Existence of Spatial Economic Equilibrium:
A Proof Utilizing the Borsuk-Ulam Theorem

Pascal Stiefenhofer
Newcastle University, UK

Andros Gregoriou
University of Brighton, UK

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Abstract
In this paper, we study spatial economic equilibrium of a single good economy distributed over the earth’s equator using a representative agent model with linear demand and supply functions. We show using Borsuk Ulam’s theorem that there exist antipodal economic activities with associated equilibrium prices.

Mathematics Subject Classification: 00A69, 32A70

Keywords: Borsuk Ulam Theorem, Spatial Equilibrium, Existence

1 Introduction
Spatial economic equilibrium modelling remains a vibrant field of theoretical and applied economics research since its celebrated formulation within the general economic equilibrium framework [7] seven decades ago. Ever since, several spatial economic equilibrium models have emerged in various areas of economics, including regional and development economics [1, 4, 25], financial economics [5, 3, 24, 17], and economic applications of game and search theory [23, 26]. Predating the formulation of a spatial economic equilibrium within the general equilibrium framework is the model introduced by Samuelson who studied spatial prices in multi markets within a partial equilibrium context.
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He uses linear programming techniques to show existence of equilibrium [9]. Arrow and Debreu differentiate economic commodities by their physical, temporal, and spatial properties and utilize convexity and fixed point theory, in particular the Brower fixed point, to show existence of equilibrium prices [7]. McKenzie, within a similar economic framework, utilizes Kakutani’s fixed point theorem to establish existence [9]. Gale, Nikaido and Debreu independently established fixed point-like lemmata [18, 19, 20] from which Uzawa proved Brower’s and Kakutani’s fixed point theorems [21]. The relationship between fixed point theory and Borsuk-Ulam’s theorem suggests new ways of establishing existence of spatial economic equilibrium [17, 22] with positive implications for economic modelling. In particular, Nyman and Su show that Borsuk Ulam’s theorem implies Brower’s fixed point theorem [13]. In this paper we directly utilize Borsuk-Ulam’s theorem to establish existence of spatial economic equilibrium for the first time.

Let $S^n$ denote the unit sphere in $\mathbb{R}^{n+1}$ and let $f : S^n \rightarrow \mathbb{R}^n$ be a continuous map. Then there exists a pair of antipodal points on $S^n$ that are mapped by $f$ on the same point in $\mathbb{R}^n$. This statement was conjectured by S. Ulam and proved by K. Borsuk in 1933 [11]. A topological construction shows how a Brower fixed point follows from Borsuk Ulam antipodes [13]. Similar implications were shown using alternative constructive proofs by [12, 14]. These results have profound implications for economics. Several important results in economics rely on Brower fixed points, including game theory [15], general equilibrium theory [7], and economic goods division problems [16]. However, the study of antipodal economic activities and their fixed points in spatial economies remains unexplored. In this communiqué we utilise Borsuk-Ulam’s theorem to establish this connection providing researchers with a new theoretical perspective on the relationship between spatial economic activities and equilibrium prices.

In order to establish this result within the simplest economic scenario, we consider the class of linear single good economies within a representative agent model, where the spatial economy is represented by the earth’s equator, $S^1$. A spatial economic equilibrium consists of a triple $(\pm x^*, \theta^*, q^*)$, where $\pm x^* \in S^1$ represent the antipodal geographical locations of economic activities, $\theta^* \in \Theta \subseteq \mathbb{R}$ is an equilibrium price with associated quantity $q^* \in \mathbb{R}_0$ produced and consumed at the antipodes in equilibrium. We consider a continuous and odd mapping $g : S^1 \rightarrow \Theta$, where $S^1$ is a circle representing the earth’s equator defined by the set $\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = r^2\}$ with centre $x_0 = (0, 0)$ and $\Theta \subseteq \mathbb{R}$ is a set of prices. We consider linear supply and demand functions $f^s$ and $f^d$, with $\alpha, \beta, \gamma, \delta > 0$, given by
**Existence of spatial economic equilibrium**

\[ f^s(\theta) = -\alpha + \beta \theta, \text{ for } \theta \leq \theta \leq \bar{\theta} \]  
\[ f^d(\theta) = \gamma - \delta (\bar{\theta} - \theta), \text{ for } \bar{\theta} \geq \bar{\theta} - \theta \geq \theta. \]  

