Degenerate Solutions and New Mixed Solutions of (2+1)-Dimensional Potential Boiti-Leon-Manna-Pempinelli Equation

Miao Li ¹, Wei Tan ²,* and Hou-Ping Dai ³

¹College of Mathematics and Statistics, Jishou University
Jishou, Hunan 416000, P.R. China

²College of Mathematics and Statistics, Jishou University
Jishou, Hunan 416000, P.R. China

*Corresponding author

³College of Mathematics and Statistics, Jishou University
Jishou, Hunan 416000, P.R. China

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Abstract

The N-soliton solution, M-order lump solution, M-order breather solution, high-order semi-rational solutions and their mixed solutions of the potential Boiti-Leon-Manna-Pempinelli (pBLMP) equation are obtained by using Hirota bilinear method and long wave limit method, and the mixed solution composed of soliton, lump and breather is obtained for the first time. By selecting different test functions, the mixed solutions of different types of lump solitons and triple solitons are studied. Secondly, we give the differences and reasons of the evolution behavior of these mixed solutions with time t for the first time. Finally, the spatial structure diagram is used to show the nonlinear phenomenon.

Keywords: long wave limit method; lump solitons; mixed solutions

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1 Introduction

Many nonlinear phenomena in natural science can be represented by nonlinear equations. Soliton theory can be used to solve a class of nonlinear partial differential equations (NLPDE), and the soliton solutions obtained can be used to simulate and describe nonlinear phenomena. Therefore, it is of great significance to seek and construct the soliton solutions of nonlinear equations and their structures and properties. In recent years, soliton solutions, such as rogue waves, lump solutions [1], breather solutions [2], mixed solutions [3], semi-rational solutions [4], especially the interaction between solitons [5], have become a research hotspot because of their application value in the field of nonlinear natural science. The results of these exact solutions make the research results of NLPDE more and more abundant. These solutions are attributed to the continuous development of solving methods, such as Hirota’s bilinear method [6], Darboux transformation method, polynomial expansion method, Parameter limit method [7] and so on. In this paper, we consider the (2+1)-dimensional BLMP equation:

\[ q_t + q_{xxx} - 3(q^2 - 1)q_x x = 0. \] 

(1)

The Eq.(1) was first proposed by Boiti et al. We set \( q = \alpha + u_y \) and \( q_x = u_{xy} \) in the Eq.(1) and then (1) will be transformed into the (2+1)-dimensional pBLMP equation to be studied in this paper:

\[ u_{yt} + u_{xxxy} - 3\alpha u_{xx} - 3u_y u_{xx} - 3u_{xy} u_x = 0, \]

(2)

where \( u \) is a multivariate function of \( x, y, t \), and \( \alpha \) is an arbitrary constant.

As a typical high-dimensional NEPDE, Many scholars have studied different forms of exact solutions of Eq.(2) by different methods. In [8], two exact traveling wave solutions of Eq.(2) were obtained by the exponential function method. In [9], the resonance interaction of line solitons in Eq.(2) is analyzed by the separation of variables method. Ref.[10] obtained the conservation law of Eq.(2) and the correlation between symmetry and conservation law. In [11], the rational breather soliton and kink solitary wave solutions of Eq.(2) were obtained by Hirota bilinear method. Ref.[12] used the three-wave method to prove the periodic lump solution of Eq.(2). However, as far as we know, the test function adopted in this paper has not been applied to Eq.(2), and the results of M-lumps, M-breathers, and high-order semi-rational solutions in this paper are obtained for the first time. Secondly, the mixed solution composed of solitons, lump and breather obtained in this paper is new. Finally, the differences and reasons of the evolution behaviors of two forms of lump soliton and tripe soliton mixed solutions are given for the first time.
2 N-soliton solution

Based the Hirota bilinear method, we study the N-soliton solution of the Eq.(2) in this section. According to Ref.[9], we can assume that:

\[
\begin{align*}
    u(x, y, t) &= -2(\ln f(x, y, t))_x, \\
    q(x, y, t) &= \alpha - 2(\ln f(x, y, t))_{xy}.
\end{align*}
\]

