Option Pricing Formulas with Random Interest Rate under Uncertainty Theory

Yeyu Wu

School of Science
Nanjing University of Science and Technology
Nanjing 210094, Jiangsu, China

Tianyi Wang

School of Science
Nanjing University of Science and Technology
Nanjing 210094, Jiangsu, China

This article is distributed under the Creative Commons by-nc-nd Attribution License. Copyright © 2021 Hikari Ltd.

Abstract

As an application of uncertainty theory in the field of finance, uncertain finance plays an important role in solving the financial problems. This paper proposes a new kind of uncertain mean-reverting stock model with random interest rate. The European call and put option pricing formulas of the stock model are derived. Besides, some numerical algorithms are designed to compute the prices of these options based on the pricing formulas.

Mathematics Subject Classification: 34A30, 60H20

Keywords: uncertainty theory, uncertain differential equation, stock model, option pricing formulas

1 Introduction

An option is a contract that gives the owner the right to trade the underlying asset at a specified price on a specified date, depending on the form of the option.
Since Black and Scholes [1] proposed a stock model based on geometric Brownian motion. The issue of option pricing has become a hot topic in financial markets. Afterwards, Merton [2] extended Black-Scholes model in several ways. These landmark article paved the way for many subsequent academic studies, and used the well-known Black-Scholes models to evaluate many other financial derivatives.

As we all know, when using probability theory, a fundamental premise is that the estimated distribution function is close enough to the long-run cumulative frequency. Otherwise, the law of large numbers is no longer valid so that the probability theory is no longer applicable. However, in many situations, there are not enough historical data. Moreover, the price of options in reality often depends on the complex social environment and has a large uncertainty, probability way not give a completely reasonable explanation.

So Liu [4] proposed a class of uncertainty theory, which has advantages in dealing with the problem of uncertainty in human behavior, thereby making up for the short comings of existing pricing models with human uncertainty. Since then, many scholars have studied the pricing of financial derivatives in the framework of uncertainty theory. For example, Peng and Yao [11] proposed a new stock model, which was later called the Peng-Yao model. Under this uncertain model, the pricing formulas for European and American options were given. An uncertain stock model with periodic dividends is also proposed in [12].

Uncertain differential equations are widely used in options pricing research. They were first introduced into the financial field by Liu [7]. Uncertain stock model with geometric Liu process was used to study pricing problems. References [3] and [10] then derived pricing formulas for American and Asian options, respectively. Moreover, Peng and Yao [11] introduced a mean recovery process and derived some option pricing formulas. Subsequently, Yao [15] provided a model of uncertain inventory floating interest rates, in which stock prices and interest rates follow uncertain differential equations. Later, Sun and Su [9] proposed a stock model with mean return to floating interest rates.

In this paper, we presents a mean-reverting interest rate model and stock model based on Peng-Yao model, which with the random floating interest rate in order to describe the stock price in a long time. The rest of this paper is organized as follows: In Section 2 we present some basic concepts and properties in uncertainty theory, including Liu’s model and Peng-Yao’s model in Section 3. And we introduce a random floating interest rate mean-reverting model in uncertain environment. And we derive the European call and put option pricing fromulas and give the Monte Carlo method to solve the equations obtained in Section 4. An example is shown in Section 5. Finally, some conclusions are proposed in Section 6.
2 Preliminary

In this section, we review some concepts and results in uncertainty theory. For more details, refer to [4].

2.1 Uncertain Measure

Let \((\Gamma, \mathcal{L})\) be a measurable space. \(\mathcal{L}\) is a \(\sigma\)-algebra on a nonempty set \(\Gamma\). And the uncertain measure \(M\) is defined as a set function on \(\mathcal{L}\) satisfying the following axioms:

- **Axiom 1.** (Normality Axiom) \(M\{\Gamma\} = 1\) for the universal set \(\Gamma\).
- **Axiom 2.** (Duality Axiom) \(M(\Lambda) + M(\Lambda^c) = 1\) for any event \(\Lambda \in \mathcal{L}\).
- **Axiom 3.** (Subadditivity Axiom) For every countable sequence of events \(\Lambda_1, \Lambda_2, ..., \Lambda_n, ...\) we have

\[
M\left(\bigcup_{i=1}^{\infty} \Lambda_i\right) \leq M\{\Lambda_i\}.
\]

The triplet \((\Gamma, \mathcal{L}, M)\) is called an uncertainty space. In addition, Liu [7] defined a product uncertain measure as the following axiom in order to describe the set function \(M\) on the product \(\sigma\)-algebra \(\mathcal{L}\).

