The Impact of Fear Effect on the Dynamics of a Double Delays Predator-Prey Model with Stage Structure, Cooperation and Refuge

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Abstract

In this paper, we consider a predator-prey model with fear effect, cooperation, stage structure and refuge. The local stability and Hopf bifurcation of positive equilibrium point with time delay as parameter are discussed under different conditions of time delay. When the time delay is equal to the critical value, Hopf bifurcation occurs at the positive equilibrium point of the model. Finally, the previous findings are verified by numerical simulation.

Keywords: Predator-prey model; Holling type III; Stage structure; Delay; Stability

1 Introduction

In the ecosystem, each population plays a vital role and has the nature of predator and prey. Predator-prey relationship in food chain is one of the important relationships in ecosystem. Since the 1920s, Lotka [1] and Volterra [2] put forward the most classical predator-prey model, from which the predator-prey model has been widely studied and concerned, and many different results

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have been obtained [3–5]. With the deepening of research and better expression of predation process, scholars have put forward many functional responses to reflect the predation process. In 1959, Holling [6] proposed three different functional responses. In 1975, Beddington [7] and DeAngelis [8] proposed the Beddington-DeAngelis functional response. In [9], a Crowley-Martin functional response is proposed, which increases the interference of predators to the system compared with the Beddington-DeAngelis functional response. These functional responses make the predator-prey model rich in biological characteristics.

As we all know, the development of the current population is closely related to the population in the past period or at a moment. Therefore, the predator-prey model with time delay can better represent the development of biological population and get more complicated dynamic results than that without time delay. Time delay is one of the important factors affecting the predator-prey model, which is manifested in the pregnancy, maturity and feedback of the population. On the other hand, in the process of biological population growth, it has to go through at least juvenile stage and adult stage, and it can show different properties at different stages. Therefore, the predator-prey model with stage structure is more in line with the development law of biological population.

For this stage structure of juvenile population and adult population, the predation ability of juvenile prey is weaker than that of adult prey. In order to ensure the number of prey, juvenile bait and adult bait can cooperate. In the literature [10–12], scholars have studied the predator-prey model with cooperation, and the research shows that cooperation can greatly promote the stability of the system.

In the process of predation, it is a common phenomenon in nature that prey eludes predators. Therefore, shelters have an important impact on the population. On the one hand, shelters will restrict predators from killing prey, which will reduce the predator population and increase the prey population. On the other hand, the prey in the shelter will reduce the chances of eating, thus reducing the prey population. Refuge, an important factor, has attracted the attention of scholars [13–17], and the research shows that refuge plays an important role in the stability of predator-prey model. In addition, the fear of predators is also an important factor. Because the prey is afraid of predators, it not only affects the birth rate of prey, but also affects the mortality rate of prey. Cao et al. [18] considered a time-delayed predator-prey model in which the fear effect affects both the birth rate and the mortality rate of the prey:

\[
\begin{align*}
\frac{dx(t)}{dt} &= \frac{ry}{1+ky} - d_1x - \alpha x, \\
\frac{dy(t)}{dt} &= \alpha x - d_1 \left[1 + \eta \left(1 - \frac{1}{1+ky}\right)\right] y - \sigma_1 y^2 - \beta (1 - r_0) yz, \\
\frac{dz(t)}{dt} &= e\beta (1 - r_0) y(t - \tau) z(t - \tau) - \sigma_2 z^2 - d_2 z,
\end{align*}
\]
Among the three functional responses proposed by Holling [6], Holling type III functional response \( \frac{ax^2}{\beta^2+x^2} \) is suitable for spinal cord animals, so the study of Holling type III functional response is an important direction of predator-prey model [19–21]. Xie et al. [22] considered a predator-prey model with Holling type III with two time delays, in which the time delay is manifested in the pregnancy period of adult prey and predator at time \( t \), and the pregnancy delay of predator respectively.

\[
\begin{align*}
\frac{dx_1(t)}{dt} &= \frac{rx_2}{1+Ky} - d_1x_1 - \beta x_1(t - \tau_1) + \zeta_1 x_1 x_2, \\
\frac{dx_2(t)}{dt} &= \beta x_1(t - \tau_1) - d_1 \left[ 1 + g \left( 1 - \frac{1}{1+Ky} \right) \right] x_2 - \alpha x_2^2 + \zeta_2 x_1 x_2 - \frac{\beta_1 (1-m)^2 x_2^2 y}{p^2 + (1-m)^2 x_2^2}, \\
\frac{dy(t)}{dt} &= \frac{\beta_2 \beta_1 (1-m)^2 x_2^2 (t-\tau_2) y(t-\tau_2)}{p^2 + (1-m)^2 x_2^2 (t-\tau_2)} - d_2 y,
\end{align*}
\]

where \( x_1(t), x_2(t) \) and \( y(t) \) represent the population densities of juvenile prey, adult prey and predator at time \( t \), respectively; \( r \) is the birth rate of juvenile prey; \( K \) is the degree of fear of predators by juvenile prey; \( d_1 \) and \( d_2 \) represent the death rates of juvenile prey and predators, respectively; \( \beta \) D represents the maturity rate of juvenile prey; \( \zeta_1 \) and \( \zeta_2 \) represent the cooperation coefficients of juvenile and adult prey respectively, assuming \( \zeta_1 > \zeta_2 \), because the energy given by adult prey to juvenile prey is greater than that given by juvenile prey to adult bait; \( g \) is the maximum cost of fear; \( \alpha \) represents the intraspecific competition coefficient of adult prey; \( \beta_1 \) is the capture rate of predator; \( \beta_2 \) is the conversion rate of nutrients; \( m \in (0,1) \) is the refuge rate of adult prey; \( p \) is a semi-saturation constant; \( \tau_1 \) and \( \tau_2 \) are the mature delay of juvenile prey and the pregnancy delay of predator respectively.

