Use of Variational Method to Evaluate Well-Posedness of an Advection-Diffusion Reaction Equation of Solute Transport

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Abstract

Although various forms of the solute transport equations model the Spatio-temporal distribution of contaminants in soil or groundwater systems using the corresponding analytical or numerical solutions, the qualitative analysis of well-posed solution properties of these models is rarely considered. This paper investigates the existence and uniqueness of classical solutions to an advection-diffusion reaction equation (ADRE). We derive the findings by applying the method of variation formulation (VF) in the Sobolev space.

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1 Introduction

Traditionally, when modelling heat and mass transport in soil or groundwater systems, studies focus on developing numerical and analytical solutions for partial differential equations (PDEs) [1, 2]. Unlike the case of ordinary differential equations (ODEs), as applied to disease models [3, 4], qualitative analysis of the formulated PDEs to heat and mass transport is rarely considered. Instead, the focus is on the dimensional homogeneity principle [5] of the models formulated. Yet, for the cause-and-effect phenomena such as heat transport, the models are often inhomogeneous [6, 7]. Recently, Hasan-Zadeh [8] examined the existence and uniqueness of the weak solution to the time-dependent advection-diffusion equation (ADE) using advanced components of Sobolev spaces [9], weak solutions and some important integral inequalities, but for an ODE version of the original PDE. Alzate et al. [10] described the variational method to qualitatively study the existence, uniqueness and regularity of the solution to a PDE and applied it to the Poisson’s equation but on the basis of the conditions of the Lax-Milgram Theorem [11, 12]. However, following Clason [13], the Banach-Nečas-Babuška Theorem [14, 15], which generalizes the Lax-Milgram Theorem (see Appendix 4), is particularly well suited for the study of the global existence of solutions to the parabolic PDEs. In this paper, we investigate the existence and uniqueness of solutions to the advection-diffusion reaction equation (ADRE) using the Banach-Nečas-Babuška Theorem in the Sobolev space. The novelty in this case is not conceptual; rather, it is in the manner we set up a transport problem to illuminate a relevant topic further while utilising abstract space analysis and outlining potential futures. The rest of the paper is organized as follows: in Section 2, we present the investigated PDE and list the mathematical preliminaries from the abstract space, relevant for the analysis framework. The result and analysis are presented in Section 3, and in Section 4 the conclusion is made.

2 Methodology

2.1 The model

A two-dimensional analytical solution to the ADRE of bilateral uniform flow with first-order decay, absorption, and sink to model the transport of dissolved solute in a homogeneous and isotropic non-fractured porous medium $\Psi_p$ was developed by [16], but not qualitatively evaluated for the existence and uniqueness of solution properties. To evaluate the solution properties of the model [16]
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\[ R_m \frac{\partial C}{\partial t} + \phi^2 \left( u \frac{\partial C}{\partial x} + w \frac{\partial C}{\partial z} \right) = \frac{\theta_w}{Pe} \left( \frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial z^2} \right) + \zeta \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right) + (\dot{\psi}_3 + \dot{\psi}_4) C + \dot{\psi}_5, \]

(1)

where \( C \) and \( T \) are solute concentration and soil temperature, respectively; \( R_m \) is matrix retardation factor; \( \zeta \) is thermophoresis parameter; \( \dot{\psi}_3, \dot{\psi}_4 \) and \( \dot{\psi}_5 \) are first-order reaction, adsorption and sink coefficients, respectively; \( \phi \) is soil porosity, \( \theta_w \) is soil water content, and \( Pe \) is Peclet number; together with the initial and boundary conditions

\[
\begin{aligned}
C(z,x,0) &= 0 \\
C(0,x,t) &= 1 \\
\lim_{x \to \infty} \frac{\partial C}{\partial x} &= 0 \\
C(\infty,x,t) &= 0 
\end{aligned}
\]

(2)

an investigation of well-posedness in abstract space was needed. Since (1)-(2) is a linear parabolic PDE, we can use the method of variational formulation (VF) [10, 17] or the energy method (EM) [18, 19] to achieve that. But, first, we assume \( \zeta \) is negligible, and transform (2) into Dirichlet boundary condition \( C(z,x,t) = 0 \) for \( x, z \in \partial \Psi_p \). Furthermore, let \( \bar{e}_1 = \phi^2 \) and \( \bar{e}_2 = \theta_w / Pe \).

