Domination Defect
for the Join and Corona of Graphs

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Abstract

The domination number of a graph $G$ denoted by $\gamma(G)$ is the minimum number of vertices required to dominate all the vertices of $G$. The minimality of $\gamma(G)$ implies that if $W \subseteq V(G)$ such that $|W| < \gamma(G)$, then there is at least one vertex of $G$ that is not dominated by $W$. Given a positive integer $k < \gamma(G)$, where $\gamma(G) \geq 2$, the $k$-domination defect of $G$ denoted by $\zeta_k(G)$ is the minimum number of vertices of $G$ left undominated by any subset of vertices of $G$ with cardinality $\gamma(G) - k$. In this paper, we study further the concept of $k$-domination defect of a graph $G$ and investigate it for graphs resulting from some binary operations. In particular, the $k$-domination defect sets of the join and corona of two nontrivial graphs are examined. Consequently, the corresponding $k$-domination defect of these graphs are obtained.

Mathematics Subject Classification: 05C69, 05C76

Keywords: $k$-domination defect, join of graphs, corona of graphs
1 Introduction

Let $G = (V(G), E(G))$ be a graph. For each $x \in V(G)$, the set $N_G(x) = \{y \in V(G) \mid xy \in E(G)\}$ is the open neighborhood of $x$ in $G$ while the set $N_G[x] = N_G(x) \cup \{x\}$ is the closed neighborhood of $x$ in $G$. For a nonempty set $S \subseteq V(G)$, the open neighborhood of $S$ in $G$ and the closed neighborhood of $S$ in $G$ are given by the sets $N_G(S) = \cup_{x \in S} N_G(x)$ and $N_G[S] = N_G(S) \cup S$, respectively.

A nonempty set $S \subseteq V(G)$ is a dominating set of $G$ if $N_G[S] = V(G)$. The minimum cardinality of a dominating set in $G$ is the domination number of $G$, denoted by $\gamma(G)$. The minimality of $\gamma(G)$ implies that if a particular set $W$ of vertices of $G$ has cardinality less than $\gamma(G)$, then there is at least one vertex of $G$ that is not dominated by $W$. This led to the introduction of a new graph parameter called the $k$-domination defect of a graph in [1].

To illustrate, imagine a network of facilities inside a big compound and a set of security guards providing security to the facilities in this network. Suppose that in this network every security guard stationed in a given facility can also guard all other facilities within a fixed distance $r$ from his position, so that two distinct facilities are considered adjacent if and only if they are within distance $r$ from each other. The minimum number $\gamma$ of security guards needed to guard the network corresponds to the domination number of this network. Due to some budgetary constraints, suppose the owner hired exactly $\gamma$ security guards, and suppose that in any given day $k$ of these guards are absent from duty. How should the owner position the remaining $\gamma - k$ guards on duty so that the number of facilities left unguarded is kept to a minimum?

Interestingly, a number of researchers attempted to also study many graphs where the dominating set exhibits some flaws. Slater and Roden [2] introduced the liar’s domination in graphs and this was further studied by Balandra, Canoy, and Aniversario [3], where the concept was said to provide a way of modeling protection devices where one may be faulty. In 2018 Das and Desormeaux [1] introduced the aforementioned domination defect of a graph which allows us to study the vulnerability of a facility if it will be guarded with fewer than the minimum number of guards necessary.

In this paper, we study further the concept of $k$-domination defect of a graph $G$ and investigate it for graphs resulting from some binary operations. In particular, the $k$-domination defect sets of the join and corona of two nontrivial graphs shall be examined using some properties possessed by each operation and some inherent properties possessed by the constituents or the individual graphs involved in the operation. Our final objective is to see how the corresponding $k$-domination defects of these graphs are dictated by some inherent properties of the constituents, a subtheme reminiscent of the works in [4] and [5].

In this work, all graphs shall be understood in the context of being finite,
undirected, and simple. To avoid triviality, only graphs with domination number at least 2 will be considered. For basic graph theoretic terminologies not specifically described in this paper, please refer to either [6] or [7].

