Energy Balance Method for Solving Nonlinear Oscillators with Non-rational Restoring Force

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Abstract

In this study, an extended energy balance method for solving nonlinear oscillators with non-rational restoring force has been presented. The classical energy balance methods cannot apply to solve nonlinear oscillators with non-rational restoring force. To overcome this difficulty, a laborious study is to be conducted to determine the solution technique that provides better results as well as requires minimum computational efforts. An example of complicated nonlinear oscillator with non-rational restoring force arising in the field of engineering and other disciplines has been provided. The accuracy of the approximate results has been compared with other existing results.

Keywords: Extended energy balance method; classical energy balance methods; complicated nonlinear oscillator; accuracy

1. Introduction

The study of nonlinear oscillators is important because the natural and scientific phenomena are mostly occurred nonlinearly and are described by nonlinear oscillators. These oscillators are especially important in engineering because many engineering components consist of vibrating systems which are modeled by oscillatory systems such as elastic beam, nonlinear pendulum, vibration of piano harmer, milling machine [4,6] etc. Also, many others application of the generalized nonlinear oscillator are available such as relativistic oscillator [7,14,18], Duffing-relativistic oscillator [7], Dynamical modeling of cables with an attached midpoint mass [5,7,10], Plasma physics [19], Dynamic modeling of vehicle suspension [7,3] etc. A large number of analytical methods have been developed to find approximate solutions of these nonlinear problems. The most
popular method is the Lindstedt-Poincare method as well as modified Lindstedt-Poincare methods [15,17]. It is widely used for weak and strong nonlinear problems when the regular perturbation approaches fail. Harmonic balance method (HBM) [2,5,7,8,12,20] is another powerful method in which truncated Fourier series is used. Mickens [20] showed that all solutions to the relativistic oscillator was periodic and introduced a method to investigate the equation. Belendez et al. [1] used He's homotopy perturbation method (HPM) to solve relativistic harmonic oscillator. Recently developed method is the energy balance method [7,9,11,13,16] for solving weak and strong nonlinear oscillators. Though, these analytical methods have been developed for handling nonlinear oscillator, they provide almost similar results for a particular approximation.

In this research, an extended energy balance method has been provided to solve the nonlinear oscillators with non-rational restoring force in which the solution rapidly converges toward its exact solution. The method can be easily exerted to solve other nonlinear oscillators.

2. Methodology

2.1 Belendez’s Technique [1]

The governing equation of relativistic oscillator is [20]

\[ \ddot{x} + [1 - (\dot{x})^2]^{3/2} x = 0, \quad x(0) = a, \ \dot{x}(0) = 0. \]  

(1)

Where over dots denote differentiation with respect to time. Belendez et al. introduced the phase space variable \((x, y)\) and Eq. (1) was written as

\[ \dot{x} = y, \ \dot{y} = -(1 - y^2)^{3/2} x. \]  

(2)

The trajectories in phase space were given by

\[ \frac{dy}{dx} = -\frac{(1-y^2)^{3/2} x}{y}. \]  

(3)

Since the physical solutions of Eq. (1) are real and \(x, y\) vary as

\[ -\infty < x < +\infty, \quad -1 < y < +1. \]  

(4)

A transformation was used as

\[ y = \frac{u}{\sqrt{1+u^2}}. \]  

(5)

It follows that

\[ \dot{u} = -x. \]  

(6)

Then, differentiating Eq. (6) with respect to \(t\) and utilizing Eqs. (2) and (5), it was followed

\[ \ddot{u} + \frac{u}{\sqrt{1+u^2}} = 0. \]  

(7)

It was considered the following initial conditions in Eq. (7)

\[ u(0) = B \text{ and } \dot{u}(0) = 0. \]  

(8)

Eq. (7) was rewritten as

\[ (1 + u^2) \ddot{u}^2 = u^2 \]

Or, \(\ddot{u}^2 + u^2 \ddot{u}^2 = u^2\)

Or, \(u - u^{-1} \ddot{u}^2 + u \ddot{u}^2 = 0.\)  

