Option Pricing in the Black Scholes Model:
A Fair Price of a European Call

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Abstract
In this paper, we review the Black-Scholes formula for the fair price of the European call option using a risk-neutral pricing methodology. To achieve this, we use the Girsanov’s theorem, Feynman-Kac theorem, and the principles of equivalent martingale measure (EMM) to formulate the said fair price.

Keywords: European call, Option, risk-neutral valuation.

1 Introduction
Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space i.e., \(\Omega\) is the sample space of a random experiment, \(\mathcal{F}\) a \(\sigma\)-algebra of events in \(\Omega\) and \(\mathbb{P}\) is a probability measure on \(\mathcal{F}\). If we assume that \(B\) is a fixed event in a probability measurable space \((\Omega, \mathcal{F})\), then the indicator function on \(B\) defined by \(1_B\) is defined for all variables \(\omega \in \Omega\) by \(1_B(\omega) = 1\) if \(\omega \in B\) and 0 otherwise as highlighted in [4], [1], and [7].

Definition 1.1 (Gaussian/Normal distribution). Let \(\beta, \alpha \in \mathbb{R}\) such that \(\alpha > 0\). An absolutely continuous random variable \(X\) on a probability space
$(\Omega, \mathcal{F}, \mathbb{P})$ has a Gaussian or Normal distribution with parameters $\beta$ and $\alpha$ denoted $X \sim N(\beta, \alpha^2)$. $X$ has a range $X(\Omega) = \mathbb{R}$ and its probability density function $f_{\beta,\alpha}$ given by

$$f_{\beta,\alpha}(x) = \frac{1}{\sqrt{2\pi\alpha}} e^{-\frac{(x-\beta)^2}{2\alpha^2}}, \ x \in \mathbb{R} \quad (1)$$

as outlined in [4]

Also it is important to note that $\Phi(-x) = 1 - \Phi(x)$ is true for all $x \in [0, \infty)$.

**Proposition 1.2** Let $X$ be any random variable. We say that $X$ follows a normal distribution with mean $\beta$ and standard deviation $\alpha$ i.e. $X \sim N(\beta, \alpha)$, then we have

$$E[e^{X} f_X(x)] = e^{(\beta+\alpha^2/2)} E[f_X(X + \alpha^2)]$$

for any non-negative random function $f_X$.

**Definition 1.3** (Standard Brownian Motion). Let $W = (W_t)_{t \geq 0}$ be a continuous time stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $W = (W_t)_{t \geq 0}$ is called a one-dimensional standard Brownian motion or a standard Weiner process if it holds:

(i) $W_0 = 0$ a.s

(ii) For all times $0 \leq s < t$, the increment is normally distributed with mean $0$ and variance $t - s$ i.e $W_t - W_s \sim \mathcal{N}(0, t - s)$,

(iii) For times $0 < t_1 < t_2 < \ldots, < t_n$, the increment $W_n - W_{n-1}, n = 1, 2, 3, \ldots$ of the process are independent of each other meaning that for any time $0 \leq s < t$, the corresponding increment $W_t - W_s$ is independent of the $\sigma$-algebra, $\sigma(W_k : k \leq s)$,

(iv) All the sample paths $X(\cdot, \omega) : \mathbb{R}^+ \to \mathbb{R}, \omega \in \Omega$ are continuous.

as indicated in [1], [3], and [5].
1.1 Change of Measure and Girsanovs Theorem

Girsanovs theorem permits the change of probability measure from physical to risk adjusted measure as outlined in [3], and [1].

**Definition 1.4** Let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures on $(\Omega, \mathcal{F})$. The measures are said to be equivalent denoted by $\mathbb{P} \sim \mathbb{Q}$ i.e. $\mathbb{P}(A) = 0$ if and only if $\mathbb{Q}(A) = 0$ on the same $\sigma$-algebra of $A \in \mathcal{F}$.

If $\mathbb{P} \sim \mathbb{Q}$, we say that $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$ also denoted by $\mathbb{Q} \ll \mathbb{P}$. In fact there exists a random variable $\gamma: \Omega \to \mathbb{R}$ on $(\Omega, \mathcal{F})$ for which $\mathbb{Q}(A) = E^\mathbb{P} [\gamma 1_A]$ for all events $A \in \mathcal{F}$. This random variable $\gamma$ is called the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ that is

$$
\gamma = \frac{d\mathbb{Q}}{d\mathbb{P}}. \tag{2}
$$

**Theorem 1.5** (Girsanov’s Theorem as outlined in [2] Let $(W_t)_{0 \leq t \leq T}$ be a standard Brownian motion with respect to physical measure $\mathbb{P}$ and a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$. We say the process $(\gamma_t)_{0 \leq t \leq T}$ is adapted to $\mathbb{F}$ for a given $T > 0$.