(Product Axiom) Let \((\Gamma_k, \mathcal{L}_k, M_k)\) be uncertainty spaces for \(k = 1, 2, \cdots\). Then the product uncertain measure \(M\) is an uncertain measure satisfying

\[
M\left(\prod_{i=1}^{\infty} \Lambda_k\right) = \bigwedge_{k=1}^{\infty} M\{\Lambda_k\}.
\]

where \(\Lambda_k\) are arbitrarily chosen events from \(\mathcal{L}_k\) for \(k = 1, 2, \cdots\), respectively.

2.2 Uncertain Differential Equation

An uncertain process \(C_t\) is said to be a Liu process if \(C_0 = 0\) and almost all sample paths are Lipschitz continuous, \(C_t\) has stationary and independent increments and every increment \(C_{s+t} - C_s\) is a normal uncertain variable with expected value 0 and variance \(t^2\). The uncertainty distribution of \(C_t\) [7] is

\[
\Phi_t(x) = (1 + \exp(-\frac{\pi x}{\sqrt{3} t}))^{-1}
\]

and inverse uncertainty distribution is

\[
\Phi_t^{-1}(\alpha) = \frac{t\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}.
\]
As the popular topic of uncertain integral, Liu integral allows us to integrate an uncertain process with respect to Liu process. The result of Liu integral is another uncertain process.

**Definition 2.1.** [5] Suppose $C_t$ is a Liu process, and $f$ and $g$ are measurable functions. Then

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t$$

is called an uncertain differential equation. A solution is an uncertain process $X_t$ that satisfies the uncertain integral equation

$$X_s = X_0 + \int_0^s f(t, X_t)dt + \int_0^s g(t, X_t)dC_t.$$

**Definition 2.2.** [16] The $\alpha$-path $(0 < \alpha < 1)$ of an uncertain differential equation

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t$$

with initial value $X_0$ is a deterministic function $X_t^\alpha$ with respect to $t$ that solves the corresponding ordinary differential equation

$$dX_t^\alpha = f(t, X_t^\alpha)dt + |g(t, X_t^\alpha)|\Phi^{-1}(\alpha)dt$$

where $\Phi^{-1}(\alpha)$ is the inverse uncertainty distribution of standard normal uncertain variable, i.e.

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}, 0 < \alpha < 1.$$

**Definition 2.3.** [4] The expected value of an uncertain variable $\xi$ is defined by

$$E[\xi] = \int_0^\infty M\{\xi \geq x\}dx - \int_{-\infty}^0 M\{\xi \leq x\}dx.$$

**Theorem 2.4.** [4] Let $u_t$ and $v_t$ be two integrable uncertain processes. Then the uncertain differential equation

$$dX_t = u_tX_tdt + v_tX_tdC_t$$

has a solution

$$X_t = X_0\exp\left(\int_0^t u_sds + \int_0^t v_sC_t\right).$$
Theorem 2.5. [17] Let $u_{1t}, u_{2t}, v_{1t}, v_{2t}$ be integrable uncertain processes. Then the linear uncertain differential equation
\[
 dX_t = (u_{1t}X_t + u_{2t}) \, dt + (v_{1t}X_t + v_{2t}) \, dC_t
\]
has a solution
\[
 X_t = U_t \cdot V_t
\]
where
\[
 U_t = \exp \left( \int_0^t u_{1s} \, ds + \int_0^t v_{1s} \, dC_s \right), \quad V_t = X_0 + \int_0^t \frac{u_{2s}}{U_s} \, ds + \int_0^t \frac{v_{2s}}{U_s} \, dC_s.
\]