(9)

This equation was re-written as
\[
\ddot{u} + \omega^2 u = \ddot{\omega} + \omega^2 u + u - u^{-1}\dddot{u}^2 + u \dddot{u},
\]

where \(\omega\) is the unknown angular frequency of the nonlinear oscillator. Homotopy was established as
\[
\ddot{u} + \omega^2 u = p[\ddot{u} + \omega^2 u + u - u^{-1}\dddot{u}^2 + u \dddot{u}],
\]

where \(p\) is the homotopy parameter. According to the homotopy perturbation method, the parameter \(p\) was used to expand the solution \(u(t)\) in powers of the parameter \(p\):
\[
u(t) = u_0(t) + pu_1(t) + p^2u_2(t) + \cdots.
\]

Substituting Eq. (12) into Eq. (11) and equating the terms with identical powers of \(p\), a series of linear equations was obtained, of which only the first two equations was written as
\[
\begin{align*}
\ddot{u}_0 + \omega^2 u_0 &= 0, \quad u_0(0) = B, \quad \dot{u}_0(0) = 0, \\
\ddot{u}_1 + \omega^2 u_1 &= \ddot{u}_0 + \omega^2 u_0 + u_0 - u_0^{-1}\dddot{u}_0^2 + u_0\dddot{u}_0^2, \quad u_1(0) = B, \quad \dot{u}_1(0) = 0. 
\end{align*}
\]

The solution of Eq. (13) was considered as
\[
u_0(t) = B \cos \omega t,
\]

where \(B\) was a constant to be determined subsequently as a function of amplitude of oscillations \(a\). Substituting Eq. (15) into Eq. (14), the following differential equation for \(u_1\) was obtained as:
\[
\ddot{u}_1 + \omega^2 u_1 = \left(\frac{1}{4} B^3 \omega^4 - \frac{3}{4} B^3 \omega^4 \right) \cos \omega t - \frac{1}{4} B^3 \omega^4 \cos 3\omega t.
\]

Removal of secular terms in \(u_1(t)\) required eliminating contributions proportional to \(\cos \omega t\) on the right-hand side of Eq. (16)
\[
B - B \omega^4 - \frac{3}{4} B^3 \omega^4 = 0.
\]

Which was solved for the first analytical approximate frequency \(\omega_a\) as a function of \(B\)
\[
\omega_a(B) = \left(1 + \frac{3}{4} B^2\right)^{-1/4}.
\]

From Eq. (12), a reasonable and simple initial approximation was taken as
\[
u_a(t) \approx u_0(t) = B \cos \omega_a t.
\]

The corresponding approximate solution for \(y\) was gotten from Eq. (5) as
\[
y_a(t) = \frac{B \cos \omega_a t}{\sqrt{1+B^2 \cos^2 \omega_a t}}.
\]

Likewise, \(x_a(t)\) was calculated by integrating equation \(y = \dot{x}\) subject to the restrictions
\[
x(0) = 0, \quad y(0) = \frac{B}{\sqrt{1+B^2}}.
\]

Which was obtained from Eqs. (5) and (7) as
\[
x_a(t) = \left(1 + \frac{3}{4} B^2\right)^{1/4} \sin^{-1}\left[\frac{B}{\sqrt{1+B^2}} \sin[\omega_a(B) t]\right].
\]

It was necessary to find a relation between oscillation amplitude \(a\) and parameter \(B\). From Eq. (3) and (7) it was written as
\[
\int \frac{y}{(1-y^2)^{3/2}} dy + x dx = 0.
\]

