Applied Mathematical Sciences, Vol. 15, 2021, no. 12, 587 - 594
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ams.2021.914561

Triangular Index of Some Graph Products

Remarl Joseph M. Damalerio and Rolito G. Eballe

Department of Mathematics
College of Arts and Sciences
Central Mindanao University
Musuan, Maramag, Bukidnon
8714 Philippines

This article is distributed under the Creative Commons by-nc-nd Attribution License. Copyright © 2021 Hikari Ltd.

Abstract

The number of triangles in a graph G is called the triangular index of G, denoted by $T_i(G)$. In this paper we give the exact expressions for the triangular indices of the complete product $G \vee H$, corona product $G \circ H$, cartesian product $G \square H$, and tensor product $G \times H$ of two finite undirected simple graphs G and H using properties that these binary operations possessed, including some inherent properties that are retained by the factors.

Mathematics Subject Classification: 05C09, 05C38, 05C76

Keywords: triangular index, complete product, corona product, cartesian product, tensor product

1 Introduction

Let $G = (V(G), E(G))$ be a finite undirected simple graph. A *triangle* in a graph G is a cycle in G of length 3. We say that triangles T_1 and T_2 are the same triangles if and only if both triangles involve exactly the same subset of three vertices of $V(G)$. If T_1 and T_2 differ in at least one vertex, then T_1 and T_2 are considered distinct triangles.

In this paper, we denote the set of all distinct triangles in a graph G by $\mathbb{T}(G)$. However, we may need to use other notations for sets of triangles that are not necessarily equal to the entire $\mathbb{T}(G)$. The triangular index of a graph G, denoted by $T_i(G)$, is the number of distinct triangles in G, that is, $T_i(G) = |\mathbb{T}(G)|$. For any graph G of order $n \geq 3$, $0 \leq T_i(G) \leq \binom{n}{3}$.

Let $u \in V(G)$ and $v \in V(H)$. For any binary operation involving G and H that has vertex set equal to $V(G) \times V(H)$, we define the induced subgraphs $\langle V(H^u)\rangle = \langle\{(u, y) : y \in V(H)\}\rangle$ and $\langle V(G^v)\rangle = \langle\{(x, v) : x \in V(G)\}\rangle$ as the u^{th}*-section* of H and v^{th}*-section* of G, respectively.

The field of sociology gives us at least one theory as to the importance of counting triangles in a community of people. The tighter the community (a network or a graph), the more likely it is for any given member (a node or a vertex) to have interacted with and to have known the reputation of any other member. As cited by Suri and Vassilvitskii in [6], Coleman and Portes [2, 5] argue that if a person were to do something against the social norm, the consequences in a tightly-knit community would be higher because more people would know about the offending action and more people would be able to penalize the individual who committed it. Accordingly, this phenomenon helps foster a higher degree of trust in the community while helping in the formation of positive social norms [6]. Thus, Suri and Vassilvitskii argued that if we have a tightly-knit community (for instance, a graph with large triangular index), the triangular index of an individual (the number of triangles in the subgraph induced by the closed neighborhood of a vertex) indicates that this individual may be able to better take advantage of the higher level of trust amongst his/her peers and so with the more positive social norms in his/her community.

The problem of determining the triangular index of a social network may also be considered within the big topic of network centrality since Estrada and Velazquez's subgraph centrality proposes counting closed paths, triangles, squares and others [3].

In this paper, we shall determine the triangular indices of the complete product, corona product, cartesian product, and tensor product of any two graphs using some properties that each binary operation possesses and some inherent characteristics possessed by the factors or constituents. Graphs considered in this paper are in the context of finite undirected simple graphs. For basic graph theory terminologies not specifically described nor defined in this paper, please refer to either [1] or [4].

2 Complete and Corona Product of Graphs

Recall that the sum $G \vee H$ of two graphs G and H, referred to as *complete product* in this paper, has the vertex set $V(G \vee H) = V(G)\dot{\cup}V(H)$ and edge set $E(G \vee H) = E(G)\dot{\cup}E(H)\dot{\cup}\{uv : u \in V(G), v \in V(H)\}$, where the symbol $\dot{\cup}$ denotes disjoint union. The following lemma will be useful; it can be verified directly from the definition of complete product of graphs.

