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Option Pricing in the Black Scholes Model: A Fair Price of a European Call

Calvine Odiwuor

Department of Statistics and Actuarial Science
Tom Mboya University College
P.O. Box 199-40300, Homa Bay, Kenya

Apaka Rangita

School of Mathematics, Statistics and Actuarial Science
Maseno University, Kenya
P.O. Box 333, Maseno, Kenya

Abstract

In this paper, we review the Black-Scholes formula for the fair price the European call option using a risk-neutral pricing methodology. To achieve this, we use the Girsanovs theorem, Feynman-Kac theorem, and the principles of equivalent martingale measure (EMM) to formulate the said fair price.

Keywords: European call, Option, risk-neutral valuation.

1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space i.e., Ω is the sample space of a random experiment, $\mathcal{F}$ a σ-algebra of events in Ω and $\mathbb{P}$ is a probability measure on $\mathcal{F}$. If we assume that B is a fixed event in a probability measurable space (Ω, $\mathcal{F}$), then the indicator function on B defined by $\mathbf{1}_B$ is defined for all variables $\omega \: \epsilon \: \Omega$ by $\mathbf{1}_B(\omega) = 1$ if $\omega \: \epsilon$ B and 0 otherwise as highlighted in [4], [1], and [7].

Definition 1.1 *(Gaussian/Normal distribution). Let $\beta, \alpha \in \mathbb{R}$ such that $\alpha > 0$. An absolutely continuous random variable X on a probability space*

$(\Omega, \mathcal{F}, \mathbb{P})$ has a Gaussian or Normal distribution with parameters β and α denoted $X \sim N\ (\beta, \alpha^2)$. X has a range $X(\Omega) = \mathbb{R}$ and its probability density function $f_{\beta,\alpha}$ given by

$$f_{\beta,\alpha}(x) = \frac{1}{\sqrt{2\pi}\alpha} e^{-\frac{(x-\beta)}{2\alpha^2}},\ x \in \mathbb{R} \tag{1}$$

as outlined in [4]

Also it is important to note that $\Phi(-x) = 1 - \Phi(x)$ is true for all $x \in [0, \infty)$.

Proposition 1.2 *Let X be any random variable. We say that X follows a normal distribution with mean β and standard deviation α i.e. $X \sim N(\beta, \alpha)$, then we have*

$$\mathbb{E}[e^X fx(X)] = e^{\left(\beta + \frac{\alpha^2}{2}\right)} \mathbb{E}[f_X(X + \alpha^2)]$$

for any non-negative random function f_X.

Definition 1.3 *(Standard Brownian Motion). Let $W = (W_t)_{t\geq 0}$ be a continuous time stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $W = (W_t)_{t\geq 0}$ is called a one - dimensional standard Brownian motion or a standard Weiner process if it holds:*

(i) $W_0 = 0$ *a.s*

(ii) For all times $0 \leq s < t$, the increment is normally distributed with mean 0 and variance $t - s$ i.e $W_t - W_s \sim \mathcal{N}(0, t - s)$,

(iii) For times $0 < t_1 < t_2 <, ..., < t_n$, the increment $W_n - W_{n-1}$, $n = 1, 2, 3, ...$ of the process are independent of each other meaning that for any time $0 \leq s < t$, the corresponding increment $W_t - W_s$ is independent of the σ-algebra, $\sigma(W_k : k \leq s)$,

(iv) All the sample paths $X(\cdot, \omega) : \mathbb{R}^+ \to \mathbb{R}$, $\omega \in \Omega$ are continuous.

as indicated in [1], [3], and [5].

1.1 Change of Measure and Girsanovs Theorem

Girsanovs theorem permits the change of probability measure from physical to risk adjusted measure as outlined in [3], and [1].

Definition 1.4 *Let* $\mathbb{P}and\mathbb{Q}$ *be two probability measures on* $(\Omega, \mathcal{F})$. *The measures are said to be equivalent denoted by* $\mathbb{P} \sim \mathbb{Q}$ *i.e.* $\mathbb{P}(A) = 0$ *if and only if* $\mathbb{Q}(A) = 0$ *on the same* σ *-algebra of* $A \in \mathcal{F}$.