By integrating Eq. (23) it was arrived at
\[
\frac{1}{(1-y^2)^{1/2}} + \frac{1}{2} x^2 = C,
\]
where C was a constant. From Eq. (21) initial conditions was used in Eq. (24), C was obtained as
\[ C = (1 + B^2)^{1/2}, \]  
and Eq. (24) was written as
\[ \frac{1}{(1-y^2)^{1/2}} + \frac{1}{2} x^2 = (1 + B^2)^{1/2}. \]  
In addition, when \( x = a \), the velocity \( y = \dot{x} \) is zero. Then from Eq. (26), the relation between amplitude \( a \) and parameter \( B \) was obtained as
\[ 1 + \frac{1}{2} a^2 = (1 + B^2)^{1/2}. \]  
From the above equation it was easily found that the solution for \( B \) as
\[ B = a \left( 1 + \frac{1}{4} a^2 \right)^{1/2}. \]  
Substituting Eq. (28) into Eq. (18) gave the approximate frequency as
\[ \omega_a(a) = \left( 1 + \frac{3}{4} a^2 + \frac{3}{16} a^4 \right)^{-1/4}. \]  
Using Eq. (28) into Eq. (22), \( x_a(t) \) was represented as
\[ x_a(t) = \left( 1 + \frac{3}{4} a^2 + \frac{3}{16} a^4 \right)^{-1/4} \sin^{-1} \left[ \frac{a^2 + \frac{1}{4} a^4}{\sqrt{1 + a^2 + \frac{1}{4} a^4}} \sin[\omega_a(a) t] \right]. \]  

2.2 Energy balance method

In this method, a variational principle established and then a Hamiltonian was constructed. Finally, the angular frequency was determined with the help of collocation technique. A nonlinear oscillator was considered [9] as
\[ \ddot{x} + f(x) = 0, \quad x(0) = a, \quad \dot{x}(0) = 0. \]  
Here, \( f(x) \) is a nonlinear function. According to the variational principle, Eq. (31) was expressed as
\[ J(x) = \int_0^{T/4} \left[ -\frac{1}{2} \dot{x}^2 + F(x) \right] dt, \]  
where \( T = 2\pi/\omega \) is a period of the oscillation, \( \omega \) is the frequency of the oscillator (to be determined) and \( F(x) = \int f(x) dx \).

The Hamiltonian of Eq. (32) becomes
\[ H(x) = \frac{1}{2} \dot{x}^2 + F(x) = F(a). \]  
which provided the following residual
\[ R(x) = \frac{1}{2} \dot{x}^2 + F(x) - F(a) = 0. \]  
Assumed, the first-order approximate solution as
\[ x(t) = a \cos \omega t. \]  
Substituting Eq. (35) into Eq. (34), then
\[ R(t) = \frac{1}{2} a^2 \omega^2 \sin^2 \omega t + F(a \cos \omega t) - F(a) = 0. \]  
When \( \omega t = \pi/4 \) (collocation principle) it became
\[ \omega = \frac{2}{A} \sqrt{F(a) - F \left( \frac{\sqrt{2}}{2} a \right)}. \]
2.3 More accurate Energy balance method [13]

Recently a more accurate energy balance method has been presented by Molla et al. [13]. In this method second approximate solution of Eq. (31) has been considered as:

\[ x(t) = a((1 - u) \cos \omega t + u \cos 3\omega t) \]  \hspace{1cm} (38)

where \( a \) is amplitude and \( u \) is an unknown constant to be determined. Eq. (38) satisfy initial conditions given in Eq. (31). By substituting Eq. (38) into Eq. (34) residual was obtained. This residual contains two unknown parameters, \( \omega \) and \( u \). To determining \( \omega \) and \( u \) two algebraic equations was obtained from the following equation

\[ \int_0^{T/4} \frac{R(t) \cos(2n-2)\omega t \, dt}{\sin^2 \omega t} = 0 \]  \hspace{1cm} (39)

for \( n = 1 \) and \( n = 2 \) respectively.

