2}{\lambda^2_n} \|Fx_n - Fy_n\|^2.
\]

Since \(F\) is pseudomonotone and \(x^* \in \Gamma\), so \(\langle Fx^*, y_n - x^* \rangle \geq 0 \Rightarrow \langle Fy_n, y_n - x^* \rangle \geq 0\). Thus, the following inequality holds
\[
\phi(x^*, z_n) \leq \phi(x^*, x_n) - \phi(y_n, x_n) + \frac{2k^2m^2}{\lambda^2_n} \|Fx_n - Fy_n\|^2.
\]

Then, it follows from (3.3) that
\[
\phi(x^*, z_n) \leq \phi(x^*, x_n) - \phi(y_n, x_n) + \frac{2k^2m^2}{\lambda^2_n} \frac{\theta^2}{4k^2\mu^2} \lambda^2_{n+1} \|x_n - y_n\|^2 \\
\leq \phi(x^*, x_n) - \phi(y_n, x_n) + \frac{m^2\theta^2}{2\mu^2} \left(\frac{\lambda_{n+1}}{\lambda_n}\right)^2 \|y_n - x_n\|^2 \\
\leq \phi(x^*, x_n) - \left[1 - \frac{m^2\theta^2}{2\mu} \left(\frac{\lambda_{n+1}}{\lambda_n}\right)^2\right] \phi(y_n, x_n).
\]

From Lemma 3.5 we know that \(\lim_{n \to \infty} \lambda_n = \lambda > 0\), so \(\lim_{n \to \infty} \frac{\lambda_n}{\lambda_{n+1}} = 1\). Without loss of generality, we may assume that \(\frac{\lambda_n}{\lambda_{n+1}} \geq \frac{1}{2}\) for all positive integer \(n\), then we have \(\frac{\lambda_{n+1}}{\lambda_n} \leq 2\). Hence, we can get that
\[
\phi(x^*, z_n) \leq \phi(x^*, x_n) - \left[1 - \frac{2m^2\theta^2}{\mu}\right] \phi(y_n, x_n).
\]

Since \(0 < m < \frac{\sqrt{2\mu}}{2\theta}\), so we have \(\left[1 - \frac{2m^2\theta^2}{\mu}\right] \phi(y_n, x_n) > 0\). Then we obtain that
\[
\phi(x^*, z_n) \leq \phi(x^*, x_n).
\]
It follows from (2.5) (3.4) and (3.14) that
\[ \phi(x^*, x_{n+1}) = V(x^*, \alpha_n Jx_1 + (1 - \alpha_n) Jz_n) \]
\[ \leq \alpha_n \phi(x^*, x_1) + (1 - \alpha_n) \phi(x^*, z_n) \]
\[ \leq \alpha_n \phi(x^*, x_1) + (1 - \alpha_n) \phi(x^*, x_n) \]
\[ \leq \max\{\phi(x^*, x_1), \phi(x^*, x_n)\} \]
\[ \leq \ldots \]
\[ = \phi(x^*, x_1). \]

Hence \( \{x_n\} \) is bounded. The Lemma 3.6 is proved.

**Lemma 3.7** Let Assumption 3.1–3.3 hold, the sequences \( \{x_n\} \) and \( \{y_n\} \) be generated by Algorithm 3.4. If \( \lim_{n \to \infty} ||x_n - y_n|| = 0 \), and \( \{x_{n_k}\} \) converges weakly to some \( z \in C \), then \( z \in \Gamma \).