If $\mathbb{P} \sim \mathbb{Q}$, we say that $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$ also denoted by $\mathbb{Q} \ll \mathbb{P}$. In fact there exists a random variable $\gamma : \Omega \to \mathbb{R}$ on $(\Omega, \mathcal{F})$ for which $\mathbb{Q}(A) = \mathbb{E}^{\mathbb{P}}[\gamma \mathbf{1}_A]$ for all events $A \in \mathcal{F}$. This random variable γ is called the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ that is

$$\gamma = \frac{d\mathbb{Q}}{d\mathbb{P}}. \tag{2}$$

Theorem 1.5 *(Girsanov's Theorem as outlined in [2] Let* $(W_t)_{0 \le t \le T}$ *be a standard Brownian motion with respect to physical measure* $\mathbb{P}$ *and a filtration* $\mathbb{F} = \{\mathcal{F}_t\}_{0 \le t \le T}$. *We say the process* $(\gamma_t)_{0 \le t \le T}$ *is adapted to* $\mathbb{F}$ *for a given* $T > 0$.

Defining

$$\rho_t := \exp\left(\int_0^t -\gamma_s dW_s - \frac{1}{2} \int_0^t |\gamma_s|^2 ds \right) \text{ for } 0 \le t \le T \tag{3}$$

and by Radon-Nikodym derivative we define $\mathbb{Q}$ *by*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right)_T = \rho_T. \tag{4}$$

Assume the square integrability condition given by

$$\mathbb{E}\left[\int_0^T |\gamma_s \rho_s|^2 ds \right] < \infty, \tag{5}$$

the process $(\widetilde{W}_t)_{0 \le t \le T}$ *defined by*

$$\widetilde{W}_t := W_t + \int_0^t \gamma_s ds \tag{6}$$

is a Brownian motion under the probability measure $\mathbb{Q}$

Definition 1.6 *(Stochastic differential equation (SDE) according to [8] and [4]). A stochastic differential equation of a one-dimensional real-valued continuous stochastic process* X_t *is an equation of the form*

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t \tag{7}$$

where $X_0 = 0,\ a(t, X_t)$ and $b(t, X_t)$ are initial condition, drift and diffusion coefficients respectively.

Theorem 1.7 *(Discounted Feynman-Kac in [1]). Consider a SDE in equation (7). Let $g(y)$ be a Borel measurable function with r, a constant. If we fix $T > 0$ and let $t \in [0, T]$ we can define*

$$f(t, x) = \mathbb{E}^{t,x}[e^{-r(T-t)} g(X_T)]. \tag{8}$$

Assuming that $\mathbb{E}^{t,x}[|\ g(X_T)] < \infty\ \forall x, t$, then equation (8) is the solution to the PDE;

$$f_t(t, x) + a(t, x) f_x(t, x) + \frac{1}{2} b^2(t, x) f_{xx}(t, x) = r f(t, x), \tag{9}$$

with the terminal condition

$$f(T, x) = g(x) \forall x \tag{10}$$

Definition 1.8 *(Equivalent Martingale Measure (EMM) as outlined in [1], [6], and [8]). Recall the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ modeling the evolution of the stock prices process S_t. Another probability $\mathbb{Q}$ on the measurable space $(\Omega, \mathcal{F})$ is said to be an equivalent martingale probability measure (or a risk-neutral probability measure) if*

- *(i) $\mathbb{Q}$ is equivalent to $\mathbb{P}$ written $(\mathbb{Q} \sim \mathbb{P})$ i.e. for every $A \epsilon \mathcal{F}, \mathbb{P}(A) = 0$ if and only if $\mathbb{Q}(A) = 0$,*

- *(ii) Under $\mathbb{Q}$, the discounted stock price process $\widetilde{S}(t) := [e^{-rt} S(t)]_{0 \le t \le T}$ is a martingale i.e., for $t \le s$*

$$\mathbb{E}^{\mathbb{Q}} \left[\widetilde{S}(s) | \mathcal{F}_t \right] = \widetilde{S}(t) \tag{11}$$

The first fundamental theorem of asset pricing states that the market model does not admit arbitrage opportunities if and only if there exist an equivalent martingale measure [1].