The initial conditions for the system (3) are

\[
x_1(\theta) = \phi_1(\theta), \quad x_2(\theta) = \phi_2(\theta), \quad y(\theta) = \phi_3(\theta),
\]

\[
\phi_i(\theta) \geq 0, \quad \phi_i(0) > 0, \quad i = 1, 2, 3, \quad \theta \in [-\tau, 0], \quad \tau = \max\{\tau_1, \tau_2\}.
\]

The rest of this paper is organized as follows. In section 2, we discuss the stability of positive equilibrium and Hopf bifurcation under different time delays. In section 3, the main results are illustrated by numerical simulation. Finally, we give a brief conclusion.
2 Local stability and Hopf bifurcation

In this section, we mainly discuss the local stability and Hopf bifurcation of system (3) at the positive equilibrium point $E^*$. If \((H0)C < 0, \beta_1 \beta_2 > d_2, d_1 + \beta > \zeta_1 x_2^*\) hold, then system (3) possesses a positive equilibrium $E^*(x_1^*, x_2^*, y^*)$,

$$x_2^* = \sqrt{\frac{d_2 p^2}{(\beta_1 \beta_2 - d_2)(1 - m)^2}, \quad x_1^* = \frac{r x_2^*}{(d_1 + \beta - \zeta_1 x_2^*)(1 + K y^*)}, \quad y^* = \frac{\sqrt{B^2 - 4AC} - B}{2A}.$$ 

where

\[ A = \frac{K \beta_1 (1 - m)^2 x_2^*}{p^2 + (1 - m)^2 x_2^*}, \quad B = \frac{\beta_1 (1 - m)^2 x_2^*}{p^2 + (1 - m)^2 x_2^*} + K(\alpha x_2^* + d_1 g + d_1), \]

\[ C = d_1 + \alpha x_2^* - \frac{\beta r + \zeta_2 r x_2^*}{d_1 + \beta - \zeta_1 x_2^*}. \]

Linearizing system (3) at the positive equilibrium point $E^*$, we have

$$
\begin{align*}
&x_1'(t) = a_{11} x_1(t) + a_{12} x_2(t) + a_{13} y(t) + b_{11} x_1(t - \tau_1), \\
&x_2'(t) = a_{21} x_1(t) + a_{22} x_2(t) + a_{23} y(t) + b_{21} x_1(t - \tau_1), \\
y'(t) = a_{33} y(t) + c_{32} x_2(t - \tau_2) + c_{33} y(t - \tau_2),
\end{align*}
$$

where

\[ a_{11} = -d_1 + \zeta_2 x_2^*, \quad a_{12} = \frac{r}{1 + K y^*} + \zeta_1 x_1^*, \quad a_{13} = -\frac{K r x_2^*}{(1 + K y^*)^2}, \quad a_{21} = \zeta_2 x_2^* \]

\[ a_{22} = -d_1 \left[ 1 + g \left( 1 - \frac{1}{1 + K y^*} \right) \right] - 2\alpha x_2^* - \frac{2 p^2 \beta_1 (1 - m)^2 x_2^* y^*}{(p^2 + (1 - m)^2 x_2^*)^2} + \zeta_1 x_1^*, \]

\[ a_{23} = -\frac{d_1 g K x_2^*}{(1 + K y^*)^2} - \frac{\beta_1 (1 - m)^2 x_2^*}{p^2 + (1 - m)^2 x_2^*}, \quad a_{33} = -d_2, \quad b_{11} = -\beta, \quad b_{21} = \beta, \]

\[ c_{32} = \frac{2 p^2 \beta_2 \beta_1 (1 - m)^2 x_2^* y^*}{(p^2 + (1 - m)^2 x_2^*)^2}, \quad c_{33} = \frac{\beta_2 \beta_1 (1 - m)^2 x_2^*}{p^2 + (1 - m)^2 x_2^*}. \]

Then, we can get that the characteristic equation of system (3) at $E^*$ is given by

$$
\lambda^3 + M_2 \lambda^2 + M_1 \lambda + M_0 + (N_2 \lambda^2 + N_1 \lambda + N_0) e^{-\lambda \tau_1} + (P_2 \lambda^2 + P_1 \lambda + P_0) e^{-\lambda \tau_2} + (Q_1 \lambda + Q_0) e^{-\lambda (\tau_1 + \tau_2)} = 0, \quad (4)
$$
where

\[ M_2 = -(a_{33} + a_{11} + a_{22}), \quad M_1 = a_{33}a_{11} + a_{33}a_{22} + a_{11}a_{22} - a_{12}a_{21}, \]

\[ M_0 = a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33}, \quad N_2 = -b_{11}, \quad N_1 = (a_{33} + a_{22})b_{11} - a_{12}b_{21}, \]

\[ N_0 = a_{12}a_{33}b_{21} - a_{22}b_{11}a_{33}, \quad P_2 = -c_{33}, \quad P_1 = a_{22}c_{33} + a_{11}c_{33} - a_{23}c_{32}, \]

\[ P_0 = a_{12}a_{21}c_{33} + a_{23}c_{32}a_{11} - a_{13}c_{32}a_{21} - a_{22}a_{11}c_{33}, \quad Q_1 = b_{11}c_{33}, \]

\[ Q_0 = -a_{22}b_{11}c_{33} - a_{13}c_{32}b_{21} + a_{23}c_{32}b_{11} + a_{12}c_{33}b_{21}. \]