Theorem 2.6. [16] Let $X_t$ and $X^\alpha_t$ be the solution and $\alpha$-path of the uncertain differential equation
\[
 dX_t = f(t, X_t) \, dt + g(t, X_t) \, dC_t
\]
respectively. Then the solution $X_t$ has an inverse uncertainty distribution
\[
 \Psi_s^{-1}(\alpha) = X^\alpha_t.
\]

Theorem 2.7. [18] Let $\eta$ be a random variable and let $\tau$ be an uncertain variable. Then
\[
 E[\eta \tau] = E[\eta] \, E[\tau].
\]

### 3 Mean-reverting option pricing model

A European call option is a financial contract that gives the owner the right without the obligation to buy an agreed quantity of stock. Let $f_c$ represent the price of the contract. Then
\[
 f_c = E \left[ \exp \left( - \int_0^T r_s \, ds \right) (X_T - K)^+ \right],
\]
where $r_t$ is the interest rate and $X_t$ is the bond’s price from the seller of the option at an expiration date $T$ with a striking price $K$. 
3.1 Liu’s Original model

Let $X_t$ be the bond’s price, and $Y_t$ be the stock’s price. And assuming stock prices follow a geometric paradigm, Liu presented the uncertain stock price model as follows:

\[
\begin{aligned}
    dX_t &= rX_t dt \\
    dY_t &= eY_t dt + \sigma Y_t dC_t
\end{aligned}
\]  

(1)

where $r$ is the risk-free interest rate, $e$ is the drift coefficient, $\sigma$ is the diffusion coefficient, and $C_t$ is Liu process.

Liu gave European-style settlement times $T$ and execution prices $K$. The price of call options and put options are

\[
\begin{align*}
    f_c &= \exp(-rT) \int_0^1 (Y_0 \exp(eT + \frac{\sqrt{3}\sigma T}{\pi} \ln \frac{\alpha}{1 - \alpha}) - K)^+ d\alpha \\
    f_p &= \exp(-rT) \int_0^1 (K - Y_0 \exp(eT + \frac{\sqrt{3}\sigma T}{\pi} \ln \frac{\alpha}{1 - \alpha}))^+ d\alpha
\end{align*}
\]

where $f_c$, $f_p$ represent the prices of European call options and European put options, respectively.

3.2 Peng-Yao model

Let $X_t$ be the price of the bond and $Y_t$ be the price of the stock, Peng and Yao gave a new class securities Model-Mean Regression Model:

\[
\begin{aligned}
    dX_t &= rX_t dt \\
    dY_t &= (m - eY_t) dt + \sigma dC_t
\end{aligned}
\]  

(2)

where $r, m, e, \sigma$ are positive constant.

Then the European call option price is

\[
f_c = \frac{\sqrt{3}s}{pa} \exp(-rT)(1 - \exp(-\alpha T)) \ln \left(1 + \exp \left(-\frac{p}{\sqrt{3}}b\right)\right)
\]

where $b = (\alpha K - m - \exp(-\alpha T)(\alpha Y_0 - m))/(s - s \exp(-\alpha T))$.

The European put option price is

\[
f_p = \frac{\sqrt{3}\sigma}{p\alpha} \exp(-rT)(1 - \exp(-\alpha T))(\ln(1 + \exp(\frac{\pi}{\sqrt{3}}\beta)) - \ln(1 + \exp(\frac{\pi}{\sqrt{3}}\gamma))
\]

where

\[
\beta = (\alpha K - m - \exp(\alpha T)(\alpha Y_0 - m))/(\sigma - \sigma \exp(-\alpha T))
\]

and

\[
\gamma = (-m - \exp(-\alpha T)(\alpha T - m))/(\sigma - \sigma \exp(-\alpha T)).
\]
4 Stock models under floating interest rate

Peng-Yao model [2] discusses the stock price at constant interest rates. However, the interest rate is usually not constant. In this section, we introduce the stock models with floating interest rate. We use stochastic differential equations to represent interest rate fluctuations, and the uncertain differential equation to represent stock prices. Next, we will give different types of stochastic differential equations to describe the interest rate model in different situations.