1.2 The Black-Scholes Model for Stock Prices

A risky asset, for example an underlying share of stock with price process S_t which is square integrable in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and is governed by a stochastic differential equation, $dS_t = \alpha S_t dt + \sigma S_t dW_t$ and $S_0 = s_0$, where, W_t is the standard Brownian motion. The stock price is modeled as a geometric Brownian motion with drift α and volatility σ.

The solution to this stochastic differential equation can be obtained by the application of Itô formula on a function $f(x) = \ln(x)$ on $C^{1,2}([0,T] \times \mathbb{R})$ so as to obtain

$$S_t = s_0 \exp\left[(\alpha - \frac{1}{2}\sigma^2)t + \sigma W_t\right] \tag{12}$$

A risk-less asset for example a cash bond with price process B_t following an initial value problem of an ordinary differential equation given by $dB(t) = rB(t)dt$ and $B_0 = 1.$, with r as the continuously compounded rate of interest and the solution to this initial value problem is given by $B_t = e^{rt}$ as outlined in [5], [8], and [1].

2 Existence of Equivalent Martingale Measure

Consider the discounted stock price $\widetilde{S}_t = e^{-rt}S_t$. By applying the Itô formula in [1] we obtain

$$\frac{d\widetilde{S}_t}{\widetilde{S}_t} = \sigma\left(\left(\frac{\mu - r}{\sigma}\right)dt + dW_t\right). \tag{13}$$

Therefore,

Lemma 2.1 *There is a probability measure* $\widetilde{\mathbb{P}}$ *equivalent to* $\mathbb{P}$ *such that the process* $\widetilde{W}_t := W_t + \left(\frac{\mu - r}{\sigma}\right)t, t\epsilon\,[0,T]$ *is a Brownian motion under* $\widetilde{\mathbb{P}}$

Consider the constant process $\gamma_t := \frac{\mu-r}{\sigma}$for all time t, hence by applying Girsanovs Theorem, there exists a probability measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ such that $\widetilde{W}_t$ defined by $\widetilde{W}_t := W_t + \int_0^t \gamma_s ds = W_t + \left(\frac{\mu-r}{\sigma}\right)t, t\epsilon\,[0,T]$ is a Brownian motion. As required.

Lemma 2.2 *The probability measure* $\widetilde{\mathbb{P}}$ *above is an equivalent martingale measure.*

we have $d\widetilde{S}_t = \sigma\widetilde{S}_t d\widetilde{W}_t$, which is an Itô process with zero drift, hence $\widetilde{S}_t$ is a martingale under the measure $\widetilde{\mathbb{P}}$.

The Black-Scholes model is arbitrage free [8], a condition which is further guaranteed by the existence of the equivalent martingale measure. The payoff of a European call option at time zero is given by $c\,(T, S_T) := \max\,(S_T - K, 0)$as outlined in [3], and the fair price of a European call option at any earlier time is $c\,(t, S_t)$. This latter price does not generate arbitrage opportunities in the model.

2.1 Risk Neutral Valuation Principle

A fundamental property of martingale in consideration to the discounted portfolio value is given by

$$e^{-rt}V_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}V_T|\mathcal{F}_t\right]$$

where V_t is the portfolio value.
Therefore the discounted portfolio price process is given by $\widetilde{V}_t = \{e^{-rt}V_t\}_{0\leq t\leq T}$ which is a $\mathbb{Q}$-martingale.

