European Option Pricing Problem with Transaction Costs in $q$–Gaussian Process Model

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Abstract

Under the assumption that the stock price model satisfies the $q$–Gaussian process, the European option pricing problem with transaction cost proportional to the transaction amount is considered. By using no arbitrage principle and constructing forward contract, the forward price expression is derived, and a new European option pricing formula with transaction cost is obtained. According to the results, we find that with the transaction costs from nothing to exist, from unilateral to bilateral, the corresponding option price shows an upward trend.

Keywords: transaction costs, European options, $q$–Gaussian process, forward contract

1 Introduction

In the process of financial market development, options have become the most potential financial derivatives in the financial market because of their outstanding role in hedging and risk aversion. Black and Scholes(1973) have derived the famous Black Scholes option pricing formula, which has made a major breakthrough in option pricing. But this pricing theory is defined in a very ideal market environment, such as no dividend, no transaction costs,

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which is very different from the real financial securities market. Because the real market charges transaction costs, many scholars have studied the option pricing problem with transaction costs. Leland(1985) obtains the option pricing model with transaction cost in discrete time by hedging strategy. Stettner(2000) gives the European option pricing with transaction cost and without transaction cost by martingale measure. Yuejiao Feng(2018) divides the European option pricing differential equation with transaction cost into several parabolic equations. By solving these parabolic equations, she gives the approximate option pricing formula. Xiaqian Liu and Jun Chai(2004) give a numerical method of European option pricing with transaction cost by using binary tree. Qian Liu and Xinping Liu(2004) construct and generalize the European option pricing model with transaction cost and subject assets obeying mixed process by means of portfolio. On the basis of Leland model, Xiaoying Zheng and Jinxian Chen(2000) put forward the short and long pricing methods with transaction cost through hedging strategy, and gave the nonlinear option pricing model in discrete time. On the basis of classical B-S model, Haoyang Qin(2015) et al. Obtained the pricing equation of rainbow option with transaction cost and dividend by using multidimensional Ito lemma and constructing investment strategy, and gave the analytical solution. Shaomin Xu and Lumin Jiang(2002) proved that the pricing equation of derivative with transaction cost can be transformed into linear partial differential equation under certain conditions.

The above mentioned stock price models are basically driven by Brownian motion, and can not effectively describe the smile problem of option volatility. Therefore, many scholars extend the stock price model, including jump diffusion model, stochastic volatility model and so on. In 2002, Borland(2002) introduced the $q-$Gaussian process in statistical physics into option pricing, and obtained the generalized form of Black Scholes differential equation through approximate substitution of distribution. Liu and Cui(2020) proved that the $q-$Gaussian process has non markovity. Although it can effectively describe the volatility of stock price process, there will be large error using approximate substitution of distribution, So Liu and Cui(2019) put forward the least square approximation method to solve the integration problem in the $q-$Gaussian process. Through numerical simulation, it is proved that the least square approximation method is more accurate. Based on the European option pricing formula derived in Liu and Cui(2019), this paper adds the factor of transaction costs, and gives the expression of forward price by constructing forward contract. Then the pricing formula of European option with transaction cost is obtained.
2 European option pricing with transaction costs

In the market, there are two types of assets, one is risk-free assets, such as risk-free bonds, and the other is risky assets such as stocks. Suppose that the risk-free bond $B_t$ satisfies
\[ dB_t = rB_t dt. \]

the stock price $S_t$ follows
\[ dS_t = \mu S_t dt + \sigma S_t d\Omega(t) \]

where $\Omega$ is non-Gaussian noise and follows the following random process
\[ d\Omega(t) = P^{1-q}(\Omega(t), t)dW_t \]

where $r$ is the risk-free interest rate and $W_t$ denotes zero mean Gaussian white noise. When $q = 1$, the stochastic process is a standard Brownian motion. Here, $P(\Omega(t), t)$ is a $q$-Gaussian distribution in the following form,
\[ P(\Omega(t), t) = \frac{1}{Z(t)} \left[ 1 - \beta(t)(1-q)\Omega^2(t) \right]^{\frac{1}{q}} \]

where
\[ \beta(t) = c^{\frac{1+q}{2}} \left[ (2-q)(3-q)t \right]^{\frac{2}{1-q}} \]
\[ Z(t) = \left[ (2-q)(3-q)ct \right]^{\frac{2}{q}} \]
\[ c = \frac{\pi}{q-1} \frac{\Gamma^2(\frac{1}{q-1} - \frac{1}{2})}{\Gamma^2(\frac{1}{q-1})}. \]