4.1 Black-Scholes stochastic differential equation with constant coefficients

First, we use the Black-Scholes stochastic differential equation with constant coefficients to describe interest rate fluctuations:

\[ dr_t = r_t (\mu dt + \sigma_1 dB_t) \]  

where \( B_t \) is a Brownian motion, \( \mu \) is the drift rate, \( \sigma_1 \) is the volatility rate. A model that uses a stochastic differential equation to describe interest rates and the uncertain differential equation to describe stock prices is as follows,

\[
\begin{align*}
\{ & dr_t = r_t (\mu dt + \sigma_1 dB_t) \\
& dX_t = (m - eX_t) dt + \sigma_2 dC_t
\end{align*}
\]

where \( \mu, \sigma_1, m, e, \sigma_2 \) are some given positive constants, \( B_t \) is a Brownian motion, and \( C_t \) is a Liu process. For the stochastic differential equation \( dr_t = r_t (\mu dt + \sigma_1 dB_t) \), its corresponding pseudo-homogeneous differential equation is \( dr_t = \sigma_1 r_t dB_t \). So we have \( r_t = ce^{\sigma_1 B_t} \). Using the constant variation method, let \( r_t = c(t)e^{\sigma_1 B_t} \). Then differentiate both sides at the same time to get

\[
dr_t = c'(t)e^{\sigma_1 B_t} dt + \sigma_1 c(t)e^{\sigma_1 B_t} dB_t + \frac{1}{2} \sigma_1^2 c(t)e^{\sigma_1 B_t} dt
\]  

We substitute the equation (5) into the original equation (3) to get

\[ c'(t)dt = \left( \mu - \frac{1}{2} \sigma_1^2 \right)c(t)dt \]

We can solve the equation (3) that

\[ r_t = r_0 e^{\left( \mu - \frac{1}{2} \sigma_1^2 \right)t + \sigma_1 B_t} \]  

In reality, interest rates will not change all the time. Therefore we usually consider the interest rate after a period of time \( \Delta t \). Let \( N \) denote the number...
of interest rate changes in a year, and \( r_j \) denote the interest rate on day \( j \). Then \( \Delta t = \frac{1}{N} \). Suppose that

\[
Y_j = \left( \mu - \frac{\sigma_1^2}{2} \right) / N + \left( \frac{\sigma_1}{\sqrt{N}} \right) Z_j
\]

where \( Z_j \sim N(0, 1) \). We have \( Y_j \sim N\left( \left( \mu - \frac{\sigma_1^2}{2} \right) / N, \frac{\sigma_1^2}{N} \right) \). Then we can construct a sequence with \( m \) days

\[
r_1 = r_0 e^{Y_1}, \quad r_2 = r_1 e^{Y_2}, \ldots, \quad r_m = r_{m-1} e^{Y_m}
\]  

(7)

where \( r_1, r_2, \ldots, r_t \) is a sequence of Monte Carlo simulation value of interest rate expressed by stochastic differential equation.

By Theorem 2.5, we have

\[
U = \exp \left( \int_0^t -ed\sigma + \int_0^t 0dC_2s \right) = \exp (-et)
\]

\[
V = X_0 + m \int_0^t \exp (es) ds + \sigma_2 \int_0^t \exp (es) dC_s
\]

\[
= X_0 + \frac{m}{e} (\exp (et) - 1) + \sigma_2 \int_0^t \exp (es) dC_s
\]

and the stock price \( X_t \) is

\[
X_t = \exp (-et) \left( X_0 + \frac{m}{e} (\exp (et) - 1) + \sigma_2 \int_0^t \exp (es) dC_s \right)
\]

(8)

Hence, we have

\[
\begin{align*}
\{ r_t &= r_0 e^{(\mu - \frac{1}{2} \sigma_1^2) t + \sigma_1 B_t} \\
X_t &= \frac{m}{e} + \exp (-et) \left( X_0 - \frac{m}{e} \right) + \sigma_2 \int_0^t \exp (es - et) dC_s.
\end{align*}
\]

(9)

According to the Definition 2.1, the \( \alpha \)-paths of \( X_t \) are the solutions of ordinary differential equations

\[
dX_t^\alpha = (m - eX) dt + \sigma_2 \Phi^{-1}(\alpha) dt
\]

(10)

where

\[
\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}.
\]

Thus

\[
X_t^\alpha = X_0 \cdot \exp (-et) + \left( \frac{m}{e} + \sigma_2 \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} (1 - \exp (-et)) \right).
\]