Examining and combining the discount factors to obtain

$$\frac{e^{-rT}}{e^{-rt}} = \frac{B(t)}{B(T)} = e^{-r(T-t)},$$

giving the equivalent martingale measure pricing formula

$$V_t = B(t)\mathbb{E}^{\mathbb{Q}}\left[\frac{V_T}{B(T)} \mid \mathcal{F}_t\right] = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[V_T|\mathcal{F}] \quad t \in [0, T] \tag{14}$$

with r as the constant rate of interest and $(T-t)$ as the total time to maturity for any equivalent martingale measure as outlined in [1] and [3] with $B(T)$ as the numeraire.

If a European call option admits a replicating/hedging portfolio and has a value process V_t then value of this option at any time t equals the value process at that particular time i.e. $c(t, S_t) = V_t \quad \forall \quad t \leq T$

Lemma 2.3 *The discounted fair price of a European call option which is a martingale with respect to canonical filtration $\mathbb{F}^W$ is given by $C(0, S_0) := e^{-rt}c(t, S_t)$ for any time $t = T$ under the equivalent martingale measure $\mathbb{Q}$.*

The expectation of the discounted European call option is given by $\mathbb{E}^Q[e^{-rt}c(t, S_t)]$ under the equivalent martingale measure $\mathbb{Q}$. We can write

$$\mathbb{E}^{\mathbb{Q}}[e^{-rt}c(t, S_t)] = \mathbb{E}^{\mathbb{Q}}[e^{-rt}V_t] = V_0.$$

Recall that $V_0 = c(0, S_0)$ $\mathbb{P}$-a.s, hence we can write that

$$\mathbb{E}^{\mathbb{Q}}[e^{-rt}c(t, S_t)] = c(0, S_0)$$

2.2 The Fair Price of A European Call Option

Next, we assume that the underlying fair-price given by $c(t,x)$ is $c^{1,2}([0,T]\mathbb{R}_+)$,the we can state without proof the Black-Scholes PDE by the theorem which follows.

Theorem 2.4 *(Black-Scholes PDE [1] and [3]). The fair price of a hedgeable European call option with a price function $c(t,x), x > 0 \in Rat$ any time $t \in [0,T]$ is usually the solution to the PDE.*

$$\frac{\partial c(t,x)}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 c(t,x)}{\partial x} + rx\frac{\partial^2 c(t,x)}{\partial x} = rc(tx) \text{ and } c(T,x) = \max(x-K,0)$$

With $c(T,x)$ as the terminal condition, r is the continuously compounding risk-free rate of interest, K is the strike price and σ is the volatility.

The proof of this theorem is omitted and can be obtained in [3].

2.3 An Alternative Proof for A European Call Option

The fair price at time $t = 0$ for a replicable European Call option is given by

$$c(0,S_0) = S_0\Phi(d_1) - Ke^{-rt}\Phi(d_2) \tag{15}$$

where

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\alpha^2}{2}\right)T}{\alpha\sqrt{T}} \tag{16}$$

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\alpha^2}{2}\right)T}{\alpha\sqrt{T}} = d_1 - \alpha\sqrt{T} \tag{17}$$

with $\Phi(x)$ as the cumulative density of a standard normal distribution as discussed in the previous functions. Also $c(t,0) = 0 \quad \forall t$ and $c(t,S) \longrightarrow S$ as $S \longrightarrow \infty$ as outlined in [5].

The following lemma is necessary before outlining the proof.

Lemma 2.5 *Let X $N(\beta,\alpha)$. If m and n are two positive constants. Then we have*

$$\mathbb{E}\left(\max\left(me^X - n, 0\right)\right) = me^{\left(\beta+\frac{\alpha^2}{2}\right)}\Phi\left(\frac{\ln(\frac{m}{n}) + \beta}{\alpha} + \alpha\right) - n\Phi\left(\frac{\ln(\frac{m}{n}) + \beta}{\alpha}\right) \tag{18}$$