Conventionally, studies widely employ the VF because of its simplicity. The EM, however, has an advantage over the VF since it does not only describe the mechanism for testing the existence of the unique solution but also helps to identify the type of boundary conditions to use to ensure a unique solution. We employ the VF method but blend it with the EM technique and term the approach ‘the VF-EM’ technique. To use the VF technique, we transform the given PDE (Eq.1) from its original domain \( \Psi_p \) into some fixed reference domain \( \Psi \) and apply the abstract techniques from functional analysis [20]. With the EM, however, we need to show that the energy integral

\[ \dot{E}(t) = \frac{\partial}{\partial t} \int_\Psi u^2, \]

(3)

is a decreasing function for every real-valued function \( u \in L^2(0,T;\mathcal{V}) \). In the VF approach, we employ the Banach-Nečas-Babuška Theorem [13, 20], a generalized Lax-Milgram theorem [12], to show that the model is well posed by investigating existence of unique, positive, and bounded weak solutions [21]. But, first, we recall some definitions and facts from functional analysis [22, 23, 24].
2.2 Preliminaries

Lemma 2.1. Hölder’s inequality [22]. For \( p, q \in \mathbb{R}, \frac{1}{p} + \frac{1}{q} = 1 \), and \( \xi_j, \zeta_j \in \mathbb{C} \),
\[
\sum_{j=1}^{n} |\xi_j| |\zeta_j| \leq \left( \sum_{j=1}^{n} |\xi_j|^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^{n} |\zeta_j|^q \right)^{\frac{1}{q}}.
\]

Lemma 2.2. Cauchy-Schwarz inequality [22]. Let \( \xi, \zeta \in L^2(\Psi) \); then,
\[
| (\xi, \zeta) | \leq \|\xi\|_{L^2(\Psi)} \|\zeta\|_{L^2(\Psi)}.
\]

Definition 2.3. Let \( T > 0 \) be some fixed time and \( \Psi \subset \mathbb{R}^n \) be the given space domain. Set \( Q = (0, T) \times \Psi \) and let \( u(t, \bar{x}) \) be a real-valued function of \( t \) on \( Q \) defined in a Banach space\(^1\) \( V \). Suppose \( V \) consists of functions dependent on \( \bar{x} \) only, then
\[
u : (0, T) \to V, \quad \text{and} \quad t \mapsto u(t) \in V.
\]

Definition 2.4. Hölder spaces \([12, 13]\).

Let \( \mathbb{N} \) be the set of non-negative integers. Let \( Q \) be an open bounded set in \( \mathbb{R}^n \) and let \( k \in \mathbb{N} \). Then, for \( k \geq 0 \), \( C^k(0, T; V) \) is the set of all \( V \)-valued functions on \( [0, T] \) which are \( k \) times continuously differentiable on \( Q \) with respect to \( t \). Further, if \( d_t^j u \) is denoted as the \( j \)th derivative of \( u \), then the Hölder space\(^2\) \( C^k(0, T; V) \) is a Banach space when equipped with the norm
\[
\|u\|_{C^k(0, T; V)} = \sum_{0 \leq j \leq k} \sup_{t \in [0, T]} \|d_t^j u\|_V.
\]