(11)
4.1.1 European call option

In this part, we introduce the European call option and derive some formulas to calculate the price of this option for the stock model (4). A European call option is a financial contract that gives the owner the right without the obligation to buy an agreed quantity of stock from the seller of the option at an expiration date $T$ with a striking price $K$. Let $f_c$ represent the price of the contract. Then we have

$$ f_c = E \left[ \exp \left( - \int_0^T r_s ds \right) (X_T - K)^+ \right] $$

**Theorem 4.1.** The price of the European call option with a strike price $K$ and an expiration date $T$ for the stock model (4) is

$$ f_c = E \left[ \exp \left( - \int_0^T r_0 e^{(\mu - \frac{1}{2} \sigma^2) s + \sigma_1 B_s ds} \right) \right] \int_0^1 (X_T^\alpha - K)^+ d\alpha $$

where

$$ X_T^\alpha = X_0 \cdot \exp (-eT) + \left( \frac{m}{e} + \frac{\sigma_2}{e} \cdot \frac{\sqrt{3}}{\pi} \ln \frac{1 - \alpha}{\alpha} \right) \left( 1 - \exp (-eT) \right) $$

**Proof.** Since $X_t$ is an uncertain process with an $\alpha$-path represented by equation (11), the uncertain process $(X_t - K)^+$ has an $\alpha$-path $(X_t^\alpha - K)^+$. According to Theorem 2.5 and Theorem 2.6, the expectation of $(X_t^\alpha - K)^+$ is

$$ E(X_T - K)^+ = \int_0^1 \left( X_0 \cdot \exp (-eT) + \left( \frac{m}{e} + \frac{\sigma_2}{e} \cdot \frac{\sqrt{3}}{\pi} \ln \frac{1 - \alpha}{\alpha} \right) \left( 1 - \exp (-eT) \right) - K \right)^+ d\alpha. $$

Because $\exp \left( - \int_0^T r_s ds \right)$ is a random variable and $(K - X_T)^+$ is an uncertain variable, by Theorem 2.7 we have

$$ f_c = E \left[ \exp \left( - \int_0^T r_s ds \right) \right] (X_T - K)^+ $$

$$ = E \left[ \exp \left( - \int_0^T r_0 e^{(\mu - \frac{1}{2} \sigma^2) s + \sigma_1 B_s ds} \right) \right] E(X_T - K)^+ $$

$$ = E \left[ \exp \left( - \int_0^T r_0 e^{(\mu - \frac{1}{2} \sigma^2) s + \sigma_1 B_s ds} \right) \right] \int_0^1 (X_T^\alpha - K)^+ d\alpha $$

The theorem is proved.

Because the expectation in (12) cannot be expressed analytically, we decide to use the Monte Carlo method to calculate it.
Algorithm 1 Algorithm for approximating expected value

1: Set initial value $r_0, N, m$ and $n$.
2: Randomly draw a sample $Z_{ij}$ from the standard normal distribution.
3: $Y_{ij} = \left( \mu - \frac{\sigma^2}{2} \right) / N + \left( \frac{\sigma_1}{\sqrt{N}} \right) Z_{ij}$.
4: Repeat 2 and 3 to get $Y_{1j}, Y_{2j}, \ldots, Y_{mj}$.
5: Get a sequence $r_{1j} = r_0 e^{Y_{1j}}, r_{2j} = r_{1j} e^{Y_{2j}}, \ldots, r_{mj} = r_{m-1,j} e^{Y_{mj}}$.
6: $R_j = \exp \left( -\int_0^T r_s ds \right) = \exp \left( -\sum_{i=1}^m \frac{1}{m} r_{ij} \right)$.
7: Repeat 2 to 6 to get a sequence $R_1, R_2, \ldots, R_n$.
8: Return $\frac{1}{n} \sum_{j=1}^n \sum_{i=1}^m \exp \left( -\frac{1}{m} \sum_{i=1}^m r_{ij} \right)$.