Lemma 2.1 *Let G and H be graphs. Then $\mathbb{T}(G\vee H)=\mathbb{T}(G)\dot{\cup}\mathbb{T}(H)\dot{\cup}\mathbb{T}_{1,2}(G\vee H)\dot{\cup}\mathbb{T}_{2,1}(G\vee H)$, where $\mathbb{T}_{1,2}(G\vee H)$ consists of all triangles with one vertex of G and two vertices of H, while $\mathbb{T}_{2,1}(G\vee H)$ consists of all triangles with two vertices of G and one vertex of H.*

Theorem 2.2 *Let G and H be graphs with $|V(G)|=n_1$, $|E(G)|=m_1$, $|V(H)|=n_2$ and $|E(H)|=m_2$. Then the triangular index of the complete product $G\vee H$ is given by the formula $T_i(G\vee H)=T_i(G)+T_i(H)+n_1m_2+n_2m_1$.*

Proof: By Lemma 2.1, we have $\mathbb{T}(G\vee H)=\mathbb{T}(G)\dot{\cup}\mathbb{T}(H)\dot{\cup}\mathbb{T}_{1,2}(G\vee H)\dot{\cup}\mathbb{T}_{2,1}(G\vee H)$. Thus,

$$\begin{aligned} T_i(G\vee H) &= |\mathbb{T}(G\vee H)| \\ &= |\mathbb{T}(G)|+|\mathbb{T}(H)|+|\mathbb{T}_{1,2}(G\vee H)|+|\mathbb{T}_{2,1}(G\vee H)| \\ &= T_i(G)+T_i(H)+|\mathbb{T}_{1,2}(G\vee H)|+|\mathbb{T}_{2,1}(G\vee H)|. \end{aligned}$$

Now, if $v\in V(G)$ and $uw\in E(H)$, then vertices v, u and w form a triangle in $G\vee H$. This implies that there are $|E(H)|=m_2$ such triangles that contain vertex v, with the other two vertices belonging to H. Since there are $|V(G)|=n_1$ vertices in G, it follows that $|\mathbb{T}_{1,2}(G\vee H)|=n_1m_2$. A similar argument shows that $|\mathbb{T}_{2,1}(G\vee H)|=n_2m_1$. Therefore, $T_i(G\vee H)=T_i(G)+T_i(H)+n_1m_2+n_2m_1$. ■

Corollary 2.3 *Let F_n be the fan of order $n\geq 3$ and W_m the wheel of order $m\geq 4$. Then $T_i(F_n)=n-2$ and $T_i(W_m)=m-1$.*

Proof: This is immediate from Theorem 2.2, since $F_n=K_1\vee P_{n-1}$, where K_1 is the trivial graph and P_{n-1} the path graph of order $n-1$, while $W_m=K_1\vee C_{m-1}$ where C_{m-1} is the cycle graph of order $m-1$. ■

Corollary 2.4 *Let $K_{r,s,t}$ be the complete tripartite graph where r, s, $t\geq 1$. Then $T_i(K_{r,s,t})=rst$.*

Proof: Consider the empty graphs $\bar{K}_r$, $\bar{K}_s$, and $\bar{K}_t$ of orders r, s, t, respectively. Let $G=\bar{K}_r\vee\bar{K}_s$ and $H=\bar{K}_t$. Then apply Theorem 2.2 to obtain the desired result. ■

Now for the *corona product* of a graph G by a graph H, denoted by $G\circ H$, recall that this is defined as the graph obtained by taking one copy of G together with $|V(G)|$ copies of H and then joining the i-th vertex of G to every vertex in the i-th copy of H. For notational simplicity, we let H^i be the i^{th} copy of H corresponding to the i^{th} vertex of G. Moreover, the subgraph of $G\circ H$ induced by $\{i\}\cup V(H^i)$, assuming that $i\in V(G)$, shall be denoted by $i\vee H^i$. The following lemma is clear from the definition of the corona product of graphs.

