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Variable-order Legendre Polynomials and Application [1]

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Abstract

A new orthogonal function named variable-order Legendre polynomial (VOLP) is proposed. Several useful formulas for VOLP are directly generalized from the classic Legendre polynomial. The fractional differential expression for VOLP of variable order is derived. As an application, it is successful to solve the variable-order fractional differential equation with initial value problem by the method of VOLP tau.

Keywords: Legendre polynomials, variable-order fractional derivative, fractional differential equation

1 Introduction

Legendre equation is a kind of differential equation often encountered in physics and other technical fields. It is used to solve the three-dimensional Laplace equation in spherical coordinates. As early as 1785, Legendre studied the attraction between spheres and the line of star motion by introducing Legendre

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equation, and its solution is obtained by series solution. The form of the solution is called Legendre polynomial. Polynomial is one of the most important function sets in mathematical physics, which has a wide range of applications in physics and engineering. As a very important application, Legendre spectral methods are very successful to solve the numerical solution to the various differential equations, including integer and fractional order. Through google scholar search, there are almost 64,000 articles from 1980 to 2020 on using Legendre spectral methods to research various problems, such as numerical method ([1]), random regression analysis ([2]), segmentation method capable ([3]), fingerprint research ([4]), optimum control ([5]), and so on. Recently, many authors ([6, 7, 8, 9, 10])devote to apply Müntz orthogonal polynomials to solve fractional differential equations (FDE). Inspired by these literatures, we defined a new orthogonal polynomials named variable-order Legendre polynomials, the order function $\alpha(x)$ of which is depend on the variable x. In particular, when $\alpha(x) = 1$, VOLPs degenerate into the classic Legendre polynomials; when $\alpha(x) = \alpha(0 < \alpha < 1)$, VOLPs are transformed into the fractional order Legendre polynomials proposed in ([10]). Further more, using the method of VOLPs tau to obtain some variable-order FDE is better than the one based on other orthogonal polynomials.

2 Preliminaries

In this section, we introduce several definitions and lemmas, which are further used in this article. First, let's make the following assumptions. For $\beta(x) > 0$, the functions $u_a(x) = (x-a)^{\beta(x)},\ x > a$ and $u_b(x) = (b-x)^{\beta(x)},\ x < b$, mean that when doing the orthogonalization for $u_a(x)$ and $u_b(x)$, $\beta(x)$ is seen as the usual function, otherwise, we treat $\beta(x)$ is a constant function. For example, $u_a'(x) = \beta(x)(x-a)^{\beta(x)-1}$; $u_b'(x) = -\beta(x)(b-x)^{\beta(x)-1}$. Such assumptions is needed due to the definition of variable-order fractional derivative introduced following.

Definition 2.1 *For $m-1 < \alpha(x) \le m$, $m \in Z_+$, then left-side Riemann-Liouville fractional derivative of variable is defined by*

$$ {}_aD_x^{\alpha(x)}u(x) = \left[\frac{1}{\Gamma(m-\alpha(x))}\frac{\partial^m}{\partial\zeta^m}\int_a^\zeta(\zeta-\xi)^{m-1-\alpha(x)}u(\xi)d\xi\right]_{\zeta=x}. \tag{1}$$

Definition 2.2 *For $m-1 < \alpha(x) \le m$, $m \in Z_+$, then right-side Riemann-Liouville fractional derivative of variable is defined by*

$$ {}_xD_b^{\alpha(x)}u(x) = \left[\frac{1}{\Gamma(m-\alpha(x))}\frac{\partial^m}{\partial\zeta^m}\int_\zeta^b(\xi-\zeta)^{m-1-\alpha(x)}u(\xi)d\xi\right]_{\zeta=x}. \tag{2}$$

Definition 2.3 *[11]For $0 < \alpha(x) \le 1$, the left-side Caputo fractional derivative of variable is defined by*

$$ {}_a^C D_x^{\alpha(x)} u(x) = \frac{1}{\Gamma(1-\alpha(x))} \int_a^x (x-\xi)^{-\alpha(x)} u'(\xi) d\xi. \tag{3} $$

Definition 2.4 *For $0 < \alpha(x) \le 1$, then right-side Caputo fractional derivative of variable is defined by*

$$ {}_x^C D_b^{\alpha(x)} u(x) = \frac{1}{\Gamma(1-\alpha(x))} \int_x^b (\xi - x)^{-\alpha(x)} u'(\xi) d\xi. \tag{4} $$

Definition 2.5 *For $0 < \alpha(x) \le 1$, the Riesz variable-order fractional derivative of Caputo is defined by*

$$ {}^C R^{\alpha(x)} u(x) = c(\alpha(x)) \left[{}_a D_x^{\alpha(x)} +_x D_b^{\alpha(x)} \right] u(x) = \frac{\partial^{\alpha(x)} u(x)}{\partial |x|^{\alpha(x)}}. \tag{5} $$

with $c(\alpha(x)) = \frac{1}{0.5\pi\alpha(x)}$.

Then, for $\alpha(x), \beta(x) > 0$ and constant C, we have

$$ {}_a^C D_x^{\alpha(x)} C = 0, \tag{6} $$

$$ {}_a^C D_x^{\alpha(x)} (x-a)^{\beta(x)} = \begin{cases} 0, \