Definition 2.5. Lebesgue/Bochner spaces \([12, 13]\). For \( 1 \leq p \leq \infty \), let \( L^p(0, T; V) \) be defined as the space of all \( V \)-valued functions on \( (0, T) \) for which \( t \mapsto \|u(t)\|_V \) is a function in \( L^p(0, T) \). Then the Lebesgue space\(^3\) \( L^p(0, T) \), with
\[
\int_0^T |u(t)|_V^p dt < \infty,
\]
is a Banach space when equipped with the norm
\(^1\)A complete normed space, i.e. for \( u(t, \bar{x}) \in V, \|u\|_V = \max|u_t|, t \in (0, T) \) whenever \( V \) is a space of continuous functions \([24]\)  
\(^2\)also defined as space of continuous function in \([12]\)  
\(^3\)also called space of integrable functions in \([12]\)
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\[\|u(t)\|_{L^p(0,T;V)} = \begin{cases} \left( \int_0^T |u(t)|_V^p dt \right)^{\frac{1}{p}} & \text{if } p < \infty, \\ \text{ess sup}_{t \in (0,T)} |u(t)|_V & \text{if } p = \infty. \end{cases}\]

Moreover, if \( p = 2 \) and the space \( L^2(0,T) \) is endowed with an inner product \((\cdot, \cdot)\), then \( L^p(0,T) \) is a Hilbert space\(^4\) \( \mathcal{H} \).

Then, we need to construct a weak formulation of the model using some standard results and definitions based on the theory of parabolic PDEs in the Sobolev space.

**Definition 2.6.** Weak derivative [20].

Let \( V \) and \( \mathcal{H} \) be Hilbert spaces\(^5\) and let \( V \subset \mathcal{H} \subset V^{*} \) such that \((V, \mathcal{H}, V^*)\) is a Hilbert triple. Then \( u \in L^2(0,T;V) \) is said to have a weak derivative \( d_t u \in L^2(0,T;V^*) \) if there exists \( \omega \in L^2(0,T;V^*) \) such that

\[
\int_0^T \omega'(t) (u(t), v)_{\mathcal{H}} = - \int_0^T \omega(t) (u(t), v)_{V^*, V} \quad \forall \omega \in D(0,T), v \in V.
\]

**Definition 2.7.** Sobolev spaces [12, 13]. Let \( u \in L^p(0,T;V) \) have a weak derivative \( d_t u \in L^p(0,T;V) \). Then \( u \in W^p_k(0,T;V) \), is defined as a Sobolev space of order \( k \), i.e.,

\[ W^p_k(0,T;V) = \{ u \in L^p(0,T;V) : d_t^j u \in L^p(0,T;V) \} \cdot \]

It is then called a Banach space\(^6\) if endowed with the norm

\[ \|u\|_{W^p_k(0,T;V)} = \|u\|_{L^p(0,T;V)} + \|d_t u\|_{L^p(0,T;V^*)}. \]

**Lemma 2.8.** Hilbert space [20]. The space

\[ W^p_k(0,T;V, V^*) = \{ u \in L^2(0,T;V) : d_t^j u \in L^2(0,T;V^*) \} \cdot \]

equipped with an inner product,

\[
(u, v)_{W^p_k(0,T;V, V^*)} = \int_0^T (u(t), v(t))_{V} + \int_0^T (d_t u, d_t v)_{V^*},
\]

is a Hilbert space, \( \mathcal{H}^1 \).

\(^4\) in general, any vector space \( \bar{X} \), equipped with an inner product \((\cdot, \cdot)_{\bar{X}}\) and the associated norm \( \|u\|_{\bar{X}} = (\cdot, \cdot)_{\bar{X}}^{\frac{1}{2}} \) is called a Hilbert space if there exists a Cauchy sequence \( \{u_m\} \) in \( \bar{X} \).

\(^5\) any vector space \( \bar{X} \), equipped with inner product \((\cdot, \cdot)_{\bar{X}}\) and associated with norm \( \|u\|_{\bar{X}} = (\cdot, \cdot)_{\bar{X}}^{\frac{1}{2}} \) for a Cauchy sequence \( \{u_m\} \) in \( \bar{X} \) [24].