2.4 The Proposed Energy balance method

In this research, the more accurate energy balance method has been extended to solve some complicated nonlinear oscillators in where nonlinear function \( f \) contains non-rational terms with damping, e.g.,

\[ \ddot{x} + f(x, \dot{x}) = 0, \hspace{0.5cm} x(0) = a, \hspace{0.2cm} \dot{x}(0) = 0. \]  \hspace{1cm} (40)

Earlier the nonlinear problems have been investigated by EBM where \( f \) is a polynomial function of \( x \). In article [1, 13], Belendez et al. and Molla et al. have studied the nonlinear oscillators in which \( f \) contains a non-rational term of \( x \) only. It is noted that, although Eq. (40) can be written in the form of Eq. (34) but it cannot be directly used in Eq. (39). After some modifications, it takes the following form [13]

\[ G(\dot{x}^2, x^2 - a^2, x^4 - a^4, \ldots ) = 0. \]  \hspace{1cm} (41)

In article [20], Mickens has studied some nonlinear oscillators in where \( f \) as well as \( G \) is rational. But in the present study, \( f \) contains a non-rational term of \( \dot{x} \) as well as \( G \) is rational.

Let us consider 2nd approximate solution in the following form

\[ x(t) = a((1 - u) \cos \omega t + u \cos 3\omega t) \]  \hspace{1cm} (42)

where \( a \) is amplitude, \( u \) is unknown constants and \( \omega \) is the frequency of the oscillator (to be determined). Equation (42) satisfies initial conditions given in Eq. (40).

Differentiating \( x \), squaring and simplifying, we obtain

\[ x^2 = a^2 \omega^2 \sin^2 \omega t \, (1 + 2u + 6u \cos 2\omega t)^2. \]  \hspace{1cm} (43)

Then, we determine an expression for \( x^2 - a^2 \), as

\[ x^2 - a^2 = -a^2 \sin^2 \omega t \, (1 + 4u - 2u^2 + 4u \cos 2\omega t + 2u^2 \cos 4\omega t). \]  \hspace{1cm} (44)

All other expressions \( x^4 - a^4, \ x^6 - a^6, \ldots \) of Eq. (41) have a common factor \( x^2 - a^2 \); so that a common factor \( a^2 \sin^2 \omega t \) must be cancelled when all these values are substituted in Eq. (41). It is noted that the canceling of the common (i.e., \( a^2 \sin^2 \omega t \) factor makes the solution better than usual solution. Otherwise, the solution does not converge fast.

Now, dividing Eq. (1) by \((1 - \dot{x}^2)^{3/2}\) we obtain
\[ \frac{\ddot{x}}{(1-x^2)^2} + x = 0. \] (45)

Multiplying Eq. (45) by \( 2 \ddot{x} \) and integrating we obtain,

\[ \frac{2}{\sqrt{1-x^2}} + x^2 = C, \] (46)

where, \( C \) is integrating constant. Using initial condition in Eq. (46), Eq. (46) then becomes

\[ \frac{2}{\sqrt{1-x^2}} + x^2 - 2 - a^2 = 0. \] (47)

Eq. (47) can be written as

\[ 4 - (2 + a^2 - x^2)^2 (1 - \ddot{x}) = 0, \] (48)

which is the form of Eq. (41).

Substituting Eqs. (43) and (44) into Eq. (48) then equating the constant term and coefficient of \( \cos 2\omega t \) respectively we obtain

\[ -4 - 16u + 8u^2 - a^2 (1 + 4u + 4u^2 - 12u^3 + 6u^4)/2 + \omega^2 (32 + 16a^2 + 3a^4 + 128u - 4a^4 u + 704u^2 + 256a^2 u^2 + 37a^4 u^2 + 384a^2 u^3 + 88a^4 u^3 + \ldots \cdot)/8 = 0. \] (49)

\[ -16u + a^2 (1 - 6u^2 + 2u^4)/2 - \omega^2 (8a^2 + 2a^4 - 192u - 64a^2 u - 384u^2 - 48a^2 u^2 - 128a^2 u^3 - 12a^4 u^3 - \ldots \cdot)/4 = 0. \] (50)

Eliminating \( \omega^2 \) between Eq. (49)-Eq. (50) we obtain:

\[ -6a^2 - 2a^4 - a^6/16 + u(128 + 32a^2 + a^4 + a^6/8) + u^2(892 + 496a^2 + 100a^4 + 91a^6/16) - u^3(256 + 80a^2 + 24a^4) + \ldots \cdot = 0. \] (51)

Here the coefficient of \( u \) of Eq. (51) is 128 and \( 32a^2 \) also the constant term \( 6a^2 \).