The bilinear form of the Eq.(1) is obtained by substituting (3) into (2),

\[
(D_y D_t + D_x^2 D_y - 3\alpha D_x^2)f \cdot f = 0.
\]

By the bilinear operator, the N-soliton of the Eq.(2) can be obtained as,

\[
F = f_N = \sum_{\mu=0,1}^{N} \exp(\sum_{i<j}^{(N)} \mu_i \mu_j \eta_{ij} + \sum_{i=1}^{(N)} \mu_i \eta_i), (i > N).
\]

The first \(\sum\) represents all possible combinations of \(\mu_1 = 0,1, \mu_2 = 0,1, \cdots, \mu_N = 0,1,\) and \(\sum\) denotes that for all satisfying conditions \(1 \leq i < j \leq N,\) of which

\[
\begin{align*}
    \eta_i &= k_i(a_i x + b_i y + \left(\frac{3a_i^2 \alpha}{b_i} - k_i^2 a_i^3\right)t + r_i) + \mu_i, \\
    e^{A_{ij}} &= \frac{(a_i b_j k_i k_j - a_j b_i k_i k_j)^2 \alpha + k_i^2 b_i k_j^2 \alpha}{(a_i b_j k_i k_j - a_j b_i k_i k_j)^2 \alpha + k_i^2 b_i k_j^2 \alpha}(a_i k_i - a_j k_j) + (a_i k_i + a_j k_j).
\end{align*}
\]

Where \(a_i \neq 0, b_i \neq 0, k_i, r_i, \mu_i\) are some free parameters. Substituting (5) into the transform (3), the N-soliton of the Eq.(2) is obtained (See Fig.1).

3 M-order lump solution

In this section, we will study the M-order lump of the Eq.(2) by the long-wave limit method. Taking \(\frac{k_i}{k_j} = O(1), e^{\mu_i} = -1\) in (5) and (6), let \(k_i \rightarrow 0,\) get:

\[
F_N = \sum_{\mu=0,1}^{N} \prod_{i=1}^{N} \frac{(-1)^{\mu_i} (1 + \mu_i k_i \theta_i)}{(1 + \mu_i \mu_j k_i k_j B_{ij})} + O(k^{N+1})
\]

To get the M-order lump solution of (2), select \(N = 2M\) in (8)-(9), and \(a_i = a^*_M + i, b_i = b^*_M, q_i = q_i, p_i = p_i, r_i = r^*_M, v_i = v_i, s_i = s_i, (i = 1, 2 \cdots, M),\)
where $I$ denotes the imaginary unit, $*$ denotes the imaginary conjugate.

\[
\begin{aligned}
\theta_i &= a_i x + b_i y + \frac{3a_i^2 \alpha}{b_i} t + r_i, \\
\exp(A_{ij}) &\approx 1 - \frac{2a_i a_j b_i b_j (a_i b_j + a_j b_i) k_i k_j}{(a_i b_j - a_j b_i)^2 \alpha} = 1 + B_{ij} k_i k_j.
\end{aligned}
\]  

(8)

From (3), we can obtain the following rational function solution,

\[
F_N = \prod_{i=1}^{N} \theta_i + \frac{1}{2} \sum_{i,j}^{(N)} B_{ij} \prod_{p \neq i,j}^{N} \theta_p + \frac{1}{M!2^M} \sum_{i,j,r,s}^{(N)} B_{ij} B_{rs} \cdots B_{mn} \prod_{p \neq i,j,r,s}^{N} \theta_p + \cdots
\]

(9)

As $N = 2$, substituting $a_1 = a_2^*, b_1 = b_2^*, r_1 = r_2^*$ into (9) to get,

\[
\begin{aligned}
F_2(x,y,t) &= (n_1 x + q_1 y + \frac{3\alpha(n_1^2 q_1 - m_1 q_1 - 2m_1 n_1 p_1)}{q_1^2 + p_1^2} t + v_1)^2 \\
&\quad + (m_1 x + p_1 y + \frac{3\alpha(n_1^2 p_1 - m_1 p_1 + 2m_1 n_1 q_1)}{q_1^2 + p_1^2} t + s_1)^2 \\
&\quad + \frac{(m_1 p_1 + n_1 q_1)(-m_1^2 - n_1^2)(-p_1^2 - q_1^2)}{\alpha(m_1 q_1 - n_1 p_1)^2} \\
&= (n_1 x + q_1 y + \xi_1 t + v_1)^2 + (m_1 x + p_1 y + \xi_1 t + s_1)^2 + \Delta.
\end{aligned}
\]