\(^6\) A complete normed space, i.e. for \( u(t, \bar{x}) \in V, \|u\|_V = \max|u|, t \in (0,T) \) whenever \( V \) is a space of continuous functions [24]
Lemma 2.9. Poincaré-Friedrich’s inequality. Let $\Psi$ be an open bounded set in $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Psi$. Let $u \in H_0^1(\Psi)$, then there exists a constant $k_*(\Psi)$, independent of $u$, so that
\[
\int_{\Psi} u^2(x)dx \leq k_* \sum_{i=1}^n \int_{\Psi} \left| \frac{\partial u}{\partial x_i}(x) \right|^2 dx.
\]

Then
- $H_0^1(\Psi) = \{ u \in L^2(\Psi) : d_x u \in L^2(\Psi), u = 0 \text{ on } \partial\Psi \}$,
- $X = H_0^1(0,T) = \{ u \in L^2(0,T;V) : d_t u \in L^2(0,T;V^*), u_0 = 0 \text{ in } X \}$.

Lemma 2.10. Weak-formulation [13] or Transport Theorem [20].

Let $(\mathcal{V}, \mathcal{H}, \mathcal{V}^*)$ be a Hilbert triple such that for every $u, v \in \mathcal{W}(\mathcal{V}, \mathcal{V}^*)$, the map
\[
t \mapsto \langle t; u(t), v(t) \rangle_{\mathcal{H}}.
\]
is absolutely continuous on $t \in (0,T)$ and for almost every $t \in (0,T)$,
\[
\int_0^T \langle d_t u(t), v(t) \rangle_{\mathcal{V}, \mathcal{V}^*} dt = \langle u(T), v(T) \rangle_{\mathcal{H}} - \langle u(0), v(0) \rangle_{\mathcal{H}}
- \int_0^T \{ \langle d_t v(t), u(t) \rangle_{\mathcal{V}, \mathcal{V}^*} - a(t; u(t), v(t)) \} dt. \quad (7)
\]

Using Theorem .1 in Appendix and the above mathematical preliminaries, we establish that there exists a unique solution of (1)–(2). In compressed form, let (1)–(2) be
\[
R_m \partial_t C(\vec{x}, t) - \bar{e}_2 \nabla^2 C(\vec{x}, t) + b_i \nabla C(\vec{x}, t) + \bar{g} C(\vec{x}, t) = f(\vec{x}, t), \quad \text{for } (t, \vec{x}) \text{ in } (0,T) \times \Psi,
\]
\[
C(\vec{x}, 0) = 0, \quad (9a)
C(\vec{x}, t) = 0, \quad \text{on } \partial\Psi, \quad (9b)
\]
where $\vec{x} = (x, z)$, $b_i = \{ \bar{e}_1 u, \bar{e}_1 v \}$, $\bar{g} = (\hat{\varrho}_3 + \hat{\varrho}_4)$, and $f = \hat{\varrho}_5$. Let $a(t; \cdot, \cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ bilinear on $\mathcal{V} \times \mathcal{V}$, $\bar{g} \in L^\infty(0,T;\mathcal{V})$, $b_i, \bar{e}_2 \in H_0^1$, and $f \in L^2(0,T;\mathcal{V}^*)$ be a continuous linear function so that $C_0 \in \mathcal{H}$. 
Then, since “every classical solution of (8)–(9) is a weak solution of (1)–(2)”, we show that (8)–(9) has a unique \textbf{weak} solution \( c \in \mathcal{W}(\mathcal{V}, \mathcal{V}^*) \) so that

\[
\begin{cases}
\langle d_tc(t), v(t) \rangle_{\mathcal{V}^*, \mathcal{V}} + a(t; c(t), v(t)) = \langle f(t), v(t) \rangle_{\mathcal{V}^*, \mathcal{V}}, \forall v \in \mathcal{V}, \forall t \in (0, T), \\
c_0(x) = 0,
\end{cases}
\tag{10}
\]

where \( v \)'s are test functions whose collection generates a Banach space. Eventually (1)–(2) has a unique solution.