Therefore, Eqs. (51) can be solved in power series of a parameter \( \lambda \) by choosing \( \lambda = 6a^2/(128 + 32a^2) \). It is clear that \( \lambda \) is much smaller than 1 for every value of \( a \). As \( a \to \infty \), \( \lambda \) becomes \( 6/32 \) (which is the largest). Therefore, \( u \) can be obtained in powers of \( \lambda \) of the forms \( u = l_1 \lambda + l_2 \lambda^2 + \ldots \cdot \) (see Alam et al. [16] for details). Substituting the value of \( u \) in Eqs. (51) and equating equal powers of \( \lambda \), a set of linear algebraic equations of \( l_1, l_2, \ldots \cdot \), are obtained. Solving these algebraic equations, the unknown constants, \( l_1, l_2, \ldots \cdot \), are determined. Thus \( u \) become

\[ u = \lambda + \frac{\lambda^2}{9} - \frac{52\lambda^3}{9} + \ldots \cdot \] (52)

Now substituting the value of \( u \) in Eq. (49) and then simplifying, the frequency \((i.e., \omega)\) is obtained as:

\[ \omega^2 = 1 - 8a^4 + \frac{56\lambda^2}{3} - \frac{832\lambda^3}{27} + \ldots \cdot \] (53)

Also putting \( u = 0 \) in Eq. (49) and then simplifying, the frequency \((i.e., \omega)\) is obtained as:

\[ \omega^2 = \frac{4(8 + a^2)}{32 + 16a^2 + 3a^4}. \] (54)

We obtain a series from \( 1/\omega^2 \) as

\[ 1 + \frac{3a^2}{8} + \frac{3a^4}{64} - \frac{3a^6}{512} + \ldots \cdot \] (55)

Now multiplying Eq. (53) by \( 1 + \frac{3a^2}{8} \) and obtain the frequency \((where a = \frac{8\sqrt{3}}{\sqrt{3} - 16a})\) as
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\[ \omega^2 = 1 - \frac{8\lambda^2}{3} + \frac{128\lambda^3}{27} + \cdots. \]  
(56)

Finally, the series of \( \omega^2 \) becomes

\[ \omega^2 = \left( 1 - \frac{8\lambda^2}{3} + \frac{128\lambda^3}{27} + \cdots \right) / \left( 1 + \frac{3a^2}{8} \right). \]  
(57)

The simple first approximate solution of Eq. (1) by the proposed method is given [1] as

\[ x(t) = B \tan^{-1} \left( \frac{B\sqrt{2} \cos \omega t}{\sqrt{2}B^2 + B^2 \cos 2\omega t} \right) / B \omega, \]  
(58)

Where, \( B = \frac{a}{2} \sqrt{1 + a^2} \),  
(59)

and \( \omega \) is given by Eq. (57).

In similar way, 3rd approximate solution of Eq. (1) is obtained by present method as

\[ x(t) = a \left( (1 - u - v) \cos \omega t + u \cos 3\omega t + v \cos 5 \omega t \right) \]  
(60)

where,

\[ u = \lambda - \frac{8\lambda^2}{3} - \frac{14\lambda^3}{9} + \frac{2641\lambda^4}{243} - \frac{29516\lambda^5}{243} - \cdots \]  
(61)

\[ v = \frac{25\lambda^2}{9} + \frac{11\lambda^3}{27} - \frac{769\lambda^4}{243} + \frac{2407\lambda^5}{243} - \cdots. \]  
(62)

The frequency \( \omega \) is obtained as

\[ \omega^2 = \left( 1 - \frac{8\lambda^2}{3} + \frac{128\lambda^3}{27} - \frac{88\lambda^4}{9} + \frac{256\lambda^5}{27} + \cdots \right) / \left( 1 + \frac{3a^2}{8} \right). \]  
(63)