Applying proposition 1.2
poof

$$\mathbb{E}\left(\max\left(me^{X}-n,0\right)\right)=m\mathbb{E}\left(e^{X}\mathbf{1}_{[x>\ln(\frac{n}{m})]}\right)-n\mathbb{E}\left(e^{X}\mathbf{1}_{[x>\ln(\frac{n}{m})]}\right) \tag{19}$$

from the fact that

$$\mathbb{E}\left[\mathbf{1}_{[X>\ln(\frac{n}{m})]}\right]=\mathbb{P}\left[X>\ln\left(\frac{n}{m}\right)\right] \tag{20}$$

And using equation (20) we obtain,

$$\mathbb{E}\left(\max(me^{x}-n,0)\right)=m\mathbb{E}\left[e^{X}\mathbf{1}_{\left[X>\ln\left(\frac{n}{m}\right)\right]}\right]-n\mathbb{P}\left[X>\ln\left(\frac{n}{m}\right)\right] \tag{21}$$

$$\mathbb{E}(\max(me^{x}-n,0))=me^{\left(\beta+\frac{\alpha^2}{2}\right)}\mathbb{E}\left[\mathbf{1}_{\left[X+\alpha^2>\ln\left(\frac{n}{m}\right)\right]}\right]-n\mathbb{P}\left[X>\ln\left(\frac{n}{m}\right)\right], \tag{22}$$

Again using equation (20) the first part of the right hand side we have

$$\mathbb{E}\left(\max(me^{X}-n,0)\right)=me^{(\beta+\frac{\alpha^2}{2})}\mathbb{P}\left[X+\alpha^2>\ln\left(\frac{n}{m}\right)\right]-n\mathbb{P}\left[X>\ln\left(\frac{n}{m}\right)\right] \tag{23}$$

$$\mathbb{E}\left(\max(me^{X}-n,0)\right)=me^{(\beta+\frac{\alpha^2}{2})}\mathbb{P}\left[X>\ln\left(\frac{n}{m}\right)-\alpha^2\right]-n\mathbb{P}\left[X>\ln\left(\frac{n}{m}\right)\right] \tag{24}$$

$$\mathbb{E}(\max(me^{x}-n,0))=me^{\left(\beta+\frac{\alpha^2}{2}\right)}\left(1-\mathbb{P}\left[X\leq\ln\left(\frac{n}{m}\right)-\alpha^2\right]\right)-n\left(1-\mathbb{P}\left[X\leq\ln\left(\frac{n}{m}\right)\right]\right)$$

we recall that $\mathcal{Z}:=\frac{X-\beta}{\alpha}$

$$=me^{\beta+\frac{\alpha^2}{2}}\left(1-\Phi\left(\frac{\ln(\frac{n}{m})-\alpha^2-\beta}{\alpha}\right)\right)-n\left(1-\Phi\left(\frac{\ln(\frac{n}{m})-\beta}{\alpha}\right)\right)$$

$$=me^{\beta+\frac{\alpha^2}{2}}\left(1-\Phi\left(\frac{\ln(\frac{n}{m})-\beta}{\alpha}-\alpha\right)\right)-n\left(1-\Phi\left(\frac{\ln(\frac{n}{m})-\beta}{\alpha}\right)\right)$$

from Proposition 1.2 $\Phi(-x)=1-\Phi(x)$

$$=me^{\left(\beta+\frac{\alpha^2}{2}\right)}\left[\Phi\left(\alpha-\frac{\ln(\frac{n}{m})-\beta}{\alpha}\right)\right]-n\left[\Phi\left(\frac{\beta-\ln(\frac{n}{m})}{\alpha}\right)\right]$$

$$=me^{\left(\beta+\frac{\alpha^2}{2}\right)}\left[\phi\left(\frac{\beta+\ln(\frac{m}{n})}{\alpha}+\alpha\right)\right]-n\left[\phi\left(\frac{\beta+\ln(\frac{m}{n})}{\alpha}\right)\right]$$

Feymann-Kac stated that the solution to the Black-Scholes PDE for a fair price of a European Call option at any given time $tlegT$ is given by

$$c(t,x) = \mathbb{E}^{Q}\left(e^{-r(T-t)}\max(S(T)-x,0) \mid S(t)=x\right) \tag{25}$$