### 3 Result Analysis

**Proposition 3.1.** Let \( c \in \mathcal{W}(\mathcal{V}, \mathcal{V}^*) \) be a unique \textbf{weak} solution of (8)–(9). If every property of Theorem 1.1 is satisfied, then (1)–(2) has a unique classical solution.

**Proof.** To show that there exists a weak solution that satisfies (8)–(9), we first recast (10) into a Banach space. Thus, we seek the solution \( c \in \mathcal{X} \) such that

\[
a(c, v) = \mathcal{L}(v), \quad \forall v \in \mathcal{Y}
\]

where the bi-linear function

\[
a(c, v) = \int_{t=0}^{T} \{ \langle d_tc(t), v(t) \rangle_{\mathcal{V}^*, \mathcal{V}} + a(t; c(t), v(t)) \} dt,
\tag{11}
\]

\[
\mathcal{L}(v) = \langle f, v \rangle_{\mathcal{V}^*, \mathcal{V}} = \int_{0}^{T} \langle f(t), v(t) \rangle_{\mathcal{V}^*, \mathcal{V}} dt,
\tag{12}
\]

\[
a(c, v) : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}, \quad \mathcal{X} = \{ c \in \mathcal{W}(\mathcal{V}, \mathcal{V}^*) : c(0) = 0 \}, \quad \mathcal{Y} = L^2(0, T; \mathcal{V}), \text{ and } \mathcal{Y}^* = L^2(0, T; \mathcal{V}^*)
\]

First, we show that Eq. (8) satisfies property (\( i \)) of the Theorem 1.1. Re-arranging Eq. (8), we have

\[
a(t; c(t), v(t)) = -\sum_{i,j} \int_{0}^{T} \frac{\partial}{\partial t} \left( a_{i,j} \frac{\partial c(t)}{\partial t} \right) v(t) dt + \sum_{i} \int_{0}^{T} \left( b_i \frac{\partial c(t)}{\partial t} \right) v(t) dt + \int_{0}^{T} \tilde{g}(t)v(t) dt
\tag{13}
\]

Since the functions to the RHS of Eq. (13) and their derivatives are continuous and integrable over \( L^2(0, T; \mathcal{V}) \), then by definitions (2.4) and (2.5), the mapping \( t \mapsto a(t; c(t), v(t)) \) is Lebesgue measurable for each \( t \in [0, T] \).
Then, we show that boundedness property (ii) of Theorem (1.1) is satisfied. Integrating the first integral on the RHS of Eq. (13) by parts, and using the initial condition \(c(0) = 0\) (see 9a), yields

\[
a(t, \cdot, \cdot) = \sum_{i,j} \int_0^T a_{i,j} \frac{\partial c(t)}{\partial t} \frac{\partial v(t)}{\partial t} dt + \sum_i \int_0^T \left( b_i \frac{\partial c(t)}{\partial t} \right) v(t) dt + \int_0^T \tilde{g}(t) v(t) dt.
\]

(14)

Taking the modulus of (14), we obtain

\[
|a(t; c(t), v(t))| \leq \bar{m} \left\{ \sum \left| \int_0^T \frac{\partial c(t)}{\partial t} \frac{\partial v(t)}{\partial t} dt \right| + \sum \left| \int_0^T \frac{\partial c(t)}{\partial t} v(t) dt \right| + \left| \int_0^T c(t) v(t) dt \right| \right\},
\]

(15)

where \(\bar{m} = \max \left\{ \max_{i,j \leq n; t \in [0,T]} |a_{i,j}(t)|, \max_{i \leq n; t \in [0,T]} |b_i(t)|, \max_{t \in [0,T]} |\tilde{g}(t)| \right\}\). Applying the Hölders inequality (lemma 2.1), for \(p = 2\), into (15) gives

\[
|a(t; c(t), v(t))| \leq \bar{m} \left\{ \sum \left( \int_0^T \left| \frac{\partial c(t)}{\partial t} \right|^2 dt \right)^{\frac{1}{2}} \left( \int_0^T \left| \frac{\partial v(t)}{\partial t} \right|^2 dt \right)^{\frac{1}{2}} \right\} + \bar{m} \left\{ \sum \left( \int_0^T \left| \frac{\partial c(t)}{\partial t} \right|^2 dt \right)^{\frac{1}{2}} \left( \int_0^T \left| v(t) \right|^2 dt \right)^{\frac{1}{2}} + \left( \int_0^T \left| c(t) \right|^2 dt \right)^{\frac{1}{2}} \left( \int_0^T \left| v(t) \right|^2 dt \right)^{\frac{1}{2}} \right\}.
\]