Defining

$$
\rho_t := \exp \left( \int_0^t -\gamma_s dW_s - \frac{1}{2} \int_0^t |\gamma_s|^2 ds \right) \text{ for } 0 \leq t \leq T \tag{3}
$$

and by Radon-Nikodym derivative we define $\mathbb{Q}$ by

$$
\frac{d\mathbb{Q}}{d\mathbb{P}} := \left( \frac{d\mathbb{Q}}{d\mathbb{P}} \right)_T = \rho_T. \tag{4}
$$

Assume the square integrability condition given by

$$
E \left[ \int_0^T |\gamma_s \rho_s|^2 ds \right] < \infty, \tag{5}
$$

the process $(\tilde{W}_t)_{0 \leq t \leq T}$ defined by

$$
\tilde{W}_t := W_t + \int_0^t \gamma_s ds \tag{6}
$$

is a Brownian motion under the probability measure $\mathbb{Q}$.

**Definition 1.6** (Stochastic differential equation (SDE) according to [8] and [4]). A stochastic differential equation of a one-dimensional real-valued continuous stochastic process $X_t$ is an equation of the form

$$
dX_t = a(t, X_t)dt + b(t, X_t)dW_t \tag{7}
$$
where \( X_0 = 0, a(t, X_t) \) and \( b(t, X_t) \) are initial condition, drift and diffusion coefficients respectively.

**Theorem 1.7** *(Discounted Feynman-Kac in [1]).* Consider a SDE in equation (7). Let \( g(y) \) be a Borel measurable function with \( r, a \) a constant. If we fix \( T > 0 \) and let \( t \in [0, T] \) we can define

\[
f(t, x) = \mathbb{E}^{t,x}[e^{-r(T-t)}g(X_T)].
\]

Assuming that \( \mathbb{E}^{t,x}[g(X_T)] < \infty \forall x, t \), then equation (8) is the solution to the PDE;

\[
f_t(t, x) + a(t, x)f_x(t, x) + \frac{1}{2}b^2(t, x)f_{xx}(t, x) = rf(t, x),
\]

with the terminal condition

\[
f(T, x) = g(x) \forall x
\]

**Definition 1.8** *(Equivalent Martingale Measure (EMM) as outlined in [1], [6], and [8]).* Recall the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) modeling the evolution of the stock prices process \( S_t \). Another probability \( \mathbb{Q} \) on the measurable space \((\Omega, \mathcal{F})\) is said to be an equivalent martingale probability measure (or a risk-neutral probability measure) if

- (i) \( \mathbb{Q} \) is equivalent to \( \mathbb{P} \) written \( \mathbb{Q} \sim \mathbb{P} \) i.e. for every \( A \in \mathcal{F}, \mathbb{P}(A) = 0 \) if and only if \( \mathbb{Q}(A) = 0 \),

- (ii) Under \( \mathbb{Q} \), the discounted stock price process \( \tilde{S}(t) := [e^{-rt}S(t)]_{0 \leq t \leq T} \) is a martingale i.e., for \( t \leq s \)

\[
\mathbb{E}^{\mathbb{Q}}[\tilde{S}(s)|\mathcal{F}_t] = \tilde{S}(t)
\]

The first fundamental theorem of asset pricing states that the market model does not admit arbitrage opportunities if and only if there exist an equivalent martingale measure [1].

**1.2 The Black-Scholes Model for Stock Prices**

A risky asset, for example an underlying share of stock with price process \( S_t \) which is square integrable in the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and is governed by a stochastic differential equation, \( dS_t = \alpha S_t dt + \sigma S_t dW_t \) and \( S_0 = s_0 \), where, \( W_t \) is the standard Brownian motion. The stock price is modeled as a geometric Brownian motion with drift \( \alpha \) and volatility \( \sigma \).
The solution to this stochastic differential equation can be obtained by the application of Itô formula on a function $f(x) = \ln(x)$ on $C^{1,2}([0, T] \times \mathbb{R})$ so as to obtain

$$S_t = s_0 \exp \left[ \left( \alpha - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right] \tag{12}$$

A risk-less asset for example a cash bond with price process $B_t$ following an initial value problem of an ordinary differential equation given by $dB(t) = rB(t)dt$ and $B_0 = 1.$, with $r$ as the continuously compounded rate of interest and the solution to this initial value problem is given by $B_t = e^{rt}$ as outlined in [5], [8], and [1].