2 Preliminary Notes

Two of the main concepts in this paper are formally defined below for emphasis.

Definition 2.1 [5] Let $G = (V(G), E(G))$. A subset $S$ of $V(G)$ is a dominating set of $G$ if for every $u \in V(G) \setminus S$, there exists $v \in S$ such that $uv \in E(G)$, that is, $N[S] = V(G)$. The domination number of $G$, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of $G$. A dominating set of $G$ with cardinality $\gamma(G)$ is called a $\gamma$-set of $G$.

Example 2.2 In Figure 1(a) below, set $S = \{s, t, u\}$ is a dominating set of $G$ of minimum cardinality; thus, $\gamma(G) = 3$. In Figure 1(b), graph $H$ has 8 distinct $\gamma$-sets of the form $\{u_i, v_j\}$, $i = 1, 2, 3, 4$ and $j = 1, 2$, with $\{v_1, v_2\}$ as another $\gamma$-set of $H$.

![Graphs G and H](image)

Figure 1: Two graphs $G$ and $H$, where $\gamma(G) = 3$ and $\gamma(H) = 2$

Definition 2.3 [1] Let $G$ be a graph of order $n$ with $\gamma(G) \geq 2$ and let $1 \leq k < \gamma(G)$. Let $S \subseteq V(G)$ with $|S| = \gamma(G) - k$. The $k$-defect of $S$ is $\zeta_k(S) = |V(G) \setminus N_G[S]| = n - |N_G[S]|$. The $k$-domination defect of $G$, denoted by $\zeta_k(G)$, is the minimum cardinality of the set $V(G) \setminus N_G[W]$ for $W \subseteq V(G)$ with $|W| = \gamma(G) - k$. A set $S \subseteq V(G)$ of cardinality $\gamma(G) - k$ for which $|V(G) \setminus N_G[S]| = \zeta_k(G)$ is called a $\zeta_k$-set of $G$.

Example 2.4 Consider the graphs $G$ and $H$ given in Figure 1, where $\gamma(G) = 3$ and $\gamma(H) = 2$. By inspection, $G$ has only one $\zeta_1$-set, namely $S = \{s, t\}$, while $H$ has two distinct $\zeta_1$-sets, namely $S_1 = \{v_1\}$ and $S_2 = \{v_2\}$. 
By Definition 2.3, \(\zeta_1(G) = 2\) and \(\zeta_1(H) = 1\). On the other hand, set \(\{t\}\) is the only \(\zeta_2\)-set of \(G\); thus, \(\zeta_2(G) = 5\).

From Definition 2.3 we emphasize that if \(G\) is a graph with \(\gamma(G) \geq 2\) and \(S \subseteq V(G)\) is a \(\zeta_k\)-set of \(G\), where \(1 \leq k < \gamma(G)\), then \(|S| = \gamma(G) - k\) such that \(|N_G[S]| = \max\{|N_G[W]| : W \subseteq V(G), |W| = \gamma(G) - k\}|.

**Lemma 2.5** Let \(G\) be a nontrivial graph such that \(\gamma(G) \geq 2\) and let \(k = \gamma(G) - 1\). Then \(S \subseteq V(G)\) is a \(\zeta_k\)-set of \(G\) if and only if \(S = \{v\}\) for some \(v \in V(G)\) with \(\deg_G(v) = \Delta(G)\).

*Proof:* Let \(k = \gamma(G) - 1\) and let \(S \subseteq V(G)\). Then by definition \(S\) is a \(\zeta_k\)-set of \(G\) if and only if \(|S| = \gamma(G) - k = \gamma(G) - (\gamma(G) - 1) = 1\) such that \(|N_G[S]|\) is maximum in \(G\). This means that \(S\) in this case is a \(\zeta_k\)-set of \(G\) if and only if \(S = \{v\}\) for some \(v \in V(G)\) with \(\deg_G(v) = \Delta(G)\). \qed

### 3 Main Results

Recall that the *join* or *complete product* of two graphs \(G\) and \(H\) is the graph \(G + H\) with vertex set \(V(G + H) = V(G) \cup V(H)\) and edge set \(E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}\), where the symbol \(\cup\) denotes disjoint union. The figure below provides an illustration of this binary operation.