**Proof:** It is clear that \( y_{n_k} \rightharpoonup z \) and \( J \) is norm-norm uniformly continuous. Since \( F \) is pseudomonotone and by Lemma 2.5(a) and (3.1), we can get that
\[ 0 \leq \left\langle x - y_{n_k}, Jy_{n_k} - Jx_{n_k} + \frac{m}{\lambda_{n_k}} Fx_{n_k} \right\rangle \]
\[ = \left\langle x - y_{n_k}, Jy_{n_k} - Jx_{n_k} \right\rangle + \frac{m}{\lambda_{n_k}} \left\langle x - y_{n_k}, Fx_{n_k} \right\rangle \]
\[ = \left\langle x - y_{n_k}, Jy_{n_k} - Jx_{n_k} \right\rangle + \frac{m}{\lambda_{n_k}} \left\langle x_{n_k} - y_{n_k}, Fx_{n_k} \right\rangle + \frac{m}{\lambda_{n_k}} \left\langle x - x_{n_k}, Fx \right\rangle \]
\[ \leq \left\langle x - y_{n_k}, Jy_{n_k} - Jx_{n_k} \right\rangle + \frac{m}{\lambda_{n_k}} \left\langle x_{n_k} - y_{n_k}, Fx_{n_k} \right\rangle + \frac{m}{\lambda_{n_k}} \left\langle x - x_{n_k}, Fx \right\rangle . \]  
(3.15)

As \( k \to \infty \) in (3.15), we may obtain \( \langle Fx, x - z \rangle \geq 0 \), \( \forall x \in C \). It follows from Lemma 2.7 that have \( z \in \Gamma \).

**Theorem 3.8** Let Assumption 3.1–3.3 hold, \( \{x_n\}_{n=1}^{\infty} \) be a sequence generated by Algorithm 3.4. If \( m \in (0, \frac{\sqrt{2\mu}}{2\theta}) \), \( \{\alpha_n\} \subset (0, 1) \), and \( \lim_{n \to \infty} \alpha_n = 0 \), \( \sum_{n=1}^{\infty} \alpha_n = \infty \), then the sequence \( \{x_n\} \) converges strongly to \( x^* = \Pi_{\Gamma}(x_1) \in \Gamma \).

**Proof:** By Lemma 3.6 we know the sequence \( \{x_n\} \) is bounded. Setting \( y^* = -\alpha_n(Jx_1 - Jx^*) \) and \( x^* \in \Gamma \). From Lemma 2.5(c) (3.4) and (3.14), we
have
\[ \phi(x^*, x_{n+1}) = V(x^*, \alpha_n J x_1 + (1 - \alpha_n) J z_n) \]
\[ \leq V(x^*, \alpha_n J x_1 + (1 - \alpha_n) J z_n + y^*) - 2 \langle x_{n+1} - x^*, y^* \rangle \]
\[ \leq V(x^*, \alpha_n J x_1 + J z_n - \alpha_n J z_n - \alpha_n J x_1 + \alpha_n J x^*) \]
\[ + 2\alpha_n \langle x_{n+1} - x^*, J x - J x^* \rangle \]
\[ = V(x^*, (1 - \alpha_n) J z_n + \alpha_n J x^*) + 2\alpha_n \langle x_{n+1} - x^*, J x_1 - J x^* \rangle \]
\[ \leq V(x^*, (1 - \alpha_n) J z_n) + V(x^*, \alpha_n J x^*) + 2\alpha_n \langle x_{n+1} - x^*, J x_1 - J x^* \rangle \]
\[ = (1 - \alpha_n) V(x^*, J z_n) + 2\alpha_n \langle x_{n+1} - x^*, J x_1 - J x^* \rangle \]
\[ \leq (1 - \alpha_n) V(x^*, J x_n) + 2\alpha_n \langle J x_1 - J x^*, x_{n+1} - x^* \rangle \]
\[ = (1 - \alpha_n) \phi(x^*, x_n) + 2\alpha_n \langle J x_1 - J x^*, x_{n+1} - x^* \rangle. \]  
(3.16)

The rest proof is divided into two cases:

**Case 1** Suppose that there exits \( n_0 \in \mathbb{N} \) such that \( \{\phi(x^*, x_n)\}_{n=n_0}^{\infty} \) is non-increasing, then the sequence \( \{\phi(x^*, x_n)\}_{n=1}^{\infty} \) converges, and \( \phi(x^*, x_n) \to \phi(x^*, x_{n+1}) \to 0 \) as \( n \to \infty \). Using (3.13), we get that
\[ \phi(x^*, x_{n+1}) \leq \alpha_n \phi(x^*, x_1) + (1 - \alpha_n) \phi(x^*, z_n) \]
\[ \leq \alpha_n \phi(x^*, x_1) + (1 - \alpha_n) \phi(x^*, x_n) - (1 - \alpha_n)(1 - \frac{2m^2\theta^2}{\mu}) \phi(y_n, x_n). \]

It follows that \( (1 - \alpha_n)(1 - \frac{2m^2\theta^2}{\mu}) \phi(y_n, x_n) \leq \phi(x^*, x_n) - \phi(x^*, x_{n+1}) + \alpha_n M_1 \), where, \( M_1 = \phi(x^*, x_1) + \sup \phi(x^*, x_n) \). Since \( m \in (0, \frac{\sqrt{\phi}}{\theta}) \), so we have \( 1 - \frac{2m^2\theta^2}{\mu} > 0 \). Then we obtain that \( (1 - \alpha_n)(1 - \frac{2m^2\theta^2}{\mu}) \phi(y_n, x_n) \leq \phi(x^*, x_n) - \phi(x^*, x_{n+1}) \to 0 \) as \( n \to \infty \). Hence, \( \phi(y_n, x_n) \to 0 \), as \( n \to \infty \). From Lemma 2.6, we know that
\[ \|y_n - x_n\| \to 0, \ n \to \infty. \]  
(3.17)

It follows from (3.2) and (3.17) that
\[ \|J z_n - J y_n\| = \|J y_n + \frac{m}{\lambda_n} (F x_n - F y_n) - J y_n\| \]
\[ = \frac{m}{\lambda_n} \|F x_n - F y_n\| \leq \frac{mL}{\lambda_n} \|y_n - x_n\| \to 0, \ n \to \infty. \]

In addition, since \( J \) is norm-norm uniformly continuous, we have
\[ \|z_n - y_n\| \to 0, \ n \to \infty. \]  
(3.18)

From (3.2) we can get that
\[ \|J x_{n+1} - J z_n\| = \|\alpha_n J x_1 + J z_n - \alpha_n J z_n - J z_n\| = \alpha_n \|J x_1 - J z_n\| \to 0, \ n \to \infty. \]
Further, we also have
\[ \|x_{n+1} - z_n\| \to 0, \ n \to \infty. \] (3.19)

Since
\[ \|x_{n+1} - x_n\| \leq \|x_{n+1} - z_n\| + \|z_n - y_n\| + \|y_n - x_n\|. \] (3.20)
From (3.17) (3.18) (3.19) and (3.20), we have
\[ \|x_{n+1} - x_n\| \to 0, \ n \to \infty. \] (3.21)
Since \( \{x_n\} \) is bounded, there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \), which converges weakly to some \( z \in E \), such that
\[ \lim sup_{n \to \infty} \langle Jx_1 - Jx^*, x_n - x^* \rangle = \lim_{k \to \infty} \langle Jx_1 - Jx^*, x_{n_k} - x^* \rangle = \langle Jx_1 - Jx^*, z - x^* \rangle. \] (3.22)
By Lemma 3.7 and (3.17) we have \( z \in \Gamma \). Since \( x^* = \Pi_\Gamma(x_1) \), from (3.22) and Lemma 2.5 we can get that
\[ \lim sup_{n \to \infty} \langle Jx_1 - Jx^*, x_n - x^* \rangle = \langle Jx_1 - Jx^*, z - x^* \rangle \leq 0. \] (3.23)
From (3.21) and (3.23), the following inequalities hold
\[ \lim sup_{n \to \infty} \langle Jx_1 - Jx^*, x_{n+1} - x^* \rangle \leq \lim sup_{n \to \infty} \langle Jx_1 - Jx^*, x_{n+1} - x_n \rangle 
+ \lim sup_{n \to \infty} \langle Jx_1 - Jx^*, x_n - x^* \rangle \] (3.24)
\[ = \langle Jx_1 - Jx^*, z - x^* \rangle \leq 0. \]
Thus, it follows from (3.16) (3.24) and Lemma 2.8 that \( \lim_{n \to \infty} \phi(x^*, x_n) = 0 \). Therefore, from Lemma 2.6, we can conclude that \( \{x_n\} \) converges strongly to \( x^* \).