**Case I:** \( \tau_1 > 0, \tau_2 = 0. \)

The characteristic equation (4) can be written as

\[
\lambda^3 + (M_2 + P_2)\lambda^2 + (M_1 + P_1)\lambda + (M_0 + P_0) \\
+ (N_2\lambda^2 + (N_1 + Q_1)\lambda + N_0 + Q_0)e^{-\lambda\tau_1} = 0, \tag{5}
\]

Let \( \lambda = i\omega_1(\omega_1 > 0) \) is a root of Equation (5), we have

\[
\begin{align*}
(M_2 + P_2) - (M_2 + P_2)i^2 \\
= N_2\omega_1^2 \cos\omega_1\tau_1 - (N_1 + Q_1)\omega_1 \sin\omega_1\tau_1 - (N_0 + Q_0) \cos\omega_1\tau_1, \\
(M_1 + P_1)\omega_1 - \omega_1^3 \\
= (N_0 + Q_0) \sin\omega_1\tau_1 - N_2\omega_1^2 \sin\omega_1\tau_1 - (N_1 + Q_1)\omega_1 \cos\omega_1\tau_1.
\end{align*} \tag{6}
\]

We can get

\[ \omega_1^6 + R_2\omega_1^4 + R_1\omega_1^2 + R_0 = 0, \tag{7} \]

where

\[ R_2 = (M_2 + P_2)^2 - 2(M_1 + P_1) - N_2^2, \quad R_0 = (M_0 + P_0)^2 - (N_0 + Q_0)^2, \]

\[ R_1 = (M_1 + P_1)^2 - 2(M_0 + P_0)(M_2 + P_2) + 2(N_0 + Q_0)N_2 - (N_1 + Q_1)^2. \]

Let \( u_1 = \omega_1^2 \), equation (4) can be written as

\[ u_1^3 + R_2u_1^2 + R_1u_1 + R_0 = 0. \tag{8} \]

Let \( h_1(u_1) = u_1^3 + R_2u_1^2 + R_1u_1 + R_0 \), then \( h_1'(u_1) = 3u_1^2 + 2R_2u_1 + R_1 \) with \( h_1(0) = R_0 \) and \( \lim_{u_1 \to +\infty} h_1(u_1) = +\infty. \) By using the analytical approach described in [23], we obtain the lemma as follows.

**Lemma 2.1.** For equation (8), we have

(i) if (H1) \( R_0 \geq 0, \Delta = R_2^2 - 3R_1 \leq 0 \) holds, then no positive root exists in equation (8);

(ii) if (H2) \( R_0 \geq 0, \Delta = R_2^2 - 3R_1 > 0, u_1^* = \frac{-R_2+\sqrt{\Delta}}{3} > 0, h_1(u_1^*) \leq 0 \) or (H3) \( R_0 < 0 \) is satisfied, then equation (8) possesses a positive root.
We assume equation (8) has three positive roots \( u_{1l} \) \((l = 1, 2, 3)\). Then, equation (7) has three positive roots \( \omega_{1l} = \sqrt{u_{1l}} \) \((l = 1, 2, 3)\).

For Equation (6), we have

\[
\tau_{1l}^{(j)} = \frac{1}{\omega_{1l}} \arccos \left\{ \frac{N^* \omega_{1l}^4 + M^* \omega_{1l}^2 + U^*}{N_2 \omega_{1l}^4 + L^* \omega_{1l}^2 + Q^*} \right\} + \frac{2j\pi}{\omega_{1l}}, \quad l = 1, 2, 3, \quad j = 0, 1, 2, \ldots
\]

where \( N^* = (Q_1 + N_1) - N_2(M_2 + P_2), \quad M^* = (M_0 + P_0)N_2 + (M_2 + P_2)(Q_0 + N_0) - (M_1 + P_1)(Q_1 + N_1), \quad U^* = -(M_0 + P_0)(N_0 + Q_0), \quad L^* = (Q_1 + N_1)^2 - 2(Q_0 + N_0)N_2, \quad Q^* = (Q_0 + N_0)^2. \) Thus, \( \pm i\omega_{1l} \) is a pair of purely imaginary roots of (5) with \( \tau_1 = \tau_{1l}^{(j)}. \) Define \( \tau^*_1 = \min_{l \in \{1, 2, 3\}} \{\tau_{1l}^{(0)}\} \), \( \omega^*_1 = \omega_{1l_0}. \)

**Lemma 2.2.** If \((H4)h'_1(\omega^*_1) \neq 0\) hold, then the transversality condition

\[
\left\{ \frac{d(Re\lambda)}{d\tau_1} \right\}_{\lambda = i\omega^*_1} \neq 0
\]