![Graphs P_2, C_6, and P_2 + C_6](image)

**Figure 2:** The graphs \(P_2\) and \(C_6\), and the join \(P_2 + C_6\)

**Remark 3.1** Let \(G\) and \(H\) be nontrivial graphs. Then it is straightforward to see that \(\gamma(G + H) = \begin{cases} 1, & \text{if either } G \text{ or } H \text{ has a spanning star;} \\ 2, & \text{otherwise.} \end{cases}\)
**Theorem 3.2** Let $G$ and $H$ be nontrivial graphs of orders $m$ and $n$, respectively, such that $G$ nor $H$ has a spanning star. Then the $\zeta_1-$sets of $G+H$ are exactly the singletons $\{u\}$, where either $u \in V(G)$ with $\deg_{(G+H)}(u) = \Delta(G) + n \geq \Delta(H) + m$ or $u \in V(H)$ with $\deg_{(G+H)}(u) = \Delta(H) + m \geq \Delta(G) + n$.

**Proof:** This is a direct consequence of Remark 3.1 and Lemma 2.5. ■

**Corollary 3.3** Let $G$ and $H$ be nontrivial graphs of orders $m$ and $n$, respectively, such that $G$ nor $H$ has a spanning star. Then $\zeta_1(G+H) = \min\{m - 1 - \Delta(G), n - 1 - \Delta(H)\}$.

**Proof:** Let $S$ be the $\zeta_1$-set of $G+H$. By Theorem 3.2, $S = \{u\}$, where either $u \in V(G)$ or $u \in V(H)$ such that $|N_{G+H}[u]| = \max\{\Delta(G) + n + 1, \Delta(H) + m + 1\}$. Consequently, $\zeta_1(G+H) = m + n - |N_{G+H}[u]| = m + n - \max\{\Delta(G) + n + 1, \Delta(H) + m + 1\} = \min\{m - 1 - \Delta(G), n - 1 - \Delta(H)\}$. ■

Recall that if $G$ and $H$ are graphs of orders $m$ and $n$, respectively, then the corona of $G$ by $H$ is the graph $G \circ H$ obtained by taking one copy of $G$ and $m$ copies of $H$, and then joining the $i$th vertex of $G$ to every vertex of the $i$th copy of $H$. We denote by $H^v$ the copy of $H$ whose vertices are joined with vertex $v \in V(G)$, and then by $v + H^v$ the subgraph of $G \circ H$ corresponding to the join $\{v\} + H^v$.

In Figure 3(c), one can see that the set $\{a, b, c\}$ is a dominating set of $P_3 \circ C_3$ of minimum cardinality. Thus, $\gamma(P_3 \circ C_3) = 3$. In general, for any two graphs $G$ and $H$ of orders $m$ and $n$, respectively, the domination number of the corona $G \circ H$ is always equal to $m = |V(G)|$. We mark this statement as follows.

**Lemma 3.4** [5] If $G$ and $H$ are graphs, then $\gamma(G \circ H) = |V(G)|$.

Actually, if $H$ is a nontrivial graph without a spanning star, then $W = V(G)$ is the unique $\gamma - set$ of $G \circ H$. However, if $H$ has $k_0$ distinct spanning stars and $G$ is of order $m$, then the corona $G \circ H$ has $(1 + k_0)^m$ distinct $\gamma - sets$. 

\[\text{Figure 3: The graphs } P_3, C_3, \text{ and the corona } P_3 \circ C_3\]
Theorem 3.5 Let $G$ be a graph of order $m \geq 2$ and let $H$ be any graph of order $n$. If $k \in \{1, 2, ..., m - 1\}$, then $\zeta_k(G \circ H) \geq kn$.

