The Origin Moment of $q$-Gaussian Process

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Abstract

In this paper, we discuss the origin moment of the $q$-Gaussian process, which is used to describe anomalous correlated diffusion. The origin moment are obtained using Itô formula and the definition of the $q$-gaussian process. Results of $q$-Gaussian process are compared with the standard Brownian motion.

Keywords: $q$-Gaussian process, Itô formula, The origin moment

1 Introduction

In 1998, C.Tsallis [1], the famous statistician, generalized the classical statistical mechanics and proposed the concept of non-extensive statistical mechanics from the perspective of generalized entropy, and discovered a series of non-extensive probability distribution families, shortened as $q$-distribution. Some scholars maximize the Tsallis derived distribution, that is D.A. Stariolo (1996) [2] study the nonlinear Fokker-Planck stochastic differential equation corresponding to drift term is 0 derived from the non-extensive distribution, called the $q$-Gaussian distribution.

Previously, Tsallis statistical theory was mainly devoted to the field of statistical physics, and it was recognized that it could make the basic concepts of

2 Main Results

We assume as given a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ satisfying the usual conditions. Let $W$ be a standard Brownian motion and the $(\mathcal{F}_t)_{t \geq 0}$ is the underlying filtration for the Brownian motion $W$.

**Definition 2.1.** The process $\Omega = (\Omega(t))_{t \geq 0}$ is called a $q$-gaussian process if $\Omega(t)$ satisfy the following stochastic differential equation

\[
\begin{align*}
\{ &d\Omega(t) = p(\Omega(t), t)^{\frac{1-q}{2}} dW(t), \\
&\Omega(0) = 0,
\end{align*}
\]

(1)

where

\[
p(x, t) = \frac{1}{Z_q(t)} [1 - \beta(t) (1 - q)x^2]^{\frac{1}{1-q}},
\]

\[
\beta(t) = c^{\frac{1-q}{2}} [(2 - q)(3 - q)t]^{\frac{1-q}{2}},
\]

\[
Z_q(t) = [(2 - q)(3 - q)ct]^{\frac{1}{3-q}},
\]

\[
c = \frac{\pi}{q - 1} \left( \Gamma^2\left(\frac{1}{q-1} - \frac{1}{2}\right) \right),
\]

and $\Gamma(.)$ denote as a Gamma function.

**Remark:** $q$ is a parameter reflecting the degree of diffusion. When $q = 1$, the system is normal diffusive and the process is Brownian motion. When $q > 1$, the system is superdiffusive and when $q < 1$, the system is subdiffusive.

The main results of this paper are given by the next theorem.
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**Theorem 2.2.** Let \( \Omega(t) \) be a \( q \)-gaussian process. Then

\[
E(\Omega^{2n}(t)) = \frac{(2n-1)!!}{(5-3q) \times \ldots \times ((2n+3) - (2n+1)q)\beta^n(t)}, n = 1, 2, \ldots
\]  

and

\[
E(\Omega^{2n-1}(t)) = 0, n = 1, 2, \ldots
\]

**Proof.** First, applying the Itô formula, we have

\[
d\Omega^2(t) = 2\Omega(t)d\Omega(t) + p^{1-q}(\Omega(t), t)dt
\]

\[
= 2\Omega(t)p^{1-q}(\Omega(t), t)dW(t) + Z_q(t)^{q-1}[1 - \beta(t)(1-q)\Omega^2]dt,
\]

Taking expectations we have

\[
dE\Omega^2(t) = Z_q(t)^{q-1}[1 - \beta(t)(1-q)E(\Omega^2(t))]dt.
\]

Therefore

\[
E\Omega^2(t) = \exp \left( - \int (Z_q(t))^{q-1}\beta(t)(1-q)dt \right)
\]

\[
\times \left\{ C + \int [(Z_q(t))^{q-1}] \exp \left( \int (Z_q(t))^{q-1}\beta(t)(1-q)dt \right) dt \right\}.
\]

Since

\[
\int (Z_q(t))^{q-1}\beta(t)(1-q)dt = \frac{(1-q)}{(2-q)(3-q)} \ln t,
\]

we conclude that

\[
E\Omega^2(t) = t^{-\frac{(1-q)}{(2-q)(3-q)}} \times \left\{ C + \int [(2-q)(3-q)c]^\frac{1-q}{5-q} t^{\frac{(1-q)}{(2-q)(3-q)}} dt \right\}
\]

\[
= t^{-\frac{(1-q)}{(2-q)(3-q)}} \times \left\{ C + [(2-q)(3-q)c]^\frac{1-q}{5-q} \frac{2}{5-q} t^{\frac{2}{5-q}} \frac{1}{2q(3-q)} \right\}
\]

\[
= Ct^{-\frac{(1-q)}{(2-q)(3-q)}} + \frac{t^{\frac{2}{5-q}} [(2-q)(3-q)]^{\frac{2}{5-q}} c^\frac{1-q}{5-q}}{\beta(t)(5-3q)}.
\]

Substituted \( E\Omega^2(0) = 0 \) yields

\[
E\Omega^2(t) = \frac{1}{\beta(t)(5-3q)}.
\]

Next, using the Itô formula, we have

\[
d\Omega^4(t) = 4\Omega(t)^3 p^{\frac{1-q}{2}}(\Omega(t), t)dW(t) + 6\Omega^2(t)(Z_q(t))^q[1 - \beta(t)(1-q)\Omega^2(t)]dt,
\]
Taking expectations we have
\[ dEΩ^4(t) = 6EΩ^2(t)(Z_q(t))^{q-1}dt - 6(Z_q(t))^{q-1}\beta(t)(1-q)EΩ^4(t)dt \]
\[ = \frac{6}{\beta(t)(5-3q)}(Z_q(t))^{q-1}dt - 6(Z_q(t))^{q-1}\beta(t)(1-q)EΩ^4(t)dt, \]

Combing the same argument and the integral of (4) yields
\[ E(Ω^4(t)) = \left( \exp \int -6(Z_q(t))^{q-1}\beta(t)(1-q)dt \right) \]
\[ \left\{ \int \frac{1}{\beta(t)(5-3q)}(Z_q(t))^{q-1} \left( \exp \int 6(Z_q(t))^{q-1}\beta(t)(1-q)dt \right) dt \right\}. \]
\[ = t^{-\frac{6(1-q)}{(2-q)(3-q)}} \int 6 \left( \frac{1}{\beta(t)(5-3q)}(Z_q(t))^{q-1}t^{\frac{6(1-q)}{(2-q)(3-q)}} \right) dt, \]
\[ = \frac{6}{5-3q} t^{-\frac{6(1-q)}{(2-q)(3-q)}} \int \left( \frac{-2(1-q)\beta(t)(3-q)}{3-q} t^{\frac{2}{3-q}} + \frac{6(1-q)}{(2-q)(3-q)} \right) dt, \]
\[ = \frac{6}{5-3q} c^{-\frac{2(1-q)}{3-q}} [(2-q)(3-q)]^{\frac{2}{3-q}} \frac{1}{(14-10q)} t^{\frac{4}{3-q}}, \]
\[ = \frac{3}{(5-3q)(7-5q)\beta(t)^2}. \]

A similar calculation can be obtained
\[ EΩ^n(t) = \frac{3}{(5-3q)(7-5q)(9-7q)\beta(t)^3}. \]

Therefore
\[ EΩ^{2n}(t) = \frac{(2n-1)!!}{(5-3q) \times \ldots \times (2n+3)-(2n+1)\beta^n(t)}, n = 1, 2, \ldots \]

Second, since Ω(t) is a martingale, we conclude \( EΩ(t) = 0 \). Therefore
\[ dΩ^3(t) = 3Ω(t)^2p^{\frac{1-q}{2}}(Ω(t),t)dW(t) + 3Ω(t)p^{1-q}(Ω(t),t)dt \]
\[ = 3Ω(t)^2p^{\frac{1-q}{2}}(Ω(t),t)dW(t) + 3Ω(t)(Z_q(t))^{q-1}[1-\beta(t)(1-q)Ω^2(t)]dt, \]

Taking expectations yields
\[ dEΩ^3(t) = 3EΩ(t)(Z_q(t))^{q-1}dt - 3(Z_q(t))^{q-1}\beta(t)(1-q)E(Ω^3(t))dt \]
\[ = -3(Z_q(t))^{q-1}\beta(t)(1-q)E(Ω^3(t)dt) \]
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Thus

$$E\Omega^3(t) = C \exp\left(\int (-3(Z_q(t))^{q-1}\beta(t)(1 - q)) \, dt\right) = C t^{\frac{3(q-1)}{q(q-1)}}.$$  

Since $\Omega(0) = 0$, then $E\Omega^3(0) = 0$. Therefore

$$E\Omega^3(t) = 0.$$

Using the Itô formula, we have

$$d\Omega^5(t) = 5\Omega(t)^4 p^{\frac{1-a}{2}}(\Omega(t), t)dW(t) + 10\Omega^3(t)(Z_q(t))^{q-1}[1 - \beta(t)(1 - q)\Omega^2(t)]dt,$$

Taking expectations yields

$$dE\Omega^5(t) = 10E\Omega^3(t)(Z_q(t))^{q-1}dt - 10(Z_q(t))^{q-1}\beta(t)(1 - q)E(\Omega^5(t))dt$$

$$= -10(Z_q(t))^{q-1}\beta(t)(1 - q)E(\Omega^3(t))dt$$

The same argument yields

$$E\Omega^5(t) = 0.$$  

Therefore

$$E(\Omega^{2n-1}(t)) = 0, n = 1, 2, ...$$

\[\Box\]

Remark: In order to ensure the variance of stochastic process is exist, we assume that $1 \leq q < \frac{5}{3}$. If $q = 1$, then $\beta(t) = (2t)^{-1}$. Thus the origin moment of the $q$-gaussian process is given by

$$E(\Omega^{2n}(t)) = (2n - 1)!!x^n, \quad E(\Omega^{2n-1}(t)) = 0, n = 1, 2, ...,$$

which is the same results with the standard Brownian motion $W(t)$.

References

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