**Proof.** Differentiating Equation (5) with respect to \( \tau_1 \), we get

\[
\left( \frac{d\lambda}{d\tau_1} \right)^{-1} = \frac{(3\lambda^2 + 2(M_2 + P_2)\lambda + (M_1 + P_1))e^{\lambda\tau_1} + 2N_1\lambda + (N_1 + Q_1)}{\lambda[N_2\lambda^2 + (N_1 + Q_1)\lambda + (N_0 + Q_0)]} - \frac{\tau_1}{\lambda},
\]

then

\[
Re \left[ \frac{d\lambda}{d\tau_1} \right]_{\lambda = i\omega^*_1}^{-1} = \frac{1}{\Lambda_1} \left\{ 3\omega^*_1 e^{6} + 2[(M_2 + P_2)^2 - N_2^2 - 2(M_1 + P_1)\omega^*_1] + [(M_1 + P_1)^2 - 2(M_0 + P_0)(M_2 + P_2) - (N_1 + Q_1)^2 + 2(N_0 + Q_0)N_2]\omega^*_1 \right\}^{-1}
\]

\[
= \frac{h'_1(\omega^*_1)}{(N_1 + Q_1)^2\omega^*_1 + [(N_0 + Q_0) - N_2\omega^*_1]^2},
\]

where \( \Lambda_1 = (N_1 + Q_1)^2\omega^*_1 + \omega^*_1 [(N_0 + Q_0) - N_2\omega^*_1]^2. \)

We derive from (6) that

\[
[(M_2 + P_2)\omega^*_1 - (M_0 + P_0)]^2 + [(M_1 + P_1)\omega^*_1 - \omega^*_1]^2 = (N_1 + Q_1)^2\omega^*_1 + [(N_0 + Q_0) - N_2\omega^*_1]^2.
\]

We have

\[
\text{sign} \left\{ \frac{d(Re\lambda)}{d\tau_1} \right\}_{\lambda = i\omega^*_1} = \text{sign} \left\{ Re \left( \frac{d\lambda}{d\tau_1} \right)^{-1} \right\}_{\lambda = i\omega^*_1}
\]

\[
= \text{sign} \left\{ \frac{h'_1(\omega^*_1)}{(N_1 + Q_1)^2\omega^*_1 + [(N_0 + Q_0) - N_2\omega^*_1]^2} \right\}.
\]

Therefore, if \((H4)h'_1(\omega^*_1) \neq 0\) hold, then \( \left\{ \frac{d(Re\lambda)}{d\tau_1} \right\}_{\lambda = i\omega^*_1} \neq 0 \), the transversal condition holds, thus, we have the following theorem.
Theorem 2.1. For system (3), when $\tau > 0, \tau_2 = 0$.

(1) If $(H1)$ hold, then the positive equilibrium $E^*$ is locally asymptotically stable for $\tau_1 \geq 0$.

(2) If $(H2)$ or $(H3)$ and $(H4)$ hold, when $\tau_1 \in [0, \tau_1^*)$, then the positive equilibrium $E^*$ is locally asymptotically stable, and the positive equilibrium $E^*$ is unstable when $\tau_1 > \tau_1^*$. Moreover, when $\tau_1 = \tau_1^*$, then the Hopf bifurcation occurs at the positive equilibrium $E^*$.

Case II: $\tau_1 = 0, \tau_2 > 0$.

The characteristic equation (4) can be written as

$$\lambda^3 + (M_2 + N_2)\lambda^2 + (M_1 + N_1)\lambda + (M_0 + N_0) + [P_2\lambda^2 + (P_1 + Q_1)\lambda + (P_0 + Q_0)]e^{-\lambda\tau_2} = 0,$$  \hspace{0.5cm} (9)

The discussion of case II is similar to that of case I. Therefore, we have the following conclusions.

Theorem 2.2. For system (3), when $\tau_1 = 0, \tau_2 > 0$. If $\tau_2 \in [0, \tau_2^*)$, then the positive equilibrium $E^*$ is locally asymptotically stable, and the positive equilibrium $E^*$ is unstable when $\tau_2 > \tau_2^*$. Moreover, when $\tau_2 = \tau_2^*$, then the Hopf bifurcation occurs at the positive equilibrium $E^*$, where $\tau_2^*$ represents the minimum critical value of Hopf bifurcation.

Case III: $\tau_1 = \tau_2 = \tau \neq 0$.

The characteristic equation (4) can be written as

$$e^{\lambda \tau} (\lambda^3 + M_2\lambda^2 + M_1\lambda + M_0) + S_2\lambda^2 + S_1\lambda + S_0 + e^{-\lambda \tau} (Q_1\lambda + Q_0) = 0,$$  \hspace{0.5cm} (10)

where $S_2 = N_2 + P_2, S_1 = N_1 + P_1, S_0 = N_0 + P_0$.

Let $\lambda = i\omega (\omega > 0)$ is a root of Equation (10), we have

$$\begin{cases}
    P_{21}\cos\omega\tau + P_{22}\sin\omega\tau = P_{25}, \\
    P_{23}\cos\omega\tau - P_{24}\sin\omega\tau = P_{26},
\end{cases}$$

where

$$P_{21} = \omega^3 - M_1\omega - Q_1\omega, \hspace{0.5cm} P_{22} = M_2\omega^2 - M_0 + Q_0, \hspace{0.5cm} P_{25} = S_1\omega,$$

$$P_{23} = M_2\omega^2 - M_0 - Q_0, \hspace{0.5cm} P_{24} = \omega^3 - M_1\omega + Q_1\omega, \hspace{0.5cm} P_{26} = S_0 - S_2\omega^2.$$