(16)

Algebraically, (16) can be expressed as

\[
|a(t; c(t), v(t))| \leq \bar{m} \left\{ \tilde{x} \tilde{y} + \tilde{x} \tilde{c} + \tilde{v} \tilde{c} + \tilde{v} \tilde{y} \right\},
\]

where the term \(\tilde{v} \tilde{y} = 0\). Since it can be shown, using the Poincaré-Friedrich’s inequality (lemma 2.9), that

\[
\tilde{v} \tilde{y} = \left( \int_0^T \left| c(t) \right|^2 dt \right)^{\frac{1}{2}} \sum \left( \int_0^T \left| \frac{\partial v(t)}{\partial t} \right| dt \right)^{\frac{1}{2}} \geq \frac{\left| c(t) \right|}{\sqrt{k_*}} \left( \int_0^T \left| v(t) \right|^2 dt \right)^{\frac{1}{2}} = 0,
\]

(17)

when \(c(t) = 0\) on \(\partial \Psi\) (see 9b), substituting (17) into (16) leads to
\[ |a(t; c(t), v(t))| \leq \bar{m} \left\{ \left( \int_0^T |c(t)|^2 dt \right)^{\frac{1}{2}} + \sum \left( \int_0^T \left| \frac{\partial c(t)}{\partial t} \right|^2 dt \right)^{\frac{1}{2}} \right\} \times \left\{ \left( \int_0^T |v(t)|^2 dt \right)^{\frac{1}{2}} + \sum \left( \int_0^T \left| \frac{\partial v(t)}{\partial t} \right|^2 dt \right)^{\frac{1}{2}} \right\}. \tag{18} \]

By majorization [25], it can be shown that

\[ \bar{w}^{\frac{3}{2}} + \sum_{j=1}^n (\delta_j)^{\frac{3}{2}} \leq \sqrt{2} \left( \bar{w} + \sum_{j=1}^n \delta_j \right)^{\frac{3}{2}}, \forall \bar{w} \in \mathbb{R}, \delta_j \in \mathbb{R}^n, \]

so that the RHS of (18) simplifies to

\[ \leq 2\bar{m} \left\{ \int_0^T |c(t)|^2 dt + \sum \int_0^T \left| \frac{\partial c(t)}{\partial t} \right|^2 dt \right\}^{\frac{1}{2}} \times \left\{ \int_0^T |v(t)|^2 dt + \sum \int_0^T \left| \frac{\partial v(t)}{\partial t} \right|^2 dt \right\}^{\frac{1}{2}}. \tag{19} \]

Employing the Cauchy-Schwarz inequality (lemma 2.2) under the normed Banach space for \( p = 2 \) onto (19), and letting \( \alpha_0 = 2\bar{m} \), the property (ii) of Theorem (.1)

\[ |a(t; c(t), v(t))| \leq \alpha_0 \| c(t) \|_V \| v(t) \|_V, \tag{20} \]

is satisfied.