Proof: Let $S \subseteq V(G \circ H)$ with $|S| = m - k$. Since $V(G \circ H) = \bigcup_{x \in V(G)} V(x + H^x)$, $S$ intersects $V(x + H^x)$ for at most $m - k$ of these $x_i + H^{x_i}$, $x_i \in V(G)$. Thus there exists $\{a_1, a_2, ..., a_k\} \subseteq V(G)$ such that $S \cap V(a_j + H^{a_j}) = \emptyset$ for each $j = 1, 2, ..., k$. Consequently, the set $V(G \circ H) \setminus N_{G \circ H}[S]$ contains $\bigcup_{j=1}^k V(H^{a_j})$. With all the properties of $S$ considered, it follows that $\zeta_k(G \circ H) \geq kn$. ■

Theorem 3.6 Let $G$ and $H$ be nontrivial graphs of orders $m$ and $n$, respectively, such that $H$ does not have a spanning star. If $k \in \{1, 2, ..., m - 1\}$ and $S \subseteq V(G \circ H)$ is a $\zeta_k$-set of $G \circ H$, then $S \subseteq V(G)$.

Proof: This is immediate from the adjacency of the vertices of $G \circ H$ and the fact that $S$ is a $\zeta_k$-set of $G \circ H$ where $H$ has no spanning star. ■

The following result provides a characterization of all possible $\zeta_k$-sets of $G \circ H$, where both $G$ and $H$ are nontrivial and connected, with $H$ having no spanning star.

Theorem 3.7 Let $G$ and $H$ be nontrivial connected graphs of orders $m$ and $n$, respectively, such that $H$ has no spanning star. Let $k \in \{1, 2, ..., m - 1\}$ and $S \subseteq V(G \circ H)$ such that $|S| = m - k$. Then $S$ is a $\zeta_k$-set of $G \circ H$ if and only if $S \subseteq V(G)$ and exactly one of the following holds:

(i) $|S| \in \{\gamma(G), \gamma(G) + 1, ..., m - 1\}$ and $S$ is a dominating set of $G$.

(ii) $|S| \in \{1, 2, ..., \gamma(G) - 1\}$ and $S$ is a $\zeta_r$-set of $G$, where $r = \gamma(G) - (m - k)$.

Proof: Suppose $S \subseteq V(G \circ H)$ is a $\zeta_k$-set of $G \circ H$. Then by definition $|S| = m - k$ with $k \in \{1, 2, ..., m - 1\}$. Note that $|V(G)| = m = \gamma(G \circ H)$ by Lemma 3.4. By Theorem 3.6, $S \subseteq V(G)$. We consider two cases for the value of $k$, namely $k \in \{1, 2, ..., m - \gamma(G)\}$ and $k \in \{m - \gamma(G) + 1, m - \gamma(G) + 2, ..., m - 1\}$.

Case 1. Assume $k \in \{1, 2, ..., m - \gamma(G)\}$. In this case, $m - k \in \{m - 1, m - 2, ..., m - (m - \gamma(G))\} = \{\gamma(G), \gamma(G) + 1, ..., m - 2, m - 1\}$. Since $|S| = m - k$, it follows that $|S| \geq \gamma(G)$. This inequality, together with the fact that $S \subseteq V(G)$ is a $\zeta_k$-set of $G \circ H$, implies that $N_G[S] = V(G)$; hence, $S$ is a dominating set of $G$.