Therefore, we can calculate the numerical solution of the option price $f_c$ as

$$f_c = \frac{1}{n} \sum_{j=1}^n \sum_{i=1}^m \exp \left( -\frac{1}{m} \sum_{i=1}^m r_{ij} \right) \int_0^1 (X_0^\alpha - K)^+ d\alpha$$

where

$$r_{ij} = r_{i-1,j} e^{Y_{ij}}, r_{1j} = r_0 e^{Y_{1j}}, Y_{ij} \sim N \left( \left( \mu - \frac{\sigma_1^2}{2} \right) / N, \frac{\sigma_1^2}{N} \right)$$

$$X_0^\alpha = X_0 \cdot \exp (-eT) + \left( \frac{m}{e} + \frac{\sigma_2}{e} \cdot \frac{\sqrt{3}}{\pi} \ln \frac{1 - \alpha}{\alpha} \right) \left( 1 - \exp (-eT) \right).$$

4.1.2 European put option

In this part, we introduce the European put option and derive some formulas to calculate the price of this option for the stock model. A European put option is a financial contract that gives the owner the right without the obligation to sell an agreed quantity of stock to the buyer of the option at an expiration date $T$ with a striking price $K$. Let $f_p$ represent the price of the contract. Then, we have

**Theorem 4.2.** The price of the European call option with a strike price $K$ and an expiration date $T$ for the stock model (4) is

$$f_c = E \left[ \exp \left( -\int_0^T r_0 e^{\left( \mu - \frac{\sigma_1^2}{2} \right) s + \sigma_1 B_s} ds \right) \right] \int_0^1 (K - X_0^\alpha)^+ d\alpha$$

where

$$X_0^\alpha = X_0 \cdot \exp (-eT) + \left( \frac{m}{e} + \frac{\sigma_2}{e} \cdot \frac{\sqrt{3}}{\pi} \ln \frac{1 - \alpha}{\alpha} \right) \left( 1 - \exp (-eT) \right).$$
Proof. The proof is omitted here for it is similar to that of Theorem 4.1. The expression of the expected interest rate here is exactly the same as in Theorem 4.1, so we can use Algorithm 1 to calculate

\[ f_c = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{m} \exp \left( -\frac{1}{m} \sum_{i=1}^{m} r_{ij} \right) \int_{0}^{1} (K - X_T^x)^+ d\alpha \]  

(15)

where

\[ r_{ij} = r_{i-1,j} e^{Y_{ij}}, r_{1j} = r_0 e^{Y_{ij}}, Y_{ij} \sim N \left( \left( \mu - \frac{\sigma_1^2}{2} \right) / N, \frac{\sigma_1^2}{N} \right) \]

\[ X_T^x = X_0 \cdot \exp (-eT) + \left( \frac{m}{e} + \frac{\sigma_2}{e} \cdot \frac{\sqrt{3}}{\pi} \ln \frac{1 - \alpha}{\alpha} \right) \left( 1 - \exp (-eT) \right). \]

4.2 Black-Scholes stochastic differential equation with time-varying coefficients

In this subsection, we introduce the use of the Black-Scholes stochastic differential equation with constant coefficients to describe interest rate fluctuations.

\[ dr_t = b_t dt + \sigma_t dB_t \]  

(16)

where \( B_t \) is a Brownian motion, \( b_t \) is the drift rate related to \( t \), and \( \sigma_t \) is the volatility rate related to \( t \). A model that uses a stochastic differential equation to describe interest rates and the uncertain differential equation to describe stock prices is as follows,

\[
\left\{ \begin{array}{l}
\quad dr_t = b_t dt + \sigma_t dB_t \\
\quad dX_t = (m - eX_t) dt + \sigma_2 dC_t
\end{array} \right. \]

(17)

where \( m, e, \sigma_2 \) are some given positive constants, \( b_t, \sigma_t \) are function expressions of time \( t \), \( B_t \) is a Brownian motion, and \( C_t \) is a Liu process. For the stochastic differential equation \( dr_t = b_t dt + \sigma_t dB_t \), its corresponding pseudo-homogeneous differential equation is \( dr_t = \sigma_t r_t dB_t \). So we have \( r_t = c \exp \left( \int_{0}^{T} \sigma_t dB_t \right) \). Using the constant variation method, let \( r_t = c(t) \exp \left( \int_{0}^{T} \sigma_t dB_t \right) \). Then differentiate both sides at the same time to get

\[ dr_t = c'(t) \exp \left( \int_{0}^{T} \sigma_s dB_s \right) dt + \sigma_t c(t) \exp \left( \int_{0}^{T} \sigma_s dB_s \right) dB_t + \frac{1}{2} \sigma_t^2 c(t) \exp \left( \int_{0}^{T} \sigma_s dB_s \right) dt. \]