**Case 2:** Suppose that there exists a subsequence \( \{x_{n_j}\} \) of \( \{x_n\} \) such that
\[ \phi(x^*, x_{n_j}) < \phi(x^*, x_{n_{j+1}}), \ \forall j \in N. \]
From Lemma 2.9, we know that there exists a nondecreasing sequence \( \{n_k\} \) of \( N \) such that \( \lim_{k \to \infty} n_k = \infty \), and the following inequalities hold for all positive integer \( k \)
\[ \phi(x^*, x_{n_k}) \leq \phi(x^*, x_{n_k+1}), \ \text{and} \ \phi(x^*, x_k) \leq \phi(x^*, x_{n_k+1}). \] (3.25)
By (3.14) and (3.25), we can obtain that
\[ \phi(x^*, x_{n_k}) \leq \phi(x^*, x_{n_k+1}) \leq \alpha_{n_k} \phi(x^*, x_1) + (1 - \alpha_{n_k}) \phi(x^*, z_{n_k}) 
\leq \alpha_{n_k} \phi(x^*, x_1) + (1 - \alpha_{n_k}) \phi(x^*, x_{n_k}). \] (3.26)
Since \( \lim_{k \to \infty} \alpha_{n_k} = 0 \), we have \( \phi(x^*, x_{n_k+1}) - \phi(x^*, x_{n_k}) \to 0 \) as \( k \to \infty \). Since \( \{x_{n_k}\} \) is bounded, there exists a subsequence of \( \{x_{n_k}\} \) still denoted by \( \{x_{n_k}\} \) which converges weakly to \( z \in E \). Next, the same as the arguments in Case 1, we can get that
\[
\|x_{n_k+1} - x_{n_k}\| \to 0, \quad k \to \infty,
\]
and
\[
\limsup_{k \to \infty} \langle x_{n_k+1} - x^*, Jx_1 - Jx^* \rangle = \limsup_{k \to \infty} \langle x_{n_k} - x^*, Jx_1 - Jx^* \rangle \leq 0. \tag{3.27}
\]
Combining (3.16) with (3.25), we have
\[
\phi(x^*, x_{n_k+1}) \leq (1 - \alpha_{n_k})\phi(x^*, x_{n_k}) + 2\alpha_{n_k} \langle Jx_1 - Jx^*, x_{n_k+1} - x^* \rangle \\
\leq (1 - \alpha_{n_k})\phi(x^*, x_{n_k+1}) + 2\alpha_{n_k} \langle x_{n_k+1} - x^*, Jx_1 - Jx^* \rangle.
\]
From (3.25), we get that
\[
\phi(x^*, x_{n_k}) \leq \phi(x^*, x_{n_k+1}) \leq 2 \langle x_{n_k+1} - x^*, Jx_1 - Jx^* \rangle.
\]
Moreover, from (3.27) we obtain
\[
\limsup_{k \to \infty} \phi(x^*, x_{n_k}) \leq 2 \limsup_{k \to \infty} \langle x_{n_k+1} - x^*, Jx_1 - Jx^* \rangle \leq 0. \tag{3.28}
\]
So, it follows from (3.16) (3.28), Lemma 2.8 and Lemma 2.6 that \( \{x_n\} \) converges strongly to \( x^* \). The proof is completed.

References


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