Case 2. Assume $k \in \{m - \gamma(G) + 1, m - \gamma(G) + 2, ..., m - 1\}$. In this case, $m - k \in \{\gamma(G) - 1, \gamma(G) - 2, ..., 3, 2, 1\}$. Since $|S| = m - k$, $|S| \in \{1, 2, ..., \gamma(G) - 1\}$. Clearly, $|S| < \gamma(G)$. This inequality, together with the fact that $S \subseteq V(G)$, implies that $N_G[S] \subsetneq V(G)$. Since $|N_{G \circ H}[S]| = |N_G[S] \cup \bigcup_{x \in S} V(H^x)|$
and \( |N_{G\circ H}[S]| = \max\{|N_{G[H]}[W]| : W \subseteq V(G \circ H), |W| = m - k\} \), this means that \( |N_G[S]| = \max\{|N_G[W] : W \subseteq V(G), |W| = m - k\} \), where again \( m - k < \gamma(G) \). If we let \( r = \gamma(G) - (m - k) \), then \( r > 0 \) and \( S \) is a \( \zeta_r \)-set of \( G \). For the converse, we note that we have \( k \in \{1, 2, ..., m - 1\} \) and \( S \subseteq V(G) \) where \( |S| = m - k \). Suppose first that \( |S| \in \{\gamma(G), \gamma(G) + 1, ..., m - 1\} \) and \( S \) is a dominating set of \( G \). Being a dominating set of \( G \) we immediately have \( N_G[S] = V(G) \). Now \( |N_{G\circ H}[S]| = |N_G[S] \cup_{x \in S} V(H_x^*)| = |V(G)| + \sum_{x \in S}|V(H_x^*)| = m + (m - k)n \) so that \( |V(G \circ H)| - |N_{G\circ H}[S]| = (m + mn) - (m + mn - kn) = kn \). This implies that \( \zeta_k(G \circ H) \leq kn \). But by Theorem 3.5, \( \zeta_k(G \circ H) \geq kn \). As a consequence, \( S \) is a \( \zeta_k \)-set of \( G \circ H \).

Next, suppose that \( |S| \in \{1, 2, ..., \gamma(G) - 1\} \) and \( S \) is a \( \zeta_{(G)-m+k} \)-set of \( G \).

Then we have \( |N_G[S]| = \max\{|N_G[W] : W \subseteq V(G), |W| = m - k\} \), implying that \( |N_{G\circ H}[S]| = \max\{|N_G[W] : W \subseteq V(G), |W| = m - k\} + \sum_{x \in S}|V(H_x^*)| = \max\{|N_{G\circ H}[W] : W \subseteq V(G \circ H), |W| = m - k\} \). Thus, \( S \) is a \( \zeta_k \)-set of \( G \circ H \).

\[\zeta_k(G \circ H) = \begin{cases} 
kn, & \text{if } k \in \{1, 2, ..., m - \gamma(G)\}; \\
kn + \zeta_r(G), & \text{if } k \in \{m - \gamma(G) + 1, m - \gamma(G) + 2, ..., m - 1\}, \\
& \text{with } r = \gamma(G) - (m - k). 
\end{cases}\]

**Proof:** Suppose \( S \subseteq V(G \circ H) \) is a \( \zeta_k \)-set of \( G \circ H \). By Theorem 3.6, \( S \subseteq V(G) \). By definition, \( \zeta_k(G \circ H) = (mn + m) - |N_{G\circ H}[S]| = (mn + m) - |N_G[S] \cup_{x \in S} V(H_x^*)| \). Case 1. Suppose \( k \in \{1, 2, ..., m - 1\} \). Then \( |S| = m - k \in \{\gamma(G), \gamma(G) + 1, ..., m - 1\} \). By Theorem 3.7, \( S \) is a dominating set of \( G \) so that \( N_G[S] = V(G) \). Since \( N_{G\circ H}[S] = N_G[S] \cup_{x \in S} V(H_x^*) \), it follows that

\[
\zeta_k(G \circ H) = (mn + m) - |N_{G\circ H}[S]| \\
= (mn + m) - |N_G[S] \cup_{x \in S} V(H_x^*)| \\
= (mn + m) - |N_G[S]| + (m - k)n \\
= (m - |N_G[S]|) + kn \\
= \zeta_r(G) + kn.
\]

Case 2. Suppose \( k \in \{m - \gamma(G) + 1, m - \gamma(G) + 2, ..., m - 1\} \). Then \( |S| \in \{1, 2, ..., \gamma(G) - 1\} \) where \( S \) is a \( \zeta_r \)-set of \( G \), with \( r = \gamma(G) - (m - k) \). In this case,

\[
\zeta_k(G \circ H) = (mn + m) - |N_{G\circ H}[S]| \\
= (mn + m) - |N_G[S] \cup_{x \in S} V(H_x^*)| \\
= (mn + m) - |N_G[S]| + (m - k)n \\
= (m - |N_G[S]|) + kn \\
= \zeta_r(G) + kn.
\]
Characterizing all possible $\zeta_k$-sets of the corona $G \circ H$ for the case when both $G$ and $H$ are nontrivial and connected with $H$ containing a spanning star has escaped from our efforts for now. However, computing the $k$-domination defect of $G \circ H$ in this case is conveniently facilitated by the following observation.