(18)

We substitute the equation (18) into the original equation (16) to get

\[ c'(t) dt = \left( b_t - \frac{1}{2} \sigma_t^2 \right) c(t) dt. \]
We can solve the equation (3) that

\[ r_t = r_0 \exp \left( \int_0^T \left( b_s - \frac{1}{2} \sigma_s^2 \right) ds + \int_0^T \sigma_s dB_s \right). \]  

(19)

In reality, interest rates will not change all the time. Therefore we usually consider the interest rate after a period of time \( \Delta t \). Let \( N \) denote the number of interest rate changes in a year, and \( r_j \) denote the interest rate on day \( j \), then \( \Delta t = \frac{1}{N} \). Suppose that

\[ Y_j = \int_0^T \left( b_s - \frac{\sigma_s^2}{2} \right) dt/N + \left( \int_0^T \sigma_s dt \right) Z_j \]

where \( Z_j \sim N(0,1) \), we have \( Y_j \sim N \left( \int_0^T \left( b_s - \frac{\sigma_s^2}{2} \right) ds/N, \frac{\int_0^T \sigma_s^2 ds}{N} \right) \). Then we can construct a sequence with \( m \) days

\[ r_1 = r_0 e^{Y_1}, r_2 = r_1 e^{Y_2}, \ldots, r_m = r_{m-1} e^{Y_m}. \]  

(20)

\( r_1, r_2, \ldots, r_t \) is a sequence of Monte Carlo simulation value of interest rate expressed by stochastic differential equation.

The stock price equation is the same as that of the equation (4). Hence, we have

\[ \left\{ \begin{array}{l}
    r_t = r_0 \exp \left( \int_0^T \left( b_s - \frac{1}{2} \sigma_s^2 \right) ds + \int_0^T \sigma_s dB_s \right) \\
    X_t = m e^{(\int_0^t ds) - et} (X_0 - m e^{(\int_0^t ds)}) + \sigma_2 \int_0^t e^{(s - et)} dC_s.
\end{array} \right. \]  

(21)

4.2.1 European call option

In this subsection, we introduce the European call option and derive some formulas to calculate the price of these option for the stock model (17). A European call option is a financial contract that gives the owner the right without the obligation to buy an agreed quantity of stock from the seller of the option at an expiration date \( T \) with a striking price \( K \). Let \( f_c \) represent the price of the contract. Then we have

\[ f_c = E \left[ \exp \left( - \int_0^T r_s ds \right) (X_T - K)^+ \right] \]

Theorem 4.3. The price of the European call option with a strike price \( K \) and an expiration date \( T \) for the stock model (17) is

\[ f_c = E \left[ \exp \left( - \int_0^T r_0 e^{(\mu - \frac{1}{2} \sigma_1^2) s + \sigma_1 B_s} ds \right) \right] \int_0^1 (X_T^\alpha - K)^+ d\alpha \]  

(22)
where
\[
    r_s = r_0 \exp \left( \int_0^s \left( b_p - \frac{1}{2} \sigma_p^2 \right) dp + \int_0^s \sigma_p dB_p \right)
\]
\[
    X_T^\alpha = X_0 \cdot \exp (-eT) + \left( \frac{m}{e} + \frac{\sigma_2}{e} \cdot \frac{\sqrt{3}}{\pi} \ln \frac{1 - \alpha}{\alpha} \right) (1 - \exp (-eT))
\]