We can get

$$\begin{cases}
    \cos\omega\tau = \frac{e_{14}\omega^4 + e_{15}\omega^2 + e_{16}}{\omega^6 + d_1\omega^4 + d_2\omega^2 + d_3}, \\
    \sin\omega\tau = \frac{e_{17}\omega^4 + e_{18}\omega^2 + e_{19}}{\omega^6 + d_1\omega^4 + d_2\omega^2 + d_3},
\end{cases}$$  \hspace{0.5cm} (11)

where

$$e_{11} = S_2, \hspace{0.5cm} e_{12} = S_1M_2 - S_0 - S_2(Q_1 + M_1), \hspace{0.5cm} e_{13} = S_0(M_1 + Q_1) - S_1(M_0 + Q_0),$$
\[ e_{14} = S_1 - S_2 M_2, \quad e_{15} = S_2(M_0 - Q_0) + S_0 M_2 + S_1(Q_1 - M_1), \quad e_{16} = S_0(Q_0 - M_0), \]
\[ d_{11} = M_0^2 - 2M_1, \quad d_{12} = M_1^2 - 2M_0 M_2 - Q_1^2, \quad d_{13} = M_0^2 - Q_0^2. \]

For (11), we have
\[ \omega^{12} + m_{15} \omega^{10} + m_{14} \omega^8 + m_{13} \omega^6 + m_{12} \omega^4 + m_{11} \omega^2 + m_{10} = 0, \quad (12) \]
where
\[ m_{15} = 2d_{11} - \epsilon_{11}^2, \quad m_{14} = 2d_{12} + d_{11}^2 - 2\epsilon_{11} \epsilon_{12} - \epsilon_{14}^2, \]
\[ m_{13} = 2d_{13} + 2d_{11} d_{12} - 2(\epsilon_{13} \epsilon_{11} + \epsilon_{15} \epsilon_{14}) - \epsilon_{12}^2, \]
\[ m_{12} = 2d_{11} d_{13} + d_{12}^2 - 2(\epsilon_{13} \epsilon_{12} + \epsilon_{16} \epsilon_{14}) - \epsilon_{15}^2, \]
\[ m_{11} = 2d_{12} d_{13} - \epsilon_{13}^2 - 2\epsilon_{16} \epsilon_{15}, \quad m_{10} = d_{13}^2 - \epsilon_{16}^2. \]

Let \( u = \omega^2 \), then equation (12) can be written as
\[ u^6 + m_{15} u^5 + m_{14} u^4 + m_{13} u^3 + m_{12} u^2 + m_{11} u + m_{10} = 0, \quad (13) \]

Without loss of generality, we assume equation (13) has six positive roots \( u_l \) \( (l = 1, 2, \ldots, 6) \). Then, equation (12) has six positive roots \( \omega_l = \sqrt{u_l} \) \( (l = 1, 2, \ldots, 6) \), for equation (11), we have
\[ \tau_l^{(j)} = \left\{ \frac{1}{\omega_l} \arccos \left( \frac{\epsilon_{14} \omega_l^4 + \epsilon_{15} \omega_l^2 + \epsilon_{16}}{\omega_l^6 + d_{11} \omega_l^4 + d_{12} \omega_l^2 + d_{13}} + \frac{2\pi j}{\omega_l} \right) \right\}, \quad l = 1, 2, \ldots, 6, \quad j = 0, 1, 2, \ldots \]

Thus, \( \pm i \omega_l \) is a pair of purely imaginary roots of (10) with \( \tau = \tau_l^{(j)}. \) Define
\[ \tau^* = \min_{l = 1, 2, \ldots, 6} \{ \tau_l^{(0)} \}, \quad \omega^* = \omega_{10}. \]

**Lemma 2.3.** If \((H5)I_1 I_3 + I_2 I_4 \neq 0\) hold, then the transversality condition \( \left\{ \frac{d(Re \lambda)}{d\tau} \right\}_{\lambda = i \omega^*} \neq 0 \) hold.

**Proof.** Differentiating Equation (10) with respect to \( \tau \), we get
\[ \left( \frac{d\lambda}{d\tau} \right)^{-1} = \frac{2S_2 \lambda + S_1 + (3\lambda^2 + 2M_2 \lambda + M_1) e^{-\lambda \tau} + Q_1 e^{-\lambda \tau} - \tau}{-\lambda^3 + M_2 \lambda + M_1 \lambda + M_0} \]
then
\[ Re \left[ \frac{d\lambda}{d\tau} \right]^{-1} = \frac{I_1 I_3 + I_2 I_4}{I_3^2 + I_4^2}, \]
where
\[ I_1 = (M_1 - 3\omega^* + Q_1) \cos \omega^* \tau^* - 2M_2 \omega^* \sin \omega^* \tau^* + S_1, \]
\[ I_2 = (M_1 - 3\omega^* + 2M_2 \omega^* - Q_1) \sin \omega^* \tau^* + 2S_2 \omega^*, \]
\[ I_3 = (M_1 - Q_1 - \omega^* \omega^* + (Q_0 + M_0 - M_2 \omega^* \omega^*) \omega^* \sin \omega^* \tau^*, \]
\[ I_4 = (M_1 + Q_1 - \omega^2)\omega^2 \sin \omega^\tau + (Q_0 - M_0 + M_2\omega^2)\omega^\cos \omega^\tau \]

Thus, we have

\[
\text{sign}\left\{ \frac{d(Re\lambda)}{d\tau} \right\}_{\lambda=i\omega^*} = \text{sign}\left\{ Re\left( \frac{d\lambda}{d\tau} \right)^{-1} \right\}_{\lambda=i\omega^*} = \text{sign}\left\{ \frac{I_1I_3 + I_2I_4}{I_3^2 + I_4^2} \right\}.
\]

Therefore, if \((H5)I_1I_3 + I_2I_4 \neq 0\) hold, \(\{d(Re\lambda)\}_{\lambda=i\omega^*} \neq 0\), the transversal condition holds, thus, we have the following theorem.