**Theorem 3.9** Let $G$ and $H$ be nontrivial connected graphs of orders $m$ and $n$, respectively, such that $H$ has a spanning star. Let $S \subseteq V(G \circ H)$ such that $|S| = m - k$, where $k \in \{1, 2, ..., m - 1\}$. If $S$ is a $\zeta_k$-set of $G \circ H$, then $|S \cap V(x + H^x)| \leq 1$ for any $x \in V(G)$, provided further that if $S \cap V(x + H^x) = \{a\}$, then $a$ is adjacent to all other vertices in the subgraph $x + H^x$.

**Proof:** The assertion follows from the requirement that

$$|N_{G \circ H}[S]| = \max\{N_{G \circ H}[W] : W \subseteq V(G \circ H), |W| = m - k\}$$

and the fact that for every $x \in V(G)$, $|N_{G \circ H}[x] > |N_{G \circ H}[u_x]|$ for any $u_x \in V(H^x)$.

**Corollary 3.10** Let $G$ and $H$ be nontrivial connected graphs of order $m$ and $n$, respectively, such that $H$ has a spanning star. Let $k \in \{1, 2, ..., m - 1\}$. Then,

$$\zeta_k(G \circ H) = \begin{cases} kn, & \text{if } 1 \leq k < m - \gamma(G); \\ kn + \zeta_r(G), & \text{if } m - \gamma(G) < k < m, \text{ where } r = \gamma(G) - (m-k). \end{cases}$$

**Proof:** Let $S \subseteq V(G \circ H)$ be a $\zeta_k$-set of $G \circ H$, where $k \in \{1, 2, ..., m - 1\}$ and $|S| = m - k$. By Theorem 3.9, $|S \cap V(x + H^x)| \leq 1$ for any $x \in V(G)$. Since $|N_{G \circ H}[x]| > |N_{G \circ H}[h_x]|$ for any $x \in V(G)$ and any $h_x \in V(H^x)$, and since $|N_{G \circ H}[S]| = \max\{|N_{G \circ H}[W] : W \subseteq V(G \circ H), |W| = m - k\}$, either we can have $S \subseteq V(G)$ or we can form a set $S' \subseteq V(G)$ by retaining all elements of $S$ that belong to $V(G)$ and replacing those elements in $S$ of the form $h_x$, where $h_x \in V(H^x)$, by the corresponding vertex $x \in V(G)$, with this remedy completely valid in view of Theorem 3.9, with $|S'| = |S|$. So without loss of generality we just assume that $S \subseteq V(G)$. If $|S| \geq \gamma(G)$, then necessarily $S$ is a dominating set of $G$ so that $|N_{G \circ H}[S]| = |N_G[S] \cup_{x \in S} V(x + H^x)| = |V(G)| + \sum_{x \in S} |V(H^x)| = m + (m - k)n$. In this case, $\zeta_k[S] = kn$, implying that $\zeta_k(G \circ H) = kn$ since $S$ was assumed to be a $\zeta_k$-set of $G \circ H$. On the other hand, if $|S| < \gamma(G)$, then necessarily $S$ is a $\zeta_r$-set of $G$, where $r = \gamma(G) - (m-k)$, and that $\zeta_k(G \circ H) = \zeta_r(G) + kn$.

**Acknowledgements.** The authors would like to acknowledge the valuable comments and inputs made by the anonymous referee. Likewise, the first author acknowledges with gratitude the support extended by the Commission on Higher Education (CHED) of the Philippines under the K-to-12 Transition Program.
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Received: August 29, 2021; Published: September 17, 2021