(10)

Substituting (10) into (3), 1-order lump solution of (2) is obtained (See Fig.2(a)(d)),

\[
\begin{aligned}
u(x,y,t) &= -4n_1 \xi_1 + m_1 \xi_1 \\
q(x,y,t) &= \alpha - \frac{4n_1 q_1 + m_1 p_1}{((n_1 x + q_1 y + \xi_1 t + v_1)^2 + (m_1 x + p_1 y + \xi_1 t + s_1)^2 + \Delta)^2}.
\end{aligned}
\]

(11)

When $N = 4$, the 2-order lump solution of (2) can be obtained (See Fig.2(b)(e)). When $N = 6$, the 3-order lump solution of (2) can be obtained (See Fig.2(c)(f)).

4 M-order breather solution

In order to obtain the M-order breather solution of (2), we take
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Figure 1: The space structure diagram of N-soliton solutions of (2) at t=0:
(a) N=1, (b) N=2, (c) N=3.

Figure 2: The diagram of lump solution of (2) at t=0:
(a) M=1, (b) M=2, (c) M=3.

\[
\begin{align*}
    k_{2n-1} &= k_{2n}^* = v_n + s_n \cdot I, \quad a_{2n-1} = a_{2n}^* = h_n + m_n \cdot I, \quad (n = 1, \cdots, M) \\
    b_{2n-1} &= b_{2n}^* = q_n + p_n \cdot I, \quad r_{2n-1} = r_{2n}, \quad \mu_{2n-1} = \mu_{2n} = 0, (M = \frac{N}{2}). \\
\end{align*}
\]  

(12)

By substituting (12) into (6), then \( \eta_{2n-1} = \eta_{2n}^* = R_e + iI_n \), namely,
\[ R_n = R_n(\eta_{2n-1}) = (v_nh_n - m_ns_n)x + \alpha((v_np_n - q_ns_n)(m_n^2 - h_n^2) + (v_ng_n - p_ns_n)y + 2m_nh_n(p_ns_n + q_nv_n))t \\
+ (m_ns_n - v_nh_n)(m_n^2s_n^2 - 3m_n^2v_n^2 - 8m_nh_ns_nv_n - 3h_n^2s_n^2 + h_n^2v_n^2)t. \]
\[ I_n = I_n(\eta_{2n-1}) = (v_nm_n + h_n^2s_n)x + \alpha((v_np_n + q_ns_n)(m_n^2 - h_n^2) + (v_nq_n + p_ns_n)y - 2m_nh_n(p_nv_n - q_ns_n))t \\
+ (m_nv_n + s_nh_n)(3m_n^2s_n^2 - m_n^2v_n^2 - 8m_nh_ns_nv_n - h_n^2s_n^2 + 3h_n^2v_n^2)t. \]

\[ e^{A_{2n-1,n}} = \frac{(p_nv_n + q_ns_n)(m_nv_n + h_n^2s_n) + \alpha(m_nq_n - h_n^2p_n)^2}{(p_ns_n - q_nv_n)(m_ns_n - h_n^2s_n) + \alpha(m_np_n + h_nq_n)^2} = \beta_{2n-1,2n}. \]

When \( N = 2 \), take the parameter limits in (5) and (6), then we get
\[ F_1 = 1 + 2e^{R_1} \cos I_1 + \beta_{12}e^{2R_1}. \]

Substituting (15) into (3), the 1-order breather of (2) is obtained (See Fig. 3(a)(d)).

**Figure 3:** The space structure diagram of the breather solution of (2) at \( t = 0; \tau_i = 1(i = 1, 2, \ldots, 6) \).

