Generalized Common Fixed Point Theorems for Mappings Satisfying Contractive Condition of Integral Type with D-Complete Topological Spaces

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Abstract

In this paper, we prove generalized common fixed point theorems for mappings satisfying contractive condition of integral type with D-complete topological space. Our results extend and generalize some well known previous results.

1. Introduction

Branciari [7] obtained a fixed point result for a single mapping satisfying an analogue of Banach’s contraction principle for an integral type inequality. The authors in [3], proved some common fixed point theorems involving more general contractive conditions. Recently ([10]) some fixed point theorems have been proved in non-metric setting wherein the distance function used need not satisfy triangle inequality. The purpose of this paper is to investigate some new result of fixed points in non-metric settings. In the sequel, we use contractive condition of integral type on d-complete Hausdorff topological spaces.
Sessa [24] generalized the concept of commuting mappings by calling selfmappings \( A \) and \( S \) on metric space \((X, d)\) a weakly commuting pair if and only if \(d(ASx, SAx) \leq d(Ax, Sx)\) for all \(x \in X\). He and others proved some common fixed point theorems of weakly commuting mappings [24, 25, 26]. Then, Jungck [13] introduced the concept of compatibility and he and others proved some common fixed point theorems using this concept [13, 14, 15, 29]. Clearly, commuting mappings are weakly commuting and weakly commuting mappings are compatible. Examples in [13, 24] show that neither converse is true. Recently, Jungck and Rhoades [15] defined the concept of weak compatibility.

**Definition 1.1** (see [15, 27]). Two maps \( A, S : X \rightarrow X \) are said to be weakly compatible if they commute at their coincidence points.

**Definition 1.2.** Let \((X, S)\) be an \(S\)-metric space. A sequence \(\{x_n\}\) in \(X\)

I. Said to be converges to \(x \in X\) if \(S( x_n, x) \rightarrow 0\) as \(n \rightarrow \infty\). We write \(x_n \rightarrow x\) for brevity.

II. Said to Cauchy sequence if for \(\varepsilon > 0\), there exists \(n_0 \in \mathbb{N}\) such that for all \(n, m \geq n_0\) we have \(S( x_n, x_m) < \varepsilon\).

III. Said to be complete if every Cauchy sequence in \(X\) converges in \(X\).

Let \((X, \tau)\) be a topological space and \(d : X \times X \rightarrow [0, \infty)\) be such that \(d(x, y) = 0\) if and only if \(x = y\). Then \(X\) is said to be \(d\)-complete if

\[
\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty
\]

implies that the sequence \(\{x_n\}\) is convergent in \(X\). A mapping \(T : X \rightarrow X\) is \(w\)-continuous at \(x\) if \(x_n \rightarrow x\) implies \(T( x_n) \rightarrow x\). For details on \(d\)-complete topological spaces, we refer to Iseki [11] and Kasahara [17]-[19]. In the sequel, we shall use the following: A symmetric function on a set \(X\) is a real valued \(d\) on \(X \times X\) such that for all \(x, y \in X\),

(i) \(d(x, y) \geq 0\), and \(d(x, y) = 0\) if and only if \(x = y\),

(ii) \(d(x, y) = d(y, x)\).

Let \(d\) be a symmetric function on a set \(X\), and for any \(\varepsilon > 0\) and any \(x \in X\), let \(S(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}\). From [10], we can define a topology \(\tau_d\) on \(X\) by \(U \in \tau_d\) if and only if for each \(x \in U\), some \(S(x, \varepsilon) \subset U\). A symmetric function \(d\) is a semi-metric if for each \(x \in X\) and for each \(\varepsilon > 0\), \(S(x, \varepsilon)\) is a neighborhood of \(x\) in the topology \(\tau_d\). A topological space \(X\) is said to be symmetrizable (resp. semi-metrizable) if its
topology is induced by a symmetric function (resp. semi-metric) on X. The d-complete symmetrizable spaces form an important class of d-complete topological spaces. Other examples of dcomplete topological spaces may be found in Hicks and Rhoades [10]. Hicks and Rhoades [10] proved the following theorem.

**Theorem 1.3.**

Let \((X, \tau)\) be a Hausdorff d-complete topological space and \(f, h\) be w-continuous self mappings on \(X\) satisfying
\[
d(hx, hy) \leq G(M^*(x, y)) \quad \text{for } x, y \in X,
\]
where \(M^*(x, y) = \max\{d(fx, fy), d(fx, hx), d(fy, hy)\}\)

and \(G\) is a real-valued function satisfying the following:

1. \(0 < Q(t) < t\), \(t \in (0, \infty)\);
2. \(\rho(t) = \frac{t}{t-Q(t)}\) is a decreasing function;
3. \(\int_0^{t_1} \rho(t)dt < +\infty\) for some positive number \(t_1\).
4. \(G(y)\) is non-decreasing.

Suppose also that

\(i\) \quad f and h commute,

\(ii\) \quad h(X) \subseteq f(X). \) Then \(f\) and \(h\) have a unique common fixed point in \(X\).

**2. Main results**

**Theorem 2.1.** Let \(P, T, f, g,\) be w-continuous self-maps defined on a Hausdorff topological space \((X, \tau)\) satisfying the following conditions:

1. \(P(X) \subseteq g(X), T(X) \subseteq f(X),\)
2. \(\int_0^{Q(S(\tau_1, T_1))} \rho(t)dt \leq \phi(\int_0^{M(x, y)} \varphi(t)dt\) for all \(x, y \in X,\)
   where \(\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) is a Lebesgue integrable mapping which is summable on each compact subset of \(\mathbb{R}^+,\) non-negative and such that where \(\rho(t)\) is a Lebesgue integrable function which is summable nonnegative and such that
3. \(M(x, y) = \max\{d(fx, gy)\}

and \(\varphi\) is a real valued function satisfying the condition (1)-(4). If \(P(x), T(x), f(x), g(x),\) a d-complete topological subspace of \(X,\) then

\(i\) \quad P and f have a coincidence point,
(ii) \( T \) and \( g \) have a coincidence point. Further if the pairs \( \{P, f\} \) and \( \{T, g\} \) are weakly compatible, then (iii) \( A, B, S \) and \( T \) have a unique common fixed point

(6) If \( P(X) \subseteq g(X) \) and \( T(X) \subseteq f(X) \) so we define two sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) by the rule \( y_{2n+1} = P x_{2n} = g x_{2n+1} \) and \( y_{2n+2} = T x_{2n+1} = f x_{2n+2} \). \( n = 0, 1, 2, \ldots \)

Now consider \( \int_0^{S(y_{2n+1}, y_{2n+2})} \rho(t) \, dt = \int_0^{S(P x_{2n}, T x_{2n-1})} \rho(t) \, dt \)

Using (2) we have

\[
\leq \varphi \left( \int_0^{S(f x_{2n}, g x_{2n+1})} \rho(t) \, dt \right) = \varphi \left( \int_0^{S(y_{2n+1}, y_{2n+2})} \rho(t) \, dt \right)
\]

Let \( t_n = S(y_n, y_{n+1}) \) then the above inequality take the form \( \int_0^{t_{2n+1}} \rho(t) \, dt \leq \int_0^{t_{2n}} \rho(t) \, dt \).

Now by the property of \( Q \) and and condition \( c \) we have

\[ t_{2n+1} \leq Q(t_{2n}) < t_{2n} \]

Similarly we can show that \( t_{2n} \leq Q(t_{2n-1}) < t_{2n-1} \).

Hence \( \{t_n\} \) is a nonnegative strictly decreasing sequence and hence convergent. Thus \( t_{n+1} \leq Q(t_n) < t_n \) for all \( n = 0, 1, 2, 3, \ldots \)

Now to prove that \( \{y_n\} \) is a Cauchy sequence consider for \( m \geq n \) and by triangle inequality we have

\[
S(y_m, y_n, y_n) \leq 2 \sum_{i=n}^{m-1} t_i = 2 \sum_{i=n}^{m-1} (t_i - t_{i-1}) \leq - \sum_{i=n}^{m-1} \frac{t_i(t_{i+1} - t_i)}{(t_i - Q(t_i))} \leq 2 \sum_{i=n}^{m-1} \int_{t_i}^{t_{i+1}} \rho(t) \, dt \frac{1}{(t - Q(t_i))} \, dt
\]

It follows from the convergence of the sequence \( \{t_n\} \) and by condition (3) we have

\[
\lim_{n \to \infty} \int_{t_n}^{t_m} P(t) \, dt = 0.
\]

Thus \( \{y_n\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete so there must exists \( u \in X \) such that

\[
\lim_{n \to \infty} y_n = u.
\]

Also the subsequences \( \{y_{2n+1}\} \) and \( \{y_{2n+2}\} \) converges to \( u \).

Therefore \( \lim_{n \to \infty} y_{2n+1} = \lim_{n \to \infty} P x_{2n} = \lim_{n \to \infty} g x_{2n+1} = u \).

\[
\lim_{n \to \infty} y_{2n+2} = \lim_{n \to \infty} T x_{2n+1} = \lim_{n \to \infty} f x_{2n+2} = u.
\]
Since \((P, f)\) are continuous \(\phi\)-weakly commuting pair so
\[
S(P, f) \leq \phi(S(P, f)) = \phi(0) = 0.
\]
Taking limit \(n \to \infty\) and since \((P, f)\) is continuous pair of mappings thus
\[
S(P u, f u) = \phi(S(u, u)) = \phi(0) = 0.
\]
Which implies that \(P u = fu\). Similarly from continuous \(\phi\)-weakly commuting pair \((T, g)\) we can show that \(T u = gu\).

Now by condition (5) and using other given information we have
\[
\int_0^{S(P u, T u)} \rho(t) dt \leq \int_0^{Q(S(f u, g u))} \rho(t) dt \leq \int_0^{Q(S(f u, f u, g u))} \rho(t) dt
\]
\[
S(P u, T u) \leq Q(S(f u, g u)) \leq S(f u, g u) = S(f u, f u, g u)
\]
\[
\leq S(g u, T u) + S(f u, T u) + S(1, P u) \leq S(P u, T u) = S(P u, T u).
\]
Which implies that \(f u = g u\). Thus \(f u = g u = P u = T u\) and let \(z = f u = g u = P u = T u\).

Therefore \(u\) is the common coincidence point of mappings \(P, T, f\) and \(g\).

Again since \((P, f)\) are \(\phi\)-weakly commuting pair so
\[
S(P z, f z) = S(P f u, f f u) \leq \phi(S(P u, f u)) = \phi(0) = 0.
\]
Implies that \(P z = f z\). Similarly we can show that \(T z = g z\).

Thus \(P f u = f P u\) and \(T g u = g T u\).

Again by condition (5) we have
\[
\int_0^{S(P z, g u)} \rho(t) dt = \int_0^{S(P P u, P T u)} \rho(t) dt \leq \int_0^{Q(S(f u, g u))} \rho(t) dt = \int_0^{Q(S(f u, f u, g u))} \rho(t) dt
\]
By condition \(c\) and using the property of \(Q\) we have
\[
S(P z, z) \leq Q(S(P f u, g u)) = S(P f u, g u)
\]
\[
S(P z, z) = S(P f u, g u) = S(P z, z) = S(P u, T u).
\]
Therefore \(P z = f z = g z = T z\) and \(z = z\). Thus \(z\) is a common fixed point of mappings \(P, T, f\) and \(g\).

**Uniqueness**

Assume that common fixed point of \(P, T, f\) and \(g\) is not unique i.e \(z \neq w\) be two distinct fixed points of \(P, T, f\) and \(g\). Then using condition (5) we have
By Lemma 1.12 and using the property of $Q$ we have $S(z, w) \leq Q(S(fz, gw)) \leq S(z, w)$.

Which is a contradiction hence $z = w$. Therefore, fixed point of $P$, $T$, $f$ and $g$ is unique.

**Theorem 2.3.** Let $P,T,f,g,$ be $w$-continuous self-maps defined on a Hausdorff topological space $(X, \tau)$ satisfying the following conditions:

1. $P(X) \subseteq g(X)$, $T(X) \subseteq f(X)$,
2. $\int_0^{Q(S(Px,Ty))} \rho(t)dt \leq \phi(\int_0^{M(x,y)} \varphi(t)dt)$ for all $x, y \in X$.

where $\phi : R^+ \to R^+$ is a Lebesgue integrable mapping which is summable on each compact subset of $R^+$, non-negative and such that where $\rho(t)$ is a Lebesgue integrable function which is summable nonnegative and such that

4. $\epsilon \leq \int_0^\epsilon \varphi(t)dt > 0 \ \forall \ \epsilon > 0$

5. $M(x,y) = \max \varphi(\int_0^{\max} d(Sx,By), d(Ty,By), d(Sx,SxBy) + d(TyTx,Ax)) / 2 \rho(t)dt$

and $\varphi$ is a real valued function satisfying the condition (1)-(4). If $P(x),T(x),f(x),g(x)$, a $d$-complete topological subspace of $X$, then

(iii) $P$ and $f$ have a coincidence point,

(iv) $T$ and $g$ have a coincidence point. Further if the pairs $\{P,f\}$ and $\{T,g\}$ are weakly compatible, then (iii) A, B, S and T have a unique common fixed point

6. If $(P, f)$ and $(T, g)$ are two pairs of continuous $\phi$-weakly commuting mappings.

Then $P(x),T(x),f(x)$ and $,g(x)$,have a unique common fixed point in $X$.

**Corollary 2.6.** Let $P,f,$ be $w$-continuous self-maps defined on a Hausdorff topological space $(X, \tau)$ satisfying the following conditions:

1. $P(X) \subseteq f(X)$,
2. $\int_0^{d(S(Px,Px),Py)} \rho(t)dt \leq \varphi(\int_0^{M(x,y)} \varphi(t)dt)$ for all $x, y \in X$.

where $\phi : R^+ \to R^+$ is a Lebesgue integrable mapping which is summable on each compact subset of $R^+$, non-negative and such that where $\rho(t)$ is a Lebesgue integrable function which is summable nonnegative and such that
6. \( \epsilon \leq \int_0^t \varphi(t) \, dt > 0 \ \forall \ \epsilon > 0 \)

7. \( M(x,y) = \max \{ \int_0^1 d(Sx,By), d(Ty,By), d(Sx,SxBy) + d(Ty,Tx,Ax) / 2 \} \rho(t) \, dt \)

and \( \varphi \) is a real valued function satisfying the condition (1)-(4). If \( P(x), T(x), f(x), g(x), \)

a d-complete topological subspace of \( X \), then

(v) \( P \) and \( f \) have a coincidence point,

(vi) \( T \) and \( g \) have a coincidence point. Further if the pairs \( \{ P,f \} \) and \( \{ T,g \} \) are weakly compatible, then (iii) \( A, B, S \) and \( T \) have a unique common fixed point

(6) If \( (P, f) \) and \( (T, g) \) are two pairs of continuous \( \varphi \)-weakly commuting mappings. Then \( P(x), T(x), f(x) \) and \( g(x), \) have a unique common fixed point in \( X \).

Proof. Putting \( \rho(t) = 1 \) in Theorem 2.1, one can easily obtain the proof of Corollary 2.6 from Theorem 2.1. Now we construct an example to support Corollary 2.3.

Example 2.7: Re- weakly commuting mappings are weak commuting mappings but reverse is not true. Let \( X = [0, \infty) \), \( S(x, y, z) = |x - z| + |y - z| + |z - x| \) for all \( x, y, z \in X \) with self mapping defined on \( X \) is given by \( Px = \frac{x^2}{4} \) and \( Tx = x^2 \) thus we have \( fx = \frac{x^4}{4} \), \( gfx = \frac{x^4}{16} \)

\( S(fgx,gfx,gfx) = S(\frac{3x^4}{8}) \), \( S(fx,gx,gx) = \frac{3x^2}{8} \) let \( \varphi(x) = \frac{1x^2}{2} \) then
\( G(fgx,gfx,gfx) = \frac{3x^4}{8} \leq \frac{9x^4}{8} \leq \frac{1}{2}(3x^2)^2 = \varphi(x) = \frac{3x^2}{2} = \varphi(G(fx,gx,gx)) \)

For any given \( R > 0 \), since \( \lim_{x \to +\infty} \frac{1}{4}x^2 = +\infty \) there exists \( x \) belong \( X \) such that \( \frac{1}{4}x^2 > R \),

So we get \( S(fgx,gfx,gfx) = \frac{1}{4}x^2 \) \( S(fx,gx,gx) > RS(fx,gx,gx) \) there fore, \( f \) and \( g \) are \( \varphi \)-weakly commuting mapping, but not \( R \) weakly commuting mappings.

References


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