To establish **positive definiteness** property (iii) of Theorem (.1), we use the bilinear form

\[
a(t; \cdot, \cdot) = \sum_{i,j} \int_0^T a_{i,j} \frac{\partial c(t)}{\partial t} \frac{\partial c(t)}{\partial t} dt + \sum_i \int_0^T \left( b_i \frac{\partial c(t)}{\partial t} \right) c(t) dt + \int_0^T \bar{g} c(t) c(t) dt.
\]

Using the uniform ellipticity condition [12],

\[
\sum_{i,j=1} \alpha_{i,j} |\xi_i \xi_j| \geq \tilde{d} \sum_{i=1} |\xi_i|^2 \quad \forall \xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n, x \in \bar{\Psi}
\]

where \( \tilde{d} > 0 \) is constant, independent of \( x \) and \( \xi \), we obtain
\[ a(t; c(t), c(t)) \geq \tilde{d} \sum \int_0^T \left| \frac{\partial c(t)}{\partial t} \right|^2 dt + \sum_i \int_0^T \left( b_i \frac{\partial c(t)}{\partial t} \right) c(t) dt + \int_0^T \tilde{g}(t)^2 dt. \quad (21) \]

Since, by the chain rule, we have
\[ \int_0^T \frac{\partial c(t)}{\partial t} c(t) dt = \int_0^T \frac{1}{2} \frac{\partial}{\partial t} c(t)^2 dt, \]
then applying integration by parts to the above expression and substituting the result into (21) yields
\[ a(t; c(t), c(t)) \geq \tilde{d} \sum \int_0^T \left| \frac{\partial c(t)}{\partial t} \right|^2 dt + \int_0^T \left\{ \tilde{g} - \frac{1}{2} \sum_i \frac{\partial b_i(t)}{\partial t} \right\} c(t)^2 dt. \quad (22) \]

Because \( b_i \) and \( \tilde{g} \) are non-negative advection, and reaction and adsorption coefficients, respectively
\[ \tilde{g} - \frac{1}{2} \sum_i \frac{\partial b_i(t)}{\partial t} \geq 0, \forall i = 1, 2, \ldots, n; \quad \forall \vec{x}. \]

Therefore,
\[ a(t; c(t), c(t)) \geq \tilde{d} \sum \int_0^T \left| \frac{\partial c(t)}{\partial t} \right|^2 dt. \quad (23) \]

But, by the Poincaré-Friedrich’s inequality (lemma 2.9),
\[ \sum \int_0^T \left| \frac{\partial c(t)}{\partial t} \right|^2 dt \geq \frac{1}{k_*} \int_0^T |c(t)|^2 dt, \]
so that
\[ k_* a(t; c(t), c(t)) \geq \tilde{d} \int_0^T |c(t)|^2 dt. \quad (24) \]

Summing (23) and (24) yields
\[ a(t; c(t), c(t)) \geq \frac{\tilde{d}}{1 + k_*} \left\{ \int_0^T |c(t)|^2 dt + \sum \int_0^T \left| \frac{\partial c(t)}{\partial t} \right|^2 dt \right\}, \quad (25) \]
so that
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\[ a(t; c(t), c(t)) \geq \alpha_1 \|c(t)\|_{\mathcal{V}}^2, \quad (26) \]

where \( \alpha_1 = \frac{\tilde{d}}{1 + k_s} \). Therefore, the property (iii) of Theorem .1 is satisfied.

Then, to show {**uniqueness**}, by the Lax-Milgram Theorem .2, (26) implies that

\[ \alpha_1 \|c(t)\|_{\mathcal{V}}^2 \leq a(t; c(t), c(t)) = \mathcal{L}(c(t)) = \langle f(t), c(t) \rangle \forall f(t) \in \mathcal{X}, \quad (27) \]

whenever \( \mathcal{V} = \mathcal{W}(\mathcal{V}, \mathcal{V}^*) \).