\( a) \alpha = 1, m_1 = 15, h_1 = 3, p_1 = \frac{1}{3}, q_1 = \frac{2}{3}, s_1 = \frac{1}{4}, v_1 = \frac{8}{9}; \) (b) \( \alpha = 1, h_1 = q_1 = s_1 = p_2 = 1, p_1 = v_1 = h_2 = s_2 = \frac{1}{2}, q_2 = v_2 = \frac{1}{3}, m_1 = \frac{4}{3}, m_2 = \frac{1}{5}; \) (c) \( \alpha = 100, h_1 = q_1 = s_1 = p_2 = p_3 = q_3 = s_3 = v_3 = 1, m_1 = h_2 = h_3 = 2, v_1 = q_2 = s_2 = \frac{1}{2}, m_2 = v_2 = \frac{1}{5}, p_1 = \frac{1}{6}, m_3 = \frac{1}{10}. \)

\[ \begin{cases} 
  u = -4\frac{e^{R_1}((v_1h_1 - m_1s_1)(\cos I_1 + \beta_{12}) - \sin I_1(v_1m_1 + h_1s_1))}{1 + 2e^{R_1} \cos I_1 + \beta_{12}e^{2R_1}} \\
  q = \alpha - 4\frac{\gamma \cos I_1(\cos I_1 + \beta_{12}e^{R_1}) - \sin I_1(v_1p_1 + q_1s_1))}{(1 + 2e^{R_1} \cos I_1 + \beta_{12}e^{2R_1})^2},
\end{cases} \]

where \( \gamma = (v_1q_1-p_1s_1)\gamma - e^{R_1}(v_1p_1 + q_1s_1)((\sin I_1(v_1h_1 - m_1s_1) + \cos I_1(v_1m_1 + h_1s_1)). \) When \( N = 4, M = 2, \) substituting the current (12) into (3), we will
get the 2-order breather solution of (2) (See Fig.3(b)(c)). When \( N = 6, M = 3 \), substituting the current (12) into (3), we will get the 3-order breather solution of (2) (See Fig.3(c)(f))

\[ a_1 = a_2^* = n_1 + m_1 \cdot I, b_1 = b_2^* = q_1 + p_1 \cdot I, r_1 = r_2^* = v_1 + s_1 \cdot I. \]  (17)

When \( N = 3 \), according to (5), we take \( e^{\mu_i} = -1, (i = 1, 2), k_1, k_2 \to 0 \), get

\[ f_3 = \theta_1 \theta_2 + B_{12} + e^{\eta_3} (\theta_1 \theta_2 + B_{12} + C_{13} \theta_2 + C_{23} \theta_1 + C_{13} C_{23}). \]  (18)

where \( \theta_i, B_{ij} \) are given in (8), and

\[ C_{ij} = \frac{-2a_i a_j b_i b_j (a_i b_j + a_j b_i) k_j}{a_i a_j b_i b_j k_j^2 + (a_i b_j - a_j b_i)^2 \alpha}. \]  (19)

By substituting (17)-(18) into (3), the 1-order semi-rational solution of (2) is obtained (Fig.4(a)(b)(e)(f))

When \( N = 4 \), take \( e^{\mu_i} = -1, (i = 1, 2), k_1 \to 0, k_2 \to 0 \), get

\[ f_4 = \theta_1 \theta_2 + B_{12} + e^{\eta_4} (\theta_1 \theta_2 + B_{12} + C_{13} \theta_2 + C_{23} \theta_1 + C_{13} C_{23}) + e^{\eta_4} (\theta_1 \theta_2 + B_{12} + C_{14} \theta_1 + C_{24} \theta_1 + C_{14} C_{24}) + e^{A_{14}} e^{\eta_4} (\theta_1 \theta_2 + B_{12} + (C_{13} + C_{14}) \theta_2 + (C_{23} + C_{24}) \theta_1 + C_{13} C_{23} + C_{14} C_{24} + C_{13} C_{24} + C_{14} C_{24}). \]  (20)

Among them, \( \theta_i, B_{ij}, C_{ij} \) has been given in (8),(19). By substituting (17)-(20) into (3), the 2-order semi-rational solution of (2) is obtained (Fig.4(c)(d)(g)(h)).