3. Results and discussion

We have been determined the 2nd and 3rd approximate frequency as well as solution by present method for the relativistic oscillator (Eq. (1)). Compared the present results with other existing results [1,20] as well as numerical (exact) results in Table 1 and Figures 1-5 for some different values of amplitude \( a \). Numerical results are obtained by fourth-order Runge-Kutta method. For large amplitude \( (a = 100) \) 2nd and 3rd approximate frequencies is used in Eq. (58) which provided in Figure 4 and figure 5 respectively. From the table and figures it is clear that the results of present method are much better than those of [1,20] and very close to numerical results.
Table 1: Comparison of the present frequencies with exact and other existing frequencies.

<table>
<thead>
<tr>
<th>(a)</th>
<th>Exact (\omega_c)</th>
<th>Mickens [20] (\omega_{HBM}) (% error)</th>
<th>Belendez et al. [1] (\omega_{HBM}) (% error)</th>
<th>Present method 2(^{nd}) Approximation ((\omega_2)) (% error)</th>
<th>Present method 3(^{rd}) Approximation ((\omega_3)) (% error)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.99813</td>
<td>0.998754 0.0625094</td>
<td>0.998129 0.000871865</td>
<td>0.99813 0.00000</td>
<td>0.99813 0.00000</td>
</tr>
<tr>
<td>0.5</td>
<td>0.956031</td>
<td>0.970984 1.56405</td>
<td>0.955598 0.045228</td>
<td>0.956031 0.00000</td>
<td>0.956031 0.00000</td>
</tr>
<tr>
<td>1</td>
<td>0.851301</td>
<td></td>
<td></td>
<td>0.847597 0.00099</td>
<td>0.851301 0.00000</td>
</tr>
<tr>
<td>5</td>
<td>0.30164</td>
<td></td>
<td></td>
<td>0.292327 3.08736</td>
<td>0.302655 0.33649</td>
</tr>
<tr>
<td>50</td>
<td>0.031403</td>
<td></td>
<td></td>
<td>0.0303813 3.25466</td>
<td>0.031608 0.65280</td>
</tr>
<tr>
<td>100</td>
<td>0.015706</td>
<td></td>
<td></td>
<td>0.0151952 3.25471</td>
<td>0.015809 0.6558</td>
</tr>
</tbody>
</table>

Where \(\omega\) denotes approximate frequency, \((\%\text{ error})\) denotes the absolute percentage error.

**Fig. - 1** Comparison of the present periodic solution (denoting by circle line) with Belendez et al. [1] (denoting by dashes line) and numerical solution (denoting by solid line) at \(a = 2\) for relativistic oscillator Eq. (1).

**Fig. - 2** Comparison of the present periodic solution (denoting by circle line) with Belendez et al. [1] (denoting by dashes line) and numerical solution (denoting by solid line) at \(a = 5\) for relativistic oscillator Eq. (1).
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Fig. 3 Comparison of the present periodic solution (denoting by circle line) with Belendez et al. [1] (denoting by dashes line) and numerical solution (denoting by solid line) at $a = 10$ for relativistic oscillator Eq. (1).

Fig. 4 Comparison of the present periodic solution (denoting by circle line) with Belendez et al. [1] (denoting by dashes line) and numerical solution (denoting by solid line) at $a = 100$ for relativistic oscillator Eq. (1).

Fig. 5 Comparison of the present periodic solution (denoting by circle line) with Belendez et al. [1] (denoting by dashes line) and numerical solution (denoting by solid line) at $a = 100$ for relativistic oscillator Eq. (1).
4. Conclusion

Extended energy balance method has been presented to solve relativistic oscillator. The present solution rapidly converges toward its exact solution. The results are nicely agreement to the exact results and much better than those of other existing results. We conclude that this method is very effective and convenient for analysis of non-rational restoring force nonlinear oscillator and quite accurate to nonlinear engineering problems.

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