**Theorem 2.3.** For system (3), when \(\tau_1 = \tau_2 = \tau \neq 0\), if \((H0)\) and \((H5)\) hold, when \(\tau \in [0, \tau^*]\), then the positive equilibrium \(E^*\) is locally asymptotically stable, and the positive equilibrium \(E^*\) is unstable when \(\tau > \tau^*\). Moreover, when \(\tau = \tau^*\), then the Hopf bifurcation occurs at the positive equilibrium \(E^*\).

**Case IV:** \(\tau_1 > 0, \ \tau_2 \in [0, \tau^*_1)\) and \(\tau_1 \neq \tau_2\).

Let \(\lambda = i\omega_{10}(\omega_{10} > 0)\) is a root of Equation (4), we have

\[
\begin{align*}
P_{31}\sin \omega_{10}\tau_1 + P_{32}\cos \omega_{10}\tau_1 &= P_{33}, \\
P_{31}\cos \omega_{10}\tau_1 - P_{32}\sin \omega_{10}\tau_1 &= P_{34},
\end{align*}
\]

where

\[
\begin{align*}
P_{31} &= N_1\omega_{10} - Q_0\sin \omega_{10}\tau_2 + Q_1\omega_{10}\cos \omega_{10}\tau_2, \\
P_{32} &= N_0 - N_2\omega_{10}^2 + Q_0\cos \omega_{10}\tau_2 + Q_1\omega_{10}\sin \omega_{10}\tau_2, \\
P_{33} &= M_2\omega_{10}^2 - M_0 + (P_2\omega_{10}^2 - P_0)\cos \omega_{10}\tau_2 - P_1\omega_{10}\sin \omega_{10}\tau_2, \\
P_{34} &= \omega_{10}^2 - M_1\omega_{10} - P_1\omega_{10}\cos \omega_{10}\tau_2 - (P_2\omega_{10}^2 - P_0)\sin \omega_{10}\tau_2.
\end{align*}
\]

We can get

\[
\cos \omega_{10}\tau_1 = \frac{P_{32}P_{33} + P_{31}P_{34}}{P_{32}^2 + P_{31}^2}, \quad \sin \omega_{10}\tau_1 = \frac{P_{33}P_{31} - P_{32}P_{34}}{P_{32}^2 + P_{31}^2}.
\] (14)

For (14), we have

\[
\begin{align*}
\omega_{10}^6 + n_{30}\omega_{10}^4 + n_{31}\omega_{10}^2 + n_{32} + (n_{33}\omega_{10}^4 + n_{34}\omega_{10}^2 + n_{35})\cos \omega_{10}\tau_2 \\
\quad + (n_{36}\omega_{10}^5 + n_{37}\omega_{10}\tau_2 + n_{38}\omega_{10})\sin \omega_{10}^2 = 0,
\end{align*}
\] (15)

where

\[
\begin{align*}
n_{30} &= M_2^2 - 2M_1 + P_2^2 - N_2^2, \quad n_{31} = M_1^2 + P_2^2 - N_1^2 - Q_1^2 + 2(N_0N_2 - M_0M_2 - P_0P_2), \\
n_{32} &= M_0^2 + P_0^2 - N_0^2 - Q_0^2, \quad n_{36} = -2P_2, \quad n_{38} = 2(P_1M_0 - P_0M_1 + N_1Q_0 - N_0Q_1), \\
n_{35} &= 2(M_0P_0 - N_0Q_0), \quad n_{33} = 2M_2P_2 - 2P_1, \quad n_{37} = 2(M_1P_2 + N_2Q_1 + P_0 - P_1M_2),
\end{align*}
\]
Next, we assume the following conditions.

(H6) Eq.(15) has at least finite positive root.

We denote equation (15) has positive roots $\omega^l_{10}$ ($l = 1, 2, \ldots, 6$), for every $\omega^l_{10}$ ($l = 1, 2, \ldots, 6$), we have

$$\tau^{(j)}_{10} = \frac{1}{\omega_{10}} \arccos \left\{ \frac{P_{32}P_{33} + P_{31}P_{34}}{P_{32}^2 + P_{31}^2} \right\}_{\omega_{10} = \omega^{l}_{10}} + \frac{2\pi j}{\omega_{10}}, \quad l = 1, 2, \ldots, 6, \quad j = 0, 1, 2, \ldots$$

Let $\tau_{1*} = \min \{ \tau^{(l(0))}_{10} \mid l = 1, 2, \ldots; j = 0, 1, 2, \ldots \}$, and $\omega_{1*}$ is the corresponding root of (15) with $\tau_{1*}$.

**Lemma 2.4.** If $(H7)H_{11}H_{13} + H_{12}H_{14} \neq 0$ hold, then the transversality condition $\left\{ \frac{d(\lambda \tau)}{d\tau_{1*}} \right\}_{\lambda = i\omega_{1*}} \neq 0$ hold.