2 Existence of Equivalent Martingale Measure

Consider the discounted stock price $\tilde{S}_t = e^{-rt}S_t.$ By applying the Itô formula in [1] we obtain

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = \sigma \left( \left( \frac{\mu - r}{\sigma} \right) dt + dW_t \right). \tag{13}$$

Therefore,

**Lemma 2.1** There is a probability measure $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ such that the process $\tilde{W}_t := W_t + \left( \frac{\mu - r}{\sigma} \right) t, t \in [0, T]$ is a Brownian motion under $\tilde{\mathbb{P}}$

Consider the constant process $\gamma_t := \frac{\mu - r}{\sigma}$ for all time $t$, hence by applying Girsanov’s Theorem, there exists a probability measure $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ such that $\tilde{W}_t$ defined by $\tilde{W}_t := W_t + \int_0^t \gamma_s ds = W_t + \left( \frac{\mu - r}{\sigma} \right) t, t \in [0, T]$ is a Brownian motion. As required.

**Lemma 2.2** The probability measure $\tilde{\mathbb{P}}$ above is an equivalent martingale measure.

we have $d\tilde{S}_t = \sigma \tilde{S}_td\tilde{W}_t,$ which is an Itô process with zero drift, hence $\tilde{S}_t$ is a martingale under the measure $\tilde{\mathbb{P}}$.

The Black-Scholes model is arbitrage free [8], a condition which is further guaranteed by the existence of the equivalent martingale measure. The payoff of a European call option at time zero is given by $c(T, S_T) := \max(S_T - K, 0)$as outlined in [3], and the fair price of a European call option at any earlier time is $c(t, S_t)$. This latter price does not generate arbitrage opportunities in the model.
2.1 Risk Neutral Valuation Principle

A fundamental property of martingale in consideration to the discounted portfolio value is given by

\[ e^{-rt}V_t = \mathbb{E}^Q [e^{-rT}V_T | \mathcal{F}_t] \]

where \( V_t \) is the portfolio value.

Therefore the discounted portfolio price process is given by \( \tilde{V}_t = \{e^{-rt}V_t\}_{0 \leq t \leq T} \) which is a \( \mathbb{Q} \)-martingale.

Examining and combining the discount factors to obtain

\[ \frac{e^{-rT}}{e^{-rt}} = \frac{B(t)}{B(T)} = e^{-r(T-t)}, \]

giving the equivalent martingale measure pricing formula

\[ V_t = B(t)\mathbb{E}^Q \left[ \frac{V_T}{B(T)} \mid \mathcal{F}_t \right] = e^{-r(T-t)}\mathbb{E}^Q[V_T \mid \mathcal{F}] \quad t \in [0, T] \quad (14) \]

with \( r \) as the constant rate of interest and \((T-t)\) as the total time to maturity for any equivalent martingale measure as outlined in [1] and [3] with \( B(T) \) as the numeraire.

If a European call option admits a replicating/hedging portfolio and has a value process \( V_t \) then value of this option at any time \( t \) equals the value process at that particular time i.e. \( c(t, S_t) = V_t \quad \forall \quad t \leq T \)

Lemma 2.3 The discounted fair price of a European call option which is a martingale with respect to canonical filtration \( \mathbb{F}^W \) is given by \( C(0, S_0) := e^{-rt}c(t, S_t) \) for any time \( t = T \) under the equivalent martingale measure \( \mathbb{Q} \).

The expectation of the discounted European call option is given by \( \mathbb{E}^Q[e^{-rt}c(t, S_t)] \) under the equivalent martingale measure \( \mathbb{Q} \). We can write

\[ \mathbb{E}^Q[e^{-rt}c(t, S_t)] = \mathbb{E}^Q[e^{-rt}V_t] = V_0. \]

Recall that \( V_0 = c(0, S_0) \) \( \mathbb{P} \)-a.s, hence we can write that

\[ \mathbb{E}^Q[e^{-rt}c(t, S_t)] = c(0, S_0) \]
2.2 The Fair Price of A European Call Option

Next, we assume that the underlying fair-price given by \( c(t, x) \) is \( c^{1,2}([0, T] \mathbb{R}_+) \), the we can state without proof the Black-Scholes PDE by the theorem which follows.