When \( N \geq 4 \), we take \( e^{\mu_i} = -1, (i = 1, 2) \), and let \( k_1, k_2 \to 0 \), get

\[ f_N = \theta_1 \theta_2 + B_{12} + e^{\eta_3} \Omega_3 + e^{\eta_4} \Omega_4 + \cdots + e^{\eta_N} \Omega_N, \]  (21)

where \( \eta_j, (j = 3, 4, \cdots, N), \theta_1, \theta_2, B_{12} \) satisfies (12), and \( \Omega_j = \theta_1 \theta_2 + B_{12} + C_{1j} \theta_2 + C_{2j} \theta_1 + C_{1j} C_{2j} \). Through the parameter constraints of (17), Substituting (21) into (3), the high-order semi-rational equation (2) is obtained. From Fig.4, Whether it is 1-order semi-rational solution or 2-order semi-rational solution, as time goes from positive to negative, the lump solution moves from the left to the right, while the 1-soliton and the 2-soliton move from right to left.
Figure 4: The spatial structure of high-order semi-rational solutions of (2): $\alpha = 1, m_1 = 20, n_1 = \frac{1}{6}, p_1 = \frac{1}{2}, q_1 = 100, s_1 = \frac{1}{10}, v_1 = 1, k_3 = 1, a_3 = 1, b_3 = 2, r_3 = -2, \mu_3 = 0, k_4 = 1, a_4 = 1, b_4 = 1, r_4 = 1, \mu_4 = 0$.

6 Mixed solutions

In this section, we use the long wave limit method to construct the mixed solutions by taking the conjugate relationship of some parameters.

6.1 Mixed Solution of breather and Soliton

In order to obtain the mixed solution of 1-breather and 1-soliton and the mixed solution of 1-breather and 2-soliton, we take the parameter constraints:

$\alpha = m_1 = p_1 = h_1 + j_1, a_1 = a_2 = n_1 + m_1 \cdot I, b_1 = b_2 = q_1 + p_1, r_1 = r_2 = v_1 + s_1 \cdot I.$

When $N = 3$, by substituting (22) into (5), the mixed solution of 1-breather and 1-soliton can be obtained. (See Fig.5(a)(e)). When $N = 4$, the mixed solution of 1-breather and 2-soliton can be obtained. (See Fig.5(e)(f)). And the parameter value is: $\alpha = m_1 = p_1 = h_1 = s_1 = v_1 = k_3 = a_3 = r_3 = r_4 = \mu_1 = \mu_2 = 1, n_1 = j_1 = \mu_3 = b_4 = \frac{1}{2}, q_1 = 10, \mu_4 = 2$. 
6.2 Mixed Solution of 1-lump and 1-breather and 1-soliton

At $N = 5$, we can obtain the mixed solutions of 1-lump and 1-breather and 1-soliton. We can simply simplify (5) to

$$\begin{align*}
  f_5 &= \theta_1 \theta_2 + B_{12} + \epsilon^{\eta_1}(\theta_1 \theta_2 + B_{12} + C_{13} \theta_2 + C_{23} \theta_1 + C_{13} C_{23}) \\
  &+ e^{A_{34}} \epsilon^{\eta_4}(B_{12} + (C_{13} + C_{14} + \theta_1)(C_{23} + C_{24} + \theta_2)) \\
  &+ e^{A_{35}} \epsilon^{\eta_5}(B_{12} + (C_{13} + C_{15} + \theta_1)(C_{23} + C_{25} + \theta_2)) \\
  &+ e^{A_{45}} \epsilon^{\eta_5}(B_{12} + (C_{13} + C_{14} + C_{15} + \theta_1) \\
  &\quad (C_{23} + C_{24} + C_{25} + \theta_2)) \\
  &+ e^{\eta_4}(\theta_1 \theta_2 + B_{12} + C_{14} \theta_2 + C_{24} \theta_1 + C_{14} C_{24}) \\
  &+ e^{\eta_5}(\theta_1 \theta_2 + B_{12} + C_{14} \theta_2 + C_{24} \theta_1 + C_{14} C_{24}).
\end{align*}$$

Among them, $\theta_i, B_{ij}, C_{ij}$ has been given in (8) and (19). By substituting (23) into (3), the mixed solutions of 1-lump, 1-breather and 1-soliton of (2) are obtained (see Fig.5(c)(d)(g)(h)). The parameter value is: $\alpha = 1, m_1 = 20, n_1 = \frac{1}{6}, p_1 = \frac{1}{2}, q_1 = 100, s_1 = \frac{1}{3}, v_1 = \frac{8}{5}, m_2 = 15, n_2 = 3, p_2 = \frac{1}{3}, q_2 = \frac{2}{3}, k_5 = 1, a_5 = 1, b_5 = 2, r_i = \mu_i = 0(i = 1, 2, \ldots, 5)$. As time goes from positive to negative, the lump solution moves from left to right, the soliton moves from right to left, and the breather solution is always in the middle.