If a stochastic process obeys the $q$-Gaussian distribution, then we call it $q$-Gaussian process. Liu and Cui(2019) has obtained the pricing formula of European call option without transaction costs under this stochastic process
\[ C_t = S_t M_q(\gamma_1, \gamma_2) - Ke^{-r(T-t)} N_q(\gamma_1, \gamma_2) \]
where
\[ M_q(\gamma_1, \gamma_2) = e^{-r(T-t)} \left( \int_{\gamma_1}^{\gamma_2} \exp(\sigma \Omega(T) - \frac{\sigma^2}{2} \hat{\delta}(\Omega(T), T)) P(\Omega(T), T) d\Omega(T) \right) \]
\[ N_q(\gamma_1, \gamma_2) = \int_{\gamma_1}^{\gamma_2} P(\Omega(T), T) d\Omega(T) \]
\[ P(\Omega(T), T) = \frac{1}{Z(T)} [1 - \beta(T)(1 - q)\Omega^2(T)]^{-\frac{1}{\gamma}} \]
\[ \hat{\delta}(\Omega(T), T) = d_1(q)T^{\frac{2}{\gamma}} + d_2(q)\Omega^2(T) + d_3(q)T^{\frac{2}{\gamma}}\Omega^2(T) + d_4(q) \]
\[ \gamma_1 = \frac{-a_2 - \sqrt{a_2^2 - 4a_1a_3}}{2a_1a_2} \]
\[ \gamma_2 = \frac{-a_2 + \sqrt{a_2^2 - 4a_1a_3}}{2a_1a_2} \]
\[ a_1 = -\frac{\sigma^2}{2}d_2(q) + d_3(q)T^{\frac{2}{\gamma}} \]
\[ a_2 = \sigma \]
\[ a_3 = (r - \alpha)(T-t) - \frac{\sigma^2}{2}d_1(q)(T-t)^{\frac{2}{\gamma}} - \frac{\sigma^2}{2}d_4(q) - \ln(K_S) \cdot \frac{\gamma}{S_t} \]

(4)

In this paper, we assume that the transaction costs is charged according to the proportion of the transaction amount. The transaction costs rate in proportion to the cost when buying stocks is \( \theta \), and the transaction costs rate when selling stocks is \( \rho \). If the investor buys \( n \) shares with a price of \( S \) shares, the expenses are \((1 + \theta)nS\) and if he sells \( n \) shares with a price of \( S \) shares, the income is \((1 - \rho)nS\).

Next, we introduce introduces a kind of financial derivative — forward contract, and assumes that there is no risk of default and there is no arbitrage in the market.

Buy a share stock \( S_t \) at time \( t \), on the premise of transaction costs, the amount spent is \((1 + \theta)S_t\), and the standard forward contract of short selling the stock at the price of \((1 - \rho)F(t, T)\), at the expiration date of \( T(T > t) \), the short seller hands over the stock and receives \((1 - \rho)F(t, T)\),

Since this value has been agreed upon at the beginning of the contract, its present value must be equal to \( t \) time wealth value \((1 + \theta)S_t\), that is,
\[ (1 - \rho)F(t, T)e^{-r(T-t)} = (1 + \theta)S_t \]
the forward price of the stock available from this
\[ F(t, T) = \frac{1 + \theta}{1 - \rho} S_t e^{r(T-t)} \]  

(5)
The price is at $t$

$$F_t = F(t, T)e^{-r(T-t)} = \frac{1 + \theta}{1 - \rho}S_te^{r(T-t)}e^{-r(T-t)} = \frac{1 + \theta}{1 - \rho}S_k$$

**Theorem 2.1.** The price of European call option with exercise price $K$ and maturity $T$, The transaction costs rate in proportion to the cost when buying stocks is $\theta$, and the transaction costs rate when selling stocks is $\rho$, Then the price of European option at $t$ is

$$C_t = F_tM_q(\gamma_1, \gamma_2) - Ke^{-r(T-t)}N_q(\gamma_1, \gamma_2) = \frac{1 + \theta}{1 - \rho}S_tM_q(\gamma_1, \gamma_2) - Ke^{-r(T-t)}N_q(\gamma_1, \gamma_2)$$

(6)

Where $M_q(\gamma_1, \gamma_2)$ and $N_q(\gamma_1, \gamma_2)$ are given by (4).