**Theorem 2.4** (Black-Scholes PDE [1] and [3]). The fair price of a hedgeable European call option with a price function \( c(t, x), x > 0 \in \mathbb{R} \) at any time \( t \in [0, T] \) is usually the solution to the PDE.

\[
\frac{\partial c(t, x)}{\partial t} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 c(t, x)}{\partial x^2} + r x \frac{\partial c(t, x)}{\partial x} = r c(t, x) \quad \text{and} \quad c(T, x) = \max(x - K, 0)
\]

With \( c(T, x) \) as the terminal condition, \( r \) is the continuously compounding risk-free rate of interest, \( K \) is the strike price and \( \sigma \) is the volatility.

The proof of this theorem is omitted and can be obtained in [3].

2.3 An Alternative Proof for A European Call Option

The fair price at time \( t = 0 \) for a replicable European Call option is given by

\[
c(0, S_0) = S_0 \Phi(d_1) - Ke^{-rt} \Phi(d_2)
\]

where

\[
d_1 = \frac{\ln \left( \frac{S_0}{K} \right) + \left( r + \frac{\sigma^2}{2} \right) T}{\alpha \sqrt{T}}
\]

(16)

\[
d_2 = \frac{\ln \left( \frac{S_0}{K} \right) + \left( r - \frac{\sigma^2}{2} \right) T}{\alpha \sqrt{T}} = d_1 - \alpha \sqrt{T}
\]

(17)

with \( \Phi(x) \) as the cumulative density of a standard normal distribution as discussed in the previous functions. Also \( c(t, 0) = 0 \ \forall t \) and \( c(t, S) \rightarrow S \) as \( S \rightarrow \infty \) as outlined in [5].

The following lemma is necessary before outlining the proof.

**Lemma 2.5** Let \( X \sim N(\beta, \alpha) \). If \( m \) and \( n \) are two positive constants. Then we have

\[
\mathbb{E} \left( \max \left( me^X - n, 0 \right) \right) = me^{(\beta + \frac{n^2}{2})} \Phi \left( \frac{\ln \left( \frac{m}{n} \right) + \beta}{\alpha} + \alpha \right) - n \Phi \left( \frac{\ln \left( \frac{m}{n} \right) + \beta}{\alpha} \right)
\]

(18)
Applying proposition 1.2

Proof

\[ \mathbb{E} \left( \max \left( me^X - n, 0 \right) \right) = m \mathbb{E} \left( e^X 1_{[X > \ln \left( \frac{n}{m} \right)]} \right) - n \mathbb{E} \left( e^X 1_{[X > \ln \left( \frac{n}{m} \right)]} \right) \]  \hspace{1cm} (19)

from the fact that

\[ \mathbb{E} \left[ 1_{[X > \ln \left( \frac{n}{m} \right)]} \right] = \mathbb{P} \left[ X > \ln \left( \frac{n}{m} \right) \right] \]  \hspace{1cm} (20)

And using equation (20) we obtain,

\[ \mathbb{E} \left( \max \left( me^X - n, 0 \right) \right) = m \mathbb{E} \left[ e^X 1_{[X > \ln \left( \frac{n}{m} \right)]} \right] - n \mathbb{P} \left[ X > \ln \left( \frac{n}{m} \right) \right] \]  \hspace{1cm} (21)

\[ \mathbb{E} (\max(\alpha^e - n, 0)) = m e^{\left( \beta + \frac{\alpha^2}{2} \right)} \mathbb{E} \left[ 1_{[X > \ln \left( \frac{n}{m} \right)]} \right] - n \mathbb{P} \left[ X > \ln \left( \frac{n}{m} \right) \right] , \]  \hspace{1cm} (22)

Again using equation (20) the first part of the right hand side we have

\[ \mathbb{E} \left( \max \left( me^X - n, 0 \right) \right) = m e^{\left( \beta + \frac{\alpha^2}{2} \right)} \mathbb{P} [X + \alpha^2 > \ln \left( \frac{n}{m} \right)] - n \mathbb{P}[X > \ln \left( \frac{n}{m} \right)] \]  \hspace{1cm} (23)

\[ \mathbb{E} (\max(\alpha^e - n, 0)) = m e^{\left( \beta + \frac{\alpha^2}{2} \right)} \mathbb{P}[X > \ln \left( \frac{n}{m} \right) - \alpha^2] - n \mathbb{P}[X > \ln \left( \frac{n}{m} \right)] \]  \hspace{1cm} (24)