Since \( \mathcal{L}(c(t)) \) is linear functional on \( \mathcal{X} \), applying the Reisz Representation Theorem\(^7\) (RRT) [22], we have

\[ \langle f(t), c(t) \rangle \leq |\langle f(t), c(t) \rangle| \leq \|f(t)\|_{L^2(0,T;\mathcal{V}^*)} \|c(t)\|_{L^2(0,T;\mathcal{V})}, \]

so that

\[ \|c(t)\|_{\mathcal{W}(\mathcal{V},\mathcal{V}^*)} \leq \frac{1}{\alpha_1} \|f(t)\|_{L^2(0,T;\mathcal{V}^*)} = \frac{1}{\alpha_1} \|f(t)\|_{\mathcal{Y}^*}. \quad (28) \]

Moreover, since the variable \( c \in L^2(0,T; \mathcal{V}) \), we can use Lemma (2.10) to show that

\[ -2 \int_0^T \frac{1}{2} \frac{\partial}{\partial t} \|c(t)\|^2 dt = \int_0^T a(t; c(t), c(t)) dt - \|c(T)\|^2. \quad (29) \]

Hence, substituting the result (26) into Eq. (29) results in

\[ -\frac{\partial}{\partial t} \int_0^T \|c(t)\|^2 dt \geq \alpha_1 \int_0^T \|c(t)\|^2 dt - \|c(T)\|^2 \geq 0, \quad (30) \]

so that

\[ \frac{\partial}{\partial t} \int_0^T \|c(t)\|^2 dt \leq 0. \quad (31) \]

**Remarks**

- Since the result (31) is a condition (3) set by the Energy method (EM), we have used the VF-EM approach to show that (8) is well posed.

- Results (20), (26), and (28) respectively, mean (8) has a non-negative, bounded, and unique solution.

\(^7\)any bounded linear function on a Hilbert space \( \mathcal{H} \) can be represented as an inner product with some unique vector in \( \mathcal{H} \).
4 Conclusions

In this paper, we have investigated the existence and uniqueness of a solution to the advection-diffusion reaction equation in a Sobolev space for the problem of modelling transport of contaminants in porous medium. The main result is proved by using the Banach-Nečas-Babuška theorem based on identifying weak solutions in the Sobolev space. By using the same methodology and concepts as deliberated in this manuscript, we can extend the results to partial differential equations of heat transport transport equations. In future, will explore the use of the Galerkin’s method to prove the existence of solutions of coupled PDEs, and approximate their solutions [26].

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Appendix

Theorem 1. Banach-Nečas-Babuška Theorem [14, 15]

Assume that the bilinear form $a(t; \cdot, \cdot) : V \times V \rightarrow R$ satisfies the following properties;

(i) The mapping $t \mapsto a(t; \cdot, \cdot)$ is measurable for all $u, v \in V$,

(ii) There exists $\alpha_0 > 0$ such that $|a(t; u, v)| \leq \alpha_0 \|u\|_V \|v\|_V$ for almost every $t \in (0, T)$ and all $u, v \in V$,

(iii) There exists $\alpha_1 > 0$ such that $a(t; u, u) \geq \alpha_1 \|u\|_V^2$ for almost every $t \in (0, T)$ and all $u \in V$.

Then, a given parabolic equation has a unique solution $u \in W^{1}_1(V, V^*)$ satisfying

$$\|u\|_{W^{1}_1(V, V^*)} \leq \frac{1}{\alpha_1} \|f\|_{V^*}.$$ 

Theorem 2. Lax-Milgram Theorem [11, 12]

Suppose that $V$ is a real Hilbert space equipped with norm $\| \cdot \|_V$. Let $a(t; \cdot, \cdot) : V \times V \rightarrow R$ be a bilinear form on $V \times V$ and $L(\cdot)$ be a linear form on $V$ such that;
(i) there exists $c_0 > 0$ such that $a(v, v) \geq c_0 \|v\|^2_V$ for all $v \in V$.

(ii) there exists $c_1 > 0$ such that $|a(v, w)| \leq c_0 \|v\|_V \|w\|_V$ for all $v, w \in V$.

(iii) there exists $c_2 > 0$ such that $|L(v)| \leq c_2 \|v\|_V$ for all $v \in V$.

Then, there exists a unique $u \in V$ satisfying

$$\|u\|_{H^1(\Psi)} \leq \frac{1}{c_0} \|f\|_{L^2(\Psi)}$$

such that

$$a(u, v) = L(v) \forall v \in V.$$

References


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