**Proof.** Differentiating Equation (4) with respect to $\tau_1$, we get

$$
\begin{aligned}
\left( \frac{d\lambda}{d\tau_1} \right)^{-1} &= 3\lambda^2 + 2M_2\lambda + M_1 + (2N_2\lambda + N_1)e^{-\lambda\tau_1} + (2P_2\lambda + P_1)e^{-\lambda\tau_2} \\
&\quad - \lambda(\lambda^3 + M_2\lambda^2 + M_1\lambda + M_0) - (P_2\lambda^2 + P_1\lambda + P_0)e^{-\lambda\tau_2} \\
&\quad - (P_2\lambda^2 + P_1\lambda + P_0)\tau_2 e^{-\lambda\tau_2} - Q_1 e^{-\lambda(\tau_1 + \tau_2)} \\
&\quad - (N_2\lambda^2 + N_1\lambda + N_0)\tau_2 e^{-\lambda\tau_1} + (\tau_1 + \tau_2)(Q_1\lambda + Q_0)e^{-\lambda(\tau_1 + \tau_2)} \\
&\quad - \lambda(\lambda^3 + M_2\lambda + M_1\lambda + M_0) - (P_2\lambda^2 + P_1\lambda + P_0)e^{-\lambda\tau_2},
\end{aligned}
$$

then

$$
\text{Re} \left[ \frac{d\lambda}{d\tau_1} \right]_{\lambda = i\omega_{1*}}^{-1} = \text{Re} \left( \frac{H_{11} + H_{12}i}{H_{13} + H_{14}i} \right) = \frac{H_{11}H_{13} + H_{12}H_{14}}{H_{13}^2 + H_{14}^2},
$$

where

$$
\begin{aligned}
H_{11} &= M_1 - 3\omega_{1*}^2 + 2N_2\omega_{1*} \sin \omega_{1*} \tau_{1*} + N_1 \cos \omega_{1*} \tau_{1*} \\
&\quad + (-P_1 \omega_{1*} \tau_2 + 2P_2 \omega_{1*} - Q_1 \sin \omega_{1*} \tau_{1*}) \sin \omega_{1*} \tau_2 \\
&\quad + (P_2 \omega_{1*}^2 \tau_2 + P_1 - P_0 \tau_2 + Q_1 \cos \omega_{1*} \tau_{1*}) \cos \omega_{1*} \tau_2,
\end{aligned}
$$

$$
\begin{aligned}
H_{12} &= 2M_2\omega_{1*} + N_1 \sin \omega_{1*} \tau_{1*} + 2N_2\omega_{1*} \cos \omega_{1*} \tau_{1*} \\
&\quad + (-P_1 + P_0 \tau_2 - P_2 \omega_{1*}^2 \tau_2 - Q_1 \cos \omega_{1*} \tau_{1*}) \sin \omega_{1*} \tau_2 \\
&\quad + (2P_2 \omega_{1*}^2 \tau_2 - P_1 \omega_{1*} \tau_2 - Q_1 \sin \omega_{1*} \tau_{1*}) \cos \omega_{1*} \tau_2,
\end{aligned}
$$

$$
\begin{aligned}
H_{13} &= (N_0 \omega_{1*} - N_2\omega_{1*}^3) \sin \omega_{1*} \tau_{1*} + N_1 \omega_{1*}^2 \cos \omega_{1*} \tau_{1*} \\
&\quad + (Q_0 \omega_{1*} \cos \omega_{1*} \tau_{1*} + Q_1 \omega_{1*}^2 \sin \omega_{1*} \tau_{1*}) \sin \omega_{1*} \tau_2 \\
&\quad + (Q_0 \omega_{1*} \sin \omega_{1*} \tau_{1*} - Q_1 \omega_{1*}^2 \cos \omega_{1*} \tau_{1*}) \cos \omega_{1*} \tau_2,
\end{aligned}
$$

$$
\begin{aligned}
H_{14} &= (N_0 \omega_{1*} - N_2\omega_{1*}^3) \cos \omega_{1*} \tau_{1*} + N_1 \omega_{1*}^2 \sin \omega_{1*} \tau_{1*} \\
&\quad + (-Q_0 \omega_{1*} \sin \omega_{1*} \tau_{1*} + Q_1 \omega_{1*} \cos \omega_{1*} \tau_{1*}) \sin \omega_{1*} \tau_2 \\
&\quad + (Q_0 \omega_{1*} \cos \omega_{1*} \tau_{1*} + Q_1 \omega_{1*} \sin \omega_{1*} \tau_{1*}) \cos \omega_{1*} \tau_2,
\end{aligned}
$$
Thus, we have
\[
\text{sign} \left[ \frac{d(Re\lambda)}{d\tau_1} \right]_{\lambda = i\omega_{1*}} = \text{sign} \left\{ Re \left( \frac{d\lambda}{d\tau_1} \right)^{-1} \right\}_{\lambda = i\omega_{1*}} = \text{sign} \left\{ \frac{H_{11}H_{13} + H_{12}H_{14}}{H_{13}^2 + H_{14}^2} \right\}.
\]

Therefore, if \((H7)H_{11}H_{13} + H_{12}H_{14} \neq 0\) hold, then \(\left\{ \frac{d(Re\lambda)}{d\tau_1} \right\}_{\lambda = i\omega_{1*}} \neq 0\), the transversal condition holds and we have the following result.

**Theorem 2.4.** For system (3), when \(\tau_1 > 0, \tau_2 \in [0, \tau_2^*]\) and \(\tau_1 \neq \tau_2\), if \((H6)\) and \((H7)\) hold, when \(\tau_1 < [0, \tau_{1*})\), then the positive equilibrium \(E^*\) is locally asymptotically stable, and the positive equilibrium \(E^*\) is unstable when \(\tau_1 > \tau_{1*}\). Moreover, when \(\tau_1 = \tau_{1*}\), then the Hopf bifurcation occurs at the positive equilibrium \(E^*\).

**Case V:** \(\tau_2 > 0, \tau_1 \in [0, \tau_1^*)\) and \(\tau_1 \neq \tau_2\).

The calculation is very similar to case IV, we can obtain the following theorem.