Lemma 2.5 *Let G and H be graphs. Then the following holds:* $\mathbb{T}(G \circ H) = \mathbb{T}(G) \dot{\cup} \big[\dot{\bigcup}_{u \in V(G)} \big(\mathbb{T}(u \vee H^u)\big]$.

Theorem 2.6 *Let G and H be graphs with* $|V(G)| = n_1$, $|E(G)| = m_1$, $|V(H)| = n_2$ *and* $|E(H)| = m_2$. *Then the triangular index of the corona product* $G \circ H$ *is given by the formula* $T_i(G \circ H) = T_i(G) + n_1 T_i(H) + n_1 m_2$.

Proof: By Lemma 2.5, we have $\mathbb{T}(G \circ H) = \mathbb{T}(G) \dot{\cup} \big[\dot{\bigcup}_{u \in V(G)} \big(\mathbb{T}(u \vee H^u)\big]$. By Theorem 2.2, $|\mathbb{T}(u \vee H^u)| = 1 \cdot m_2 + T_i(H)$ for each $u \in V(G)$. Consequently,

$$\begin{aligned} T_i(G \circ H) &= \left| \mathbb{T}(G) \dot{\cup} \left[\dot{\bigcup}_{u \in V(G)} \mathbb{T}(u \vee H^u) \right] \right| \\ &= T_i(G) + \sum_{j=1}^{n_1} (m_2 + T_i(H)) \\ &= T_i(G) + n_1(m_2 + T_i(H)) \\ &= T_i(G) + n_1 T_i(H) + n_1 m_2 \end{aligned}$$ ■

3 Cartesian and Tensor Product of Graphs

Recall that the *cartesian product* $G \Box H$ of two graphs G and H is the graph with vertex set $V(G \Box H) = V(G) \times V(H)$ and edge-set $E(G \Box H)$ satisfying the following conditions: $(u, v)(u', v') \in E(G \Box H)$ if and only if either $uu' \in E(G)$ and $v = v'$, or $u = u'$ and $vv' \in E(H)$.

Theorem 3.1 *A triangle* $\langle \{(a, a'), (b, b'), (c, c')\} \rangle$ *exists in the cartesian product* $G \Box H$ *if and only if* $\langle \{a, b, c\} \rangle$ *is a triangle in G with* $a' = b' = c'$, *or* $\langle \{a', b', c'\} \rangle$ *is a triangle in H with* $a = b = c$.

Proof: Suppose $\langle \{a, b, c\} \rangle$ is a triangle in G. We want to show that there exists a triangle $\langle \{(a, a'), (b, b'), (c, c')\} \rangle \in \mathbb{T}(G \Box H)$ for some $a', b', c' \in V(H)$. Since $\langle \{a, b, c\} \rangle$ is a triangle in G, it follows that $ab, ac, bc \in E(G)$. If we take a particular $v \in V(H)$ and let $a' = b' = c' = v$, then by definition of the cartesian product, $\langle \{(a, a'), (b, b'), (c, c')\} \rangle \in \mathbb{T}(G \Box H)$. Analogous argument applies when $\langle \{a', b', c'\} \rangle$ is a triangle in H.

Conversely, suppose $\langle \{(a, a'), (b, b'), (c, c')\} \rangle \in \mathbb{T}(G \Box H)$. This means that vertices (a, a'), (b, b') ,(c, c') are pairwise adjacent vertices in the product $G \Box H$. At this point we consider two cases:

Case 1: Let $a' = b'$ and $ab \in E(G)$. If $b = c$ and $b'c' \in E(H)$, then the adjacency of (a, a') and (c, c') in $G \Box H$ yields either

$$ac \in E(G) \text{ and } a' = c' \qquad (*)$$

or

$$a'c' \in E(H) \text{ and } a = c. \qquad (**)$$

The expression $a' = c'$ in $(*)$, combined with $a' = b'$ *(part of Case 1 assumption)*, yields $b' = c'$, which in turn produces a contradiction since it was assumed in this subcase that $b'c' \in E(H)$. On the other hand, the expression $a = c$ in $(**)$, combined with $b = c$ *(assumption of this subcase)*, yields $a = b$, which in turn also produces a contradiction since it was assumed in this case that $ab \in E(G)$. The aforementioned contradictions necessarily imply that $bc \in E(G)$ and $b' = c'$. Since $a' = b'$ was assumed at the start of this case, it follows now that $a' = b' = c'$ and that $\langle\{a, b, c\}\rangle \in \mathbb{T}(G)$.