Economic equilibrium is a condition defined by a continuous function \( z(\theta) \) given by

\[ z(\theta) := f^s(\theta) - f^d(\theta) = 0. \]  

**Definition 1.** A univariate function \( f(\theta) \) is said to be odd provided that \( f(-\theta) = -f(\theta) \).

**Lemma 1.** Let \( -\alpha = \gamma \), then \( z(\theta) \) is odd.

Assume that \( -\alpha \neq \gamma \). Then, \( z(\theta) = -(\alpha + \gamma) + (\beta + \delta)\theta \) is odd. But

\[ z(-\theta) = -(\alpha + \gamma) - (\beta + \delta)\theta \]
\[ = -[(\alpha + \gamma) + (\beta + \delta)\theta] \]
\[ \neq -z(\theta) \]
\[ = (\alpha + \gamma) - (\beta + \delta)\theta. \]

Hence \( z(\theta) \) is not odd, which is a contradiction and therefore, \( -\alpha = \gamma \) must hold.

**Theorem 1.** Let \( f^{s,d} \) be defined by equations (1) and (2) with \( -\alpha = \gamma \). Moreover, let \( g : S^1 \rightarrow \Theta \) be a continuous and odd function, then there exists a spatial economic equilibrium \((\pm x^*, \theta^*, q^*) \in S^1 \times \Theta \times \mathbb{R}\) satisfying

\[ \tilde{z}(g(-x)) + \tilde{z}(g(x)) = 0. \]

**Proof.** Consider the continuous functions \( f^s : \mathbb{R} \rightarrow \mathbb{R} \) and \( f^d : \mathbb{R} \rightarrow \mathbb{R} \) given by equations (1) and (2). By lemma 1, \( z(\theta) = -z(\theta) \) is odd and continuous. To obtain a map \( S^1 \rightarrow \mathbb{R} \) let \( \theta = g(x) \) and using equation (3) yields

\[ \tilde{z}(g(x)) := f^s(g(x)) - f^d(g(x)) = 0, \]  

where \( x \in S^1 \) and \( \tilde{z} \) being an excess demand function. Since \( f^{s,d} \) and \( g \) are continuous functions, it follows that \( \tilde{z} \) is also continuous. Since \( f^{s,d} \) and \( g \) are odd functions it follows that their composites \( f^{s,d} \) with \( g \) are also odd functions. Moreover, \( \tilde{z} \) is odd since the sum of odd composite functions is odd. Hence,
\[ \tilde{z}(g(x)) = f^{s}(-g(x)) - f^{d}(-g(x)) \]
\[ = -f^{s}(g(x)) - (-f^{d}(g(x))) \]
\[ = -f^{s}(g(x)) + f^{d}(g(x)) \]
\[ = -(f^{s}(g(x)) - f^{d}(g(x))) \]
\[ = -\tilde{z}(g(x)). \]

Now, if \( \tilde{z}(g(\bar{x})) = 0 \) for some \( \bar{x} \in S^1 \), then \( f^{s}(g(\bar{x})) = f^{d}(-g(\bar{x})) \). If \( \tilde{z}(g(x)) \neq 0 \) for \( x \in S^1 \), then \( 0 \in \{ \min\{\tilde{z}(g(x)), -\tilde{z}(g(x))\}, \max\{\tilde{z}(g(x)), \tilde{z}(-g(x))\} \} \). Since \( S^1 \) is connected and \( \mathbb{R} \) ordered by the intermediate value theorem there exists \( \bar{x} \in S^1 \) such that \( \tilde{z}(g(\bar{x})) = 0 \) from which \( f^{s}(g(\bar{x})) = f^{d}(g(-\bar{x})) \).

References


Received: January 9, 2023; Published: February 1, 2023