**Theorem 2.5.** For system (3), when \(\tau_2 > 0, \tau_1 \in [0, \tau_1^*)\) and \(\tau_1 \neq \tau_2\), if \(\tau_2 \in [0, \tau_2^*)\), then the positive equilibrium \(E^*\) is locally asymptotically stable, and the positive equilibrium \(E^*\) is unstable when \(\tau_2 > \tau_{2*}\). Moreover, when \(\tau_2 = \tau_{2*}\), then the Hopf bifurcation occurs at the positive equilibrium \(E^*\), where \(\tau_{2*}\) represents the minimum critical value of Hopf bifurcation.

### 3 Numeric simulations

In this section, we use numerical simulation to illustrate our results. Let \(r = 3, K = 8, d_1 = 0.1, \beta = 0.3, \zeta_1 = 0.002, g = 1, \alpha = 0.01, \zeta_2 = 0.001, \beta_1 = 0.25, m = 0.2, p = 1, \beta_2 = 0.3, d_2 = 0.025\). When \(\tau_1 > 0, \tau_2 = 0\), we get the positive equilibrium point \(E^* = (0.8228, 0.8839, 0.8866)\) and the critical value of time delay \(\tau_{1*} = 5.3884\). When \(\tau_1 = 4.8 < 5.3884 = \tau_{1*}\), the positive equilibrium point \(E^*\) of system (3) is locally asymptotically stable. When \(\tau_1 = 5.8 > 5.3884 = \tau_{1*}\), then \(E^*\) is unstable (see Figs. 1 and 2). When \(\tau_1 = \tau_{1*}\), the system (3) has Hopf bifurcation at the positive equilibrium of \(E^*\).

![Figure 1: \(\tau_1 = 4.8\), \(E^*\) is locally asymptotically stable](image-url)
When $\tau_1 = 0$, $\tau_2 > 0$, we get the critical value of time delay $\tau^*_2 = 2.3683$. When $\tau_2 = 1.8 < 2.3683 = \tau^*_2$, the positive equilibrium point $E^*$ of system (3) is locally asymptotically stable. When $\tau_2 = 2.8 > 2.3683 = \tau^*_2$, then $E^*$ is unstable (see Figs. 3 and 4). When $\tau_2 = \tau^*_2$, the system (3) has Hopf bifurcation at the positive equilibrium of $E^*$.

When $\tau_1 = \tau_2 = \tau$, we get the critical value of time delay $\tau^* = 1.2905$. When $\tau = 0.8 < 1.2905 = \tau^*$, the positive equilibrium point $E^*$ is locally asymptotically stable. When $\tau = 1.8 > 1.2905 = \tau^*$, then $E^*$ is unstable (see Figs. 5 and 6). When $\tau = \tau^*$, the system (3) has Hopf bifurcation at the positive equilibrium of $E^*$.
Figure 5: $\tau = 0.8$, $E^*$ is locally asymptotically stable

Figure 6: $\tau = 1.8$, $E^*$ is unstable

When $\tau_1 > 0, \tau_2 = 2 \in [0, \tau_2^*)$, we get $\tau_1^* = 1.2095$. When $\tau_1 = 0.6 < 1.2095 = \tau_1^*$, the positive equilibrium point $E^*$ is locally asymptotically stable (see Fig. 7). When $\tau_1 = 1.6 > 1.2095 = \tau_1^*$, the positive equilibrium point $E^*$ is unstable (see Fig. 8). When $\tau_1 = \tau_1^*$, the system (3) has Hopf bifurcation at the positive equilibrium of $E^*$.

Figure 7: $\tau_1 = 0.6$ and $\tau_2 = 2 \in [0, \tau_2^*)$, $E^*$ is locally asymptotically stable

Figure 8: $\tau_1 = 1.6$ and $\tau_2 = 2 \in [0, \tau_2^*)$, $E^*$ is unstable
When \( \tau_2 > 0, \tau_1 = 2 \in [0, \tau_1^*) \), we get \( \tau_2^* = 1.5283 \). When \( \tau_2 = 1.3 < 1.5283 = \tau_2^* \), the positive equilibrium point \( E^* \) is locally asymptotically stable (see Fig. 9). When \( \tau_2 = 1.8 > 1.5283 = \tau_2^* \), the positive equilibrium point \( E^* \) is unstable (see Fig. 10). When \( \tau_2 = \tau_2^* \), the system (3) has Hopf bifurcation at the positive equilibrium of \( E^* \).

Figure 9: \( \tau_2 = 1.3 \) and \( \tau_1 = 2 \in [0, \tau_1^*) \), \( E^* \) is locally asymptotically stable

Figure 10: \( \tau_2 = 1.8 \) and \( \tau_1 = 2 \in [0, \tau_1^*) \), \( E^* \) is unstable

4 Conclusions

In this paper, we consider a Holling type III double time delay predator-prey model with fear effect, cooperation, stage structure and refuge, and the prey has stage structure, which is manifested as juvenile prey and adult prey. The time delays \( \tau_1 \) and \( \tau_2 \) are the maturity from juvenile prey to adult prey and the pregnancy time of predator respectively. Firstly, the stability of positive equilibrium point with time delay \( \tau_i \) \((i = 1, 2)\) as parameter is discussed. When the time delay is less than the critical value, the positive equilibrium point \( E^* \) is locally asymptotically stable; when the time delay is greater than the critical value, the positive equilibrium point \( E^* \) is unstable; when the time delay is equal to the critical value, Hopf bifurcation occurs in the positive equilibrium point of the system. Finally, the analysis results are verified by numerical simulation.
References


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