Case 2: Let $a = b$ and $a'b' \in E(H)$. In a similar structure of reasoning with *Case 1* and with the adjacency of the vertices in $G\Box H$, it follows that $b = c$ and $b'c' \in E(H)$, and ultimately $a = b = c$ and that $\langle\{a', b', c'\}\rangle \in \mathbb{T}(H)$.

This completes the proof of the theorem. ■

Example 3.2 *Let G and H be two cycles both of order 3 with $V(G) = \{a, b, c\}$ and $V(H) = \{a', b', c'\}$. For the cartesian product $G\Box H$ (Figure 1), it can be seen that* $\mathbb{T}(G) = \{\langle\{(a, a'), (b, a'), (c, a')\}\rangle, \langle\{(a, b'), (b, b'), (c, b')\}\rangle, \langle\{(a, c'), (b, c'), (c, c')\}\rangle, \langle\{(a, a'), (a, b'), (a, c')\}\rangle, \langle\{(b, a'), (b, b'), (b, c')\}\rangle, \langle\{(c, a'), (c, b'), (c, c')\}\rangle\}$

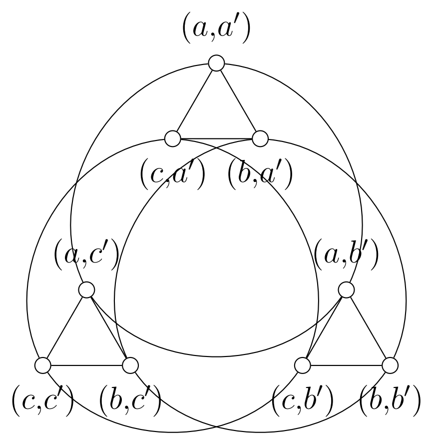

Figure 1: The graph $C_3\Box C_3$ showing all the six triangles it contains.

Theorem 3.3 *Let G and H be graphs with $|V(G)| = n_1$ and $|V(H)| = n_2$. Then the triangular index of the cartesian product $G\Box H$ is given by $T_i(G\Box H) = n_2T_i(G) + n_1T_i(H)$.*

Proof: By the adjacency relation of the vertices in $G\Box H$ and by Theorem 3.1, the collection $\mathbb{T}(G\Box H)$ of all triangles in $G\Box H$ can be partitioned into two families, namely, $\mathbb{T}_G = \dot{\cup}_{u\in V(G)}\mathbb{T}(H^u)$ and $\mathbb{T}_H = \dot{\cup}_{v\in V(H)}\mathbb{T}(G^v)$ where H^u is the u^{th}-section of H and G^v is the v^{th}-section of G. Since these are disjoint unions, it follows immediately that

$$\begin{aligned} T_i(G\Box H) &= \left|\dot{\bigcup}_{v\in V(H)}\mathbb{T}(G^v) \quad \dot{\cup} \quad \dot{\bigcup}_{u\in V(G)}\mathbb{T}(H^u)\right| \\ &= \left|\dot{\bigcup}_{v\in V(H)}\mathbb{T}(G^v)\right| + \left|\dot{\bigcup}_{u\in V(G)}\mathbb{T}(H^u)\right| \\ &= n_2T_i(G) + n_1T_i(H). \end{aligned}$$

■

Corollary 3.4 *The triangular index of the prism $K_m\Box P_2$ and the torus $C_m\Box C_n$ are given by the following:*

a.) $T_i(K_m\Box P_2) = \begin{cases} 0 & \text{if } m < 3 \\ 2\binom{m}{3} & \text{if } m \geq 3 \end{cases}$

b.) $T_i(C_m\Box C_n) = \begin{cases} 0 & \text{if } m,n > 3 \\ n & \text{if } m = 3, n > 3 \\ m & \text{if } n = 3, m > 3 \\ 6 & \text{if } m = n = 3. \end{cases}$

Proof: This is immediate from Theorem 3.3. ■

For the *tensor product* $G \times H$ of two graphs G and H, recall that this is the graph with vertex set $V(G\times H) = V(G)\times V(H)$ and edge set $E(G\times H)$ satisfying the following adjacency condition: $(u,v)(u',v') \in E(G\times H)$ if and only if $uu' \in E(G)$ and $vv' \in E(H)$.