![Figure 5: The spatial structure of the mixed solution of Eq.(2)](image-url)
6.3 The mixed solutions of lump soliton and tripe soliton

In this section, we will construct a new test function consisting of lump soliton and tripe soliton:

\[ f(x, y, t) = a_0 + \sum_{i=1}^{2} (a_i x + b_i y + d_i t + m_i)^2 + \eta_1 e^{\zeta_1}, \quad (24) \]

where \( \zeta_1 = p_1 x + q_1 y + n_1 t + s_1 \), and \( a_i, b_i, d_i, m_i(i = 1, 2) \) and \( p_1, q_1, n_1, s_1 \), are arbitrary real parameters. Substituting (24) into (4) and merge items, by solving these equations, we get the solution

\[
\begin{align*}
    a_0 &= \frac{(-b_2 p_1^4 + 4\alpha^2 a_1^2)(b_2 p_1^4 + 8\alpha^2 a_1^2 b_2 p_1^4 + 16\alpha^2 a_1^4)}{64a_1^4 b_2^2 \alpha^4 p_1^2}, \\
    a_1 &= a_1, \\
    a_2 &= \frac{-b_2^2 p_1^4 + 4\alpha^2 a_1^2}{4\alpha b_2 p_1^2}, \\
    b_1 &= 0, \\
    b_2 &= b_2, \\
    d_1 &= \frac{3a_1(-b_2 p_1^4 + 4\alpha^2 a_1^2)}{2b_2^2 p_1^2}, \\
    d_2 &= \frac{3(b_2^2 p_1^4 - 24\alpha^2 a_1^2 b_2 p_1^4 + 16\alpha^2 a_1^4)}{16\alpha b_2^2 p_1^2}, \\
    \eta_1 &= \eta_1, \\
    m_1 &= m_1, \\
    m_2 &= m_2, \\
    n_1 &= \frac{-b_2^2 p_1^4 + 12\alpha^2 a_1^2}{4p_1 b_2^2}, \\
    p_1 &= p_1, \\
    q_1 &= \frac{4\alpha^2 b_2^2 p_1^4}{b_2 p_1^4 + 4\alpha^2 a_1^2}, \\
    s_1 &= s_1. \quad (25)
\end{align*}
\]

By substituting (25) into (3), the mixed solutions of lump soliton and tripe soliton of the Eq.(2) can be obtained. ( See Fig.6). When \( t = -4 \), the lump soliton is on the left and the tripe soliton is on the right; when \( t = 0 \), the lump soliton collides with the tripe soliton; when \( t = 4 \), the lump soliton is gradually swallowed by the tripe soliton. The difference between Fig.4 and Fig.6 is analyzed. The evolution behavior of Eq.(24) is mainly reflected in the exponential form \( e^{\zeta_1} \), so the tripe soliton gradually engulfs the lump soliton. The evolution behavior of Eq.(18) is mainly reflected in the algebraic form.
$e^{it}(\theta_1\theta_2 + B_{12} + C_{13}\theta_2 + C_{23}\theta_1 + C_{13}C_{23}),$ with the development of $t$, the mixed algebraic exponential solitary wave evolves into two solitons: a rogue wave and a stripe soliton.

7 Conclusion

In this paper, based on the Hirota method and the long-wave limit method, the N-soliton, the M-order breather and the M-order lump solution of the Eq.(2) are obtained, then the semi-rational solution, the mixed solution of breather and soliton, the mixed solution of lump,soliton and breather, and different types of lump and stripe soliton are obtained by the parametric limit method. We give the spatial evolution diagrams of these solutions and some reasons for their different behaviors, we hope that these results can provide new perspectives and useful information for the field of mathematical physics.

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