we recall that \( Z := \frac{X - \beta}{\alpha} \)

\[ = m e^{\beta + \frac{\alpha^2}{2}} \left( 1 - \Phi \left( \frac{\ln \left( \frac{n}{m} \right) - \alpha^2 - \beta}{\alpha} \right) \right) - n \left( 1 - \Phi \left( \frac{\ln \left( \frac{n}{m} \right) - \beta}{\alpha} \right) \right) \]

\[ = m e^{\beta + \frac{\alpha^2}{2}} \left( 1 - \Phi \left( \frac{\ln \left( \frac{n}{m} \right) - \beta}{\alpha} \right) \right) - n \left( 1 - \Phi \left( \frac{\ln \left( \frac{n}{m} \right) - \beta}{\alpha} \right) \right) \]

from Proposition 1.2 \( \Phi(-x) = 1 - \Phi(x) \)

\[ = m e^{\beta + \frac{\alpha^2}{2}} \left[ \Phi \left( \frac{\alpha - \ln \left( \frac{n}{m} \right) - \beta}{\alpha} \right) \right] - n \left[ \Phi \left( \frac{\beta - \ln \left( \frac{n}{m} \right)}{\alpha} \right) \right] \]

\[ = m e^{\beta + \frac{\alpha^2}{2}} \left[ \phi \left( \frac{\beta + \ln \left( \frac{n}{m} \right)}{\alpha} + \alpha \right) \right] - n \left[ \phi \left( \frac{\beta + \ln \left( \frac{n}{m} \right)}{\alpha} \right) \right] \]
Feynman-Kac stated that the solution to the Black-Scholes PDE for a fair price of a European Call option at any given time $t \leq T$ is given by

$$c(t, x) = \mathbb{E}^Q \left( e^{-r(T-t)} \max(S(T) - x, 0) \mid S(t) = x \right)$$ \hspace{1cm} (25)

Next we let $\frac{n}{m} = K$ therefore we have $\mathbb{E}^Q \left( \max \left( e^X - K, 0 \right) \mid X_t = x \right)$

From Faymann-kac Theorem we have

$$c(t, x) = \mathbb{E}^Q \left( e^{-r(T-t)} \max \left( e^X - K, 0 \right) \mid X_t = x \right) = e^{(\beta + \frac{1}{2} \alpha^2)T} \phi \left( \frac{-\ln(K) + \beta + \alpha^2}{\alpha} \right) - K \phi \left( \frac{-\ln(K) + \beta}{\alpha} \right).$$

At time $t = 0$, $S_t = s_0$ hence $c(0, s_0)$. The expected market price at time $t \leq T$ is the share price at time $T$. It is given by $S_0 e^{rT} = e^{(\beta + \frac{1}{2} \alpha^2)T}$ for a Geometric Brownian motion.

Thus we have $\beta = \ln S_0 + (r - \frac{1}{2} \alpha^2)T$, also $\alpha^2 T$ is the variance of the normal distribution.

$$\mathbb{E} \left( \max(e^X - K, 0) \right) = e^{\ln S_0 + rT} \Phi \left( \frac{\ln \left( \frac{S_0}{K} + rT + \frac{1}{2} \alpha^2 T \right)}{\alpha \sqrt{T}} \right) - K \Phi \left( \frac{\ln \left( \frac{S_0}{K} + rT - \frac{1}{2} \alpha^2 T \right)}{\alpha \sqrt{T}} \right)$$

Thus

$$\mathbb{E}(m(e^X - K, 0)) = C(0, S_0) e^{rT}$$

is the expectation of the market at time $t = 0$ according to the principle of arbitrage.

$$C(0, S_0) e^{rT} = S_0 e^{rT} \Phi \left( \frac{\ln \left( \frac{S_0}{K} + rT + \frac{1}{2} \alpha^2 T \right)}{\alpha \sqrt{T}} \right) - K \Phi \left( \frac{\ln \left( \frac{S_0}{K} + rT - \frac{1}{2} \alpha^2 T \right)}{\alpha \sqrt{T}} \right)$$

Hence,

$$C(0, S_0) = S_0 \Phi(d_1) - Ke^{-rT} \Phi(d_2).$$

In conclusion the final equation is the discounted value of a positive surplus between established stock prices and their corresponding strike in the presence of a risk free rate of interest which conforms to the Black-Scholes formula for a fair price of a European call option.

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