Theorem 3.5 *If $\langle\{(a,a'),(b,b'),(c,c')\}\rangle$ is a triangle in the tensor product $G\times H$, then $\langle\{a,b,c\}\rangle$ and $\langle\{a',b',c'\}\rangle$ are triangles in G and H, respectively.*

Proof: Assume that $\langle\{(a,a'),(b,b'),(c,c')\}\rangle \in \mathbb{T}(G\times H)$. This means that vertices $(a,a'),(b,b')$ and (c,c') are pairwise adjacent in $G\times H$. By the adjacency of the vertices in the tensor product, one can observe that vertices $a,b,c \in V(G)$ are pairwise adjacent in G and vertices $a',b',c' \in V(H)$ are pairwise adjacent in H as well. Consequently, $\langle\{a,b,c\}\rangle \in \mathbb{T}(G)$ and $\langle\{a',b',c'\}\rangle \in \mathbb{T}(H)$. ■

Corollary 3.6 *If either $T_i(G) = 0$ or $T_i(H) = 0$, then $T_i(G\times H) = 0$.*

Proof: This is immediate from Theorem 3.5 as this is the ultimate version of its contrapositive. ■

Corollary 3.7 *For any tree S and any graph G, $T_i(S \times G) = 0$.*

Proof: This follows from Corollary 3.6, considering that any tree contains no triangle. ■

Lemma 3.8 *Let G_1 and H_1 be triangles in G and H, respectively. Then the pair (G_1, H_1) produces six distinct triangles in $G \times H$.*

Proof: Assume that $G_1 = \langle\{a, b, c\}\rangle \in \mathbb{T}(G)$ and $H_1 = \langle\{a', b', c'\}\rangle \in \mathbb{T}(H)$. In the tensor product, these two triangles generate 6 distinct triangles in $G \times H$, namely, $\langle\{(a, a'), (b, b'), (b, c')\}\rangle$, $\langle\{(b, a'), (a, b'), (c, c')\}\rangle$, $\langle\{(b, a'), (c, b'), (a, c')\}\rangle$, $\langle\{(c, a'), (a, b'), (b, c')\}\rangle$, $\langle\{(c, a'), (b, b'), (a, c')\}\rangle$, and $\langle\{(a, a'), (b, b'), (c, c')\}\rangle$. ■

Example 3.9 *Let G and H be the graphs drawn in Figure 2 below. Then* $\mathbb{T}(G \times H) = \{\langle\{(a, a'), (b, b'), (c, c')\}\rangle, \langle\{(a, a'), (c, b'), (b, c')\}\rangle,$ $\langle\{(b, a'), (a, b'), (c, c')\}\rangle, \langle\{(b, a'), (c, b'), (a, c')\}\rangle, \langle\{(c, a'), (a, b'), (b, c')\}\rangle,$ $\langle\{(c, a'), (b, b'), (a, c')\}\rangle, \langle\{(a, a'), (b, b'), (d, c')\}\rangle, \langle\{(a, a'), (d, b'), (b, c')\}\rangle,$ $\langle\{(b, a'), (a, b'), (d, c')\}\rangle, \langle\{(b, a'), (d, b'), (a, c')\}\rangle, \langle\{(d, a'), (a, b'), (b, c')\}\rangle,$ $\langle\{(d, a'), (b, b'), (a, c')\}\rangle\}$. *The tensor product $G \times H$ in this case contains exactly 12 distinct triangles, produced from the pair $\langle\{a, b, c\}\rangle \in \mathbb{T}(G)$ and $\langle\{a', b', c'\}\rangle \in \mathbb{T}(H)$, as well as the pair $\langle\{a, b, d\}\rangle \in \mathbb{T}(G)$ and $\langle\{a', b', c'\}\rangle \in \mathbb{T}(H)$.*