It can be seen from (5) that if $T = t$ and no transaction costs is charged, there will be $F_t = S_t$, so use $F_t$ to replace the $S_t$ in formula (3), Then we can get the pricing formula of European call options with transaction costs.

### 3 numerical simulation

In order to better maintain the order of the financial market, the state not only provides trading places for traders, but also charges some fees. The payment of transaction costs is basically to obtain more information, so as to benefit from the subsequent transactions. In China, stock transactions need to collect "stamp tax", that is, the basis of equity transfer established by trading in the securities market, It is levied according to the proportion of the actual market price at the time of establishment, and the former bilateral levy is changed to the unilateral levy on the transferor, which means that investors do not need to pay this fee when they buy, but they need to pay the stamp duty according to the proportion when they sell shares.

The following use of MATLAB software to discuss the impact of transaction costs through numerical simulation.

The parameter $d_1(q), d_2(q), d_3(q), d_4(q)$ in (4) needs to be estimated by the least square method. Firstly, the Euler discretization of equation (1) is carried out by combining with equation (2), and the result is obtained

$$\Omega(t_i) - \Omega(t_{i-1}) = \left(\frac{1}{Z(t_{i-1})}\right)^{1-q} (1 - \beta(t_{i-1})(1-q)\Omega(t_{i-1})^2)^{1/2}(W(t_i) - W(t_{i-1})).$$
Generating random paths of $\Omega(T)$ by Euler discretization

In addition, in Liu and Cui (2019), there is

$$\delta(\Omega(t), t) = \int_0^t \left( \frac{1}{Z(s)} \right)^{1-q} (1 - \beta(s)(1 - q)\Omega(s)^2) ds$$

and according to the principle of trapezoidal integral,

$$\delta(\Omega(t), t) = \sum_{i=1}^n \left\{ \frac{1}{2}(Z(t_i)^{(q-1)}[1 - \beta(t_i)(1 - q)\Omega(t_i)^2] + Z(t_{i+1})^{(q-1)}[1 - \beta(t_{i+1})(1 - q)\Omega(t_{i+1})^2]) \times (t_{i+1} - t_i) \right\}$$

After Euler discretization and trapezoidal integral calculation, we use the least square method to estimate the value of $d_1(q), d_2(q), d_3(q), d_4(q)$. Through simulation, we get that when $q = 1.3$, $d_1(q) = 0.1692$, $d_2(q) = 0.0080$, $d_3(q) = 0.0498$, $d_4(q) = 0.7422$. After the values of each parameter are known, we can use formula (6) to calculate the theoretical European option price with transaction costs.

Next, we compare and discuss the pricing of European call options with or without transaction costs when the strike price is different.

Suppose the initial price of the stock is $S_0 = 50$, risk-free interest rate $r = 0.05$, and $q = 1.3, T = 1, \sigma = 0.2$. If only unilateral transaction costs is charged, we will take the proportion of stamp duty (unilateral transaction
costs) \( \rho = 0.001, \theta = 0 \). If bilateral transaction costs is charged, we will make \( \rho = 0.001, \theta = 0.002 \), when no transaction costs is charged, \( \rho = \theta = 0 \), and calculate the option price \( C_t \) corresponding to different strike price \( K \)

<table>
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<th>( K )</th>
<th>Unilateral</th>
<th>Bilateral</th>
<th>Without</th>
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<td>55</td>
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According to the above table, with the increase of the strike price of \( K \), the option pricing shows a downward trend regardless of whether the transaction costs is charged. It can also be seen that the option price with transaction costs is higher than that without transaction costs, And the option price with bilateral transaction costs is higher than that with unilateral transaction costs. This is consistent with the market situation. It shows that the pricing formula of European call option with transaction costs is reasonable.

4 Conclusion

In this paper, we first introduce the \( q \)-Gaussian process, then construct the forward contract. By using the idea of no arbitrage and equal wealth before and after the contract, we obtain the forward price expression of the stock with transaction cost. Based on the existing European option pricing formula without transaction cost, we obtain the option pricing formula with transaction cost. Through the simulation study, we get the conclusion that the option price with bilateral transaction costs is the highest, the option price with unilateral transaction costs is the second, and the option price without transaction costs is the lowest.

References


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