**Proof** The proof is omitted here for it is similar to that of Theorem 4.1.

We can use Algorithm 1 to calculate
\[
f_c = \frac{1}{n} \sum_{j=1}^n \sum_{i=1}^m \exp \left( -\frac{1}{m} \sum_{i=1}^m r_{ij} \right) \int_0^1 (X_T^\alpha - K)^+ d\alpha
\]
(23)

\[
    r_{ij} = r_{i-1,j} e^{Y_{ij}}, r_{1j} = r_0 e^{Y_{1j}}, Y_{ij} \sim \mathcal{N} \left( \int_0^t \left( b_s - \frac{\sigma_s^2}{2} \right) ds/N, \int_0^t \sigma_s^2 ds/N \right)
\]
\[
    X_T^\alpha = X_0 \cdot \exp (-eT) + \left( \frac{m}{e} + \frac{\sigma_2}{e} \cdot \frac{\sqrt{3}}{\pi} \ln \frac{1 - \alpha}{\alpha} \right) (1 - \exp (-eT)).
\]

### 4.2.2 European put option

In this subsection, we introduce the European put option and derive some formulas to calculate the price of this option for the stock model. A European put option is a financial contract that gives the owner the right without the obligation to sell an agreed quantity of stock to the buyer of the option at an expiration date \( T \) with a striking price \( K \). Let \( f_p \) represent the price of the contract. Then, we have

**Theorem 4.4.** The price of the European call option with a strike price \( K \) and an expiration date \( T \) for the stock model (17) is
\[
f_c = E \left[ \exp \left( -\int_0^T r_0 e^{(\mu - \frac{1}{2} \sigma^2) s + \sigma_1 B_s} ds \right) \right] \int_0^1 (K - X_T^\alpha)^+ d\alpha
\]
(24)

where
\[
    r_s = r_0 \exp \left( \int_0^s \left( b_p - \frac{1}{2} \sigma_p^2 \right) dp + \int_0^s \sigma_p dB_p \right)
\]
\[
    X_T^\alpha = X_0 \cdot \exp (-eT) + \left( \frac{m}{e} + \frac{\sigma_2}{e} \cdot \frac{\sqrt{3}}{\pi} \ln \frac{1 - \alpha}{\alpha} \right) (1 - \exp (-eT))
\]
Proof The proof is omitted here for it is similar to that of Theorem 4.2. We can use Algorithm 1 to calculate

\[ f_c = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{m} \exp \left( -\frac{1}{m} \sum_{i=1}^{m} r_{ij} \right) \int_{0}^{1} (K - X_{T}^{\alpha})^{+} d\alpha \]  

(25)

where

\[ r_{ij} = r_{i-1,j} e^{Y_{ij}}, r_{1j} = r_0 e^{Y_{1j}}, Y_{ij} \sim N \left( \int_{0}^{t} \left( b_s - \frac{\sigma_s^2}{2} \right) ds / N, \int_{0}^{t} \sigma_s^2 ds \right) / N \]

\[ X_{T}^{\alpha} = X_0 \cdot \exp (-eT) + \left( \frac{m}{e} + \frac{\sigma_2}{e} \cdot \frac{\sqrt{3}}{\pi} \ln \frac{1 - \alpha}{\alpha} \right) (1 - \exp (-eT)) \].

5 Example

Let’s take an example to show the applicability of the model. Let the parameters in model (4) be as follows, \( \mu = 1, \sigma_1 = 1.3, m = 1, e = 2, \sigma_2 = 1.4 \). These imply that the model is

\[
\begin{align*}
    dr_t &= r_t (dt + 1.3dB_t) \\
    dX_t &= (1 - 2X_t) dt + 1.4dC_t
\end{align*}
\]

With the striking price \( K = 92 \), we used Algorithm 1 to calculate the option price on the 1000th day as 10.2802 and the option price changes over time is shown in Figure 1.

Figure 1: Option Price

\[ \text{Figure 1: Option Price} \]
6 Conclusion

Different from random or uncertain European call options, in this article, we propose two new uncertain stock models with random floating interest rates and use it to calculate the price of European call options in an uncertain environment. Then, we get the Monte Carlo method to approximate the price of call options and put options. This method is more close to the actual situation, and the calculation is simple and universal. In future research, the method can be improved, and the price of American options in the stock model may be considered.

Acknowledgements. We are especially grateful to Professor Yuanguo Zhu for his guidance and help on this paper, and also thank the Nanjing University of Science and Technology Research Training Program for supporting this paper.

References


Received: March 25, 2021; Published: April 10, 2021