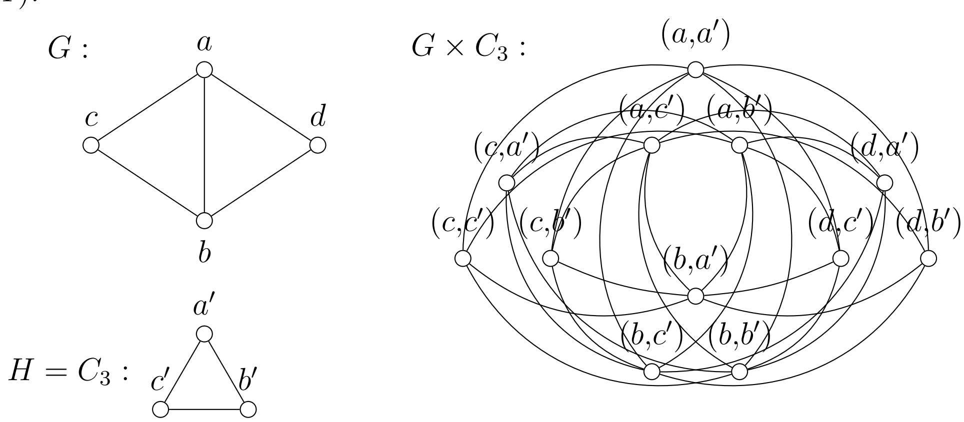

Figure 2: The graphs G, C_3, and the tensor product $G \times C_3$.

Theorem 3.10 *Let G and H be graphs. Then the triangular index of the tensor product $G \times H$ is given by $T_i(G \times H) = 6T_i(G)T_i(H)$.*

Proof: Lemma 3.8 asserts that a pair of triangles $T \in \mathbb{T}(G)$ and $T' \in \mathbb{T}(H)$ produces 6 distinct triangles in the tensor product $G \times H$. Even in the worst case scenario as shown in Example 3.9, every new pair of triangles, one from $\mathbb{T}(G)$ and another one from $\mathbb{T}(H)$, always add 6 new triangles in the tensor product $G \times H$. Applying Theorem 3.5, we now have $T_i(G \times H) = 6T_i(G)T_i(H)$. ■

4 Final Remarks

The main results in this paper indicate that the triangular index of the considered graph products completely depends on some inherent characteristics of the factors such as their individual triangular indices, orders and sizes. Although Theorem 3.10 was a bit surprising to us at first, it would probably be of interest to investigate whether or not other graph products behave similarly, and whether the simple concept of a trianguar index would gain traction within the vast area of network centrality.

Acknowledgements. The authors would like to acknowledge the valuable comments and inputs made by the anonymous referee. Likewise, the first author acknowledges the support extended by the Department of Science and Technology - Science Education Institute (DOST-SEI) of the Philippines under the Science and Technology Regional Alliance of Universities for National Development (STRAND).

References

[1] G. Chartrand, L. Lesniak, and P. Zhang, *Graphs & Digraphs (6th ed.)*, Chapman and Hall, 2015. https://doi.org/10.1201/b19731

[2] J.S. Coleman, Social capital in the creation of human capital, *American Journal of Sociology,* **94** (1988), S95-S120.

[3] E. Estrada and J. A. Rodriguez-Velazquez, Subgraph centrality in complex networks, *Physical Review E,* **71** (5) (2005). https://doi:10.1103/physreve.71.056103

[4] F. Harary, *Graph Theory*, Addisson-Wesley Publishing Company, Inc., USA, 1969. https://doi.org/10.21236/ad0705364

[5] A. Portes, Social capital: Its origins and applications in modern sociology, *Annual Review of Sociology,* **24** (1) (1998), 1-24.

[6] S. Suri and S. Vassilvitskii, Counting triangles and the curse of the last reducer, in: *Proceedings of the 20th International Conference on World Wide Web*, 2011. https://doi.org/10.1145/1963405.1963491

Received: July 27, 2021; Published: September 5, 2021