Next we let $\frac{n}{m} = K$ therefore we have $\mathbb{E}^{\mathbb{Q}}\left(\max\left(e^{X}-K,0\right)\right)$

From Faymann-kac Theorem we have

$$c(t,x) = \mathbb{E}^{\mathbb{Q}}\left(e^{-r(T-t)} max\left(e^{X}-K,0\right) \mid X_t = x\right)$$

$$= e^{(\beta+\frac{\alpha^2}{2})}\phi\left(\frac{-\ln(K)+\beta+\alpha^2}{\alpha}\right) - K\phi\left(\frac{-\ln(K)+\beta}{\alpha}\right).$$

At time $t=0$, $S_t = s_0$ hence $c(0,S_0)$. The expected market price at time $t \leq T$ is the share price at time T. It is given by $S_0e^{rT} = e^{\left(\beta+\frac{1}{2}\alpha^2T\right)}$ for a Geometric Brownian motion.

Thus we have $\beta = \ln S_0 + (r-\frac{1}{2}\alpha^2)T$, also α^2T is the variance of the normal distribution.

$$\mathbb{E}\left(\max(e^{X}-K,0)\right) = e^{\ln S_0 + rT}\Phi\left(\frac{\ln\left(\frac{S_0}{K}+rT+\frac{1}{2}\alpha^2T\right)}{\alpha\sqrt{T}}\right) - K\Phi\left(\frac{\ln\left(\frac{S_0}{K}+rT-\frac{1}{2}\alpha^2T\right)}{\alpha\sqrt{T}}\right) \tag{26}$$

Thus

$$\mathbb{E}(\mathrm{m}(e^{X}-K,0)) = C(0,S_0)e^{rT}$$

is the expectation of the market at time $t=0$ according to the principle of arbitrage.

$$C(0,S_0)e^{rT} = S_0e^{rT}\Phi\left(\frac{\ln\left(\frac{S_0}{K}+rT+\frac{1}{2}\alpha^2T\right)}{\alpha\sqrt{T}}\right) - K\Phi\left(\frac{\ln\left(\frac{S_0}{K}+rT-\frac{1}{2}\alpha^2T\right)}{\alpha\sqrt{T}}\right) \tag{27}$$

Hence,

$$C(0,S_0) = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2). \tag{28}$$

In conclusion the final equation is the discounted value of a positive surplus between established stock prices and their corresponding strike in the presence of a risk free rate of interest which conforms to the Black-Scholes formula for a fair price of a European call option.

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References

[1] Bjork, T., *Arbitrage theory in continuous time*, Oxford University Press, 2019. https://doi.org/10.1093/oso/9780198851615.001.0001

[2] Campolieti, G., Makarov, R. N., *Financial mathematics: a comprehensive treatment*, CRC Press, 2018. https://doi.org/10.1201/9781315373768

[3] Etheridge, A., & Baxter, M., *A course in financial calculus*, Cambridge University Press, 2002. https://doi.org/10.1017/cbo9780511810107

[4] Seed Ghahraman, *Fundamentals of Probability with Stochastic Processes*, Third edition, Pearson-Prantice Hall, 2015. https://doi.org/10.1201/b19602

[5] Hull J.C. Options, *Futures and Other Derivatives*, 8th edition, Pearson Education, 2010.

[6] Alexei Krouglov, Intuitive proof of Black-Scholes formula based on arbitrage and properties of lognormal distribution, arXiv preprint physics/0612022, 2006.

[7] Antoon Pelsser and Ton Vorst, Option pricing, arbitrage and martingales, *CWI Quarterly*, **10** (1) (1997), 35-53.

[8] Shreve, S. E., *Stochastic calculus for finance II: Continuous-time models*, Vol. 11, Springer Science & Business Media, 2004. https://doi.org/10.1007/978-1-4757-4296-1

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