Computational Complexity for Variants Obtained from Sums of the Same Powers

Alexander I. Nikonov
Samara State Technical University
Institute of Automation and Information Technology
Samara, 443100, Russia

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Abstract

We consider several types of records for sums of the same powers. First, it is a simple presentation of the power sums, secondly, this is the presentation of sums with binomial coefficients and powers, thirdly, this presentation of sums in the form of Faulhaber’s polynomials and, finally, it is a recurrent representation of sums of the same powers. We show that the sums under consideration have a computational complexity depending on the combinations of the numbers of the powers being added, as well as their assigned values.

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1 Introduction

There is an initial equality for sums of the same powers [1,2]:

$$\Phi(p, \nu) = \sum_{\lambda=1}^{p} \lambda^\nu,$$

(1)

where \( p, \nu \in \mathbb{N} \).

This article describes and compares the computational complexity of the following options for representing sums of the same powers: simple representation of sums of powers; representation of sums with binomial coefficients and powers; Faulhaber polynomials; recurrent representation of sums.
Expressions for the selected representations of sums of the same powers look as shown below.

The simple representation of sums of the powers [1] exactly corresponds to the expression (1).

Sums with binomial coefficients and powers [2]:

\[
\Phi(p,v) = \sum_{i=1}^{\max j(i)} \left( \sum_{q=1}^{\max i} (-1)^{i+q} C_i^q q^v \right) C_{p+1}^{i+1},
\]

\[
\max t = \min(p,v).
\]

Faulhaber’s formula [3]:

\[
\Phi(p,v) = \frac{1}{v+1} \sum_{g=1}^{v+1} C_{v+1}^g B_{v+1-g} p^g.
\]

Recurrent representation of sums of the same powers [1]:

\[
\Phi(p,v) = \sum_{r=1}^{\max j(i)} \Phi_{p(r)0} = \sum_{r=1}^{\max j(i)} \alpha_r^j C_{p+1}^{r+1},
\]

\[
\Phi_{p(i)0} = C_{p+1}^{i+1},
\]

\[
\alpha_j^i = \sum_{j(i)=j(j)+1}^{\max j(i)} C_{j(i)}^{j(i)} \alpha_j^{j(i)}.
\]

\[
j(i) \in \{0,1,\ldots, \max j(i)\}, \quad \max j(i) = v(i) = v - i; \quad i = t - 1;
\]

\[
j(i) \in \{1,\ldots, \max j(i)\}, \quad \max j(i) = v(i) = v - i;
\]

\[
\alpha_{j(i)=0}^{i=0} = \begin{cases} 0: 1 \leq j(0) \leq v-1, \\ 1: j(0) = v; \end{cases}
\]

\[
\alpha_0^j = \sum_{j(i)=1}^{\max j(i)} \alpha_j^{j(i)}.
\]

2 Transformations and complexity representation

The simple representation of the power sums (1) does not require significant initial transformations:

\[
\Phi(p,v) = \sum_{j=1}^{p} x^v \leq p \cdot p^v = p^{v+1}.
\]

The transformation of the sum (2) with binomial coefficients and powers has the following form:

\[
\Phi(p,v) = \sum_{i=1}^{\max j(i)} \left( \sum_{q=1}^{\max i} (-1)^{i+q} C_i^q q^v \right) C_{p+1}^{i+1} = \sum_{q=1}^{\max j(i)} \left( \sum_{i=1}^{\max j(i)} (-1)^{i+q} C_i^q C_{i+1}^{q+1} \right)
\]
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\[ \leq \max h \cdot \max_{q=1}^{\max i} l = \max h \cdot \max_{q=1}^{\max i+1} l, \quad h = C_{p+1}^{i+1} / C_{\max+1}^{i+1}. \]

We used here the well-known identity [4]

\[ \sum_{i=q}^{\max i} (-1)^{i+q} C_i^q C_{\max+1}^{i+1} = 1; \]

\[ \max i = \begin{cases} p: \quad p \leq v, \\ v: \quad v \leq p; \end{cases} \quad \max h = \begin{cases} 1: \quad p \leq v, \\ C_{p+1}^{v+1}: \quad v \leq p; \end{cases} \]

from here

\[ \Phi(p,v) = \sum_{j=1}^{p} \lambda_j^v \leq p^{v+1}. \quad (6) \]

Faulhaber’s formula (3) remains without initial transformations.
We are now converting the formula of the recurrent presentation of sums of the same powers:

\[ \Phi(p,v) = \sum_{i=1}^{\max i} \alpha_i^v C_{p+1}^{i+1} \leq C_{p+1}^{i+1} |_{E((\max i+1)/2)} \sum_{i=1}^{\max i} \alpha_i^i \]

\[ = C_{p+1}^{i+1} |_{E((\max i+1)/2)} \sum_{i=1}^{\max i} \sum_{j(i)=j(i)+1}^{\max j(i)} C_{j(i)}^{i(i)} \alpha_{j(i)}^i. \quad (7) \]

Here \( E(\bullet) \) denotes the integer part of the number in parentheses.
From expressions (5), (3), (6), (7), it is easy to obtain an asymptotic representation of the computational complexity [5] of the corresponding variants of sums of the same powers. We have the following order of complexity functions:

for option (1) it is

\[ \sum_{j=1}^{p} \lambda_j^v = O(p^{v+1}); \]

for option (2) it is

\[ \sum_{i=1}^{\max i} \left( \sum_{q=1}^{\max i} (-1)^{i+q} C_i^q v^v \right) C_{p+1}^{i+1} = O(p^{v+1}); \]

for variant (3), according to the theorem on the asymptotic properties of polynomial functions, this is

\[ \frac{1}{v+1} \sum_{g=1}^{v+1} C_{v+1}^{g-1} B_{v+1-g} p^g = O(p^{v+1}); \]

for option (4) it is

\[ \sum_{i=1}^{\max i} \alpha_i^i C_{p+1}^{i+1} = \begin{cases} O(vp): \quad p \leq v, \\ O(v^2): \quad v \leq p. \end{cases} \]
So, we represent the computational complexity of any option 1-4 in the following way: in the left side of such an indication we put the corresponding expression from (5), (3), (6), (7), and in its right side - the asymptotic computational complexity. We also used the fact that the addition operation takes much less time compared to the multiplication operation [6.]

3 Conclusion

So, we show that the presented variants of sums of the same powers have a specific computational complexity. It is expressed by us through an asymptotic representation. If the other conditions for the existence of options (1) - (4) are the same, then when choosing a computation option, you should be guided by the above provisions. Guided by the considerations set out in Section 2, you can choose options with less complexity when calculating sums of the same powers. So, for cases \( p \leq v \) it can be the top line of variant (4), and for cases \( v \leq p \) it can be the bottom line of the same variant. We remind that further studies of this issue may lead to a decrease in the values in the left-hand sides of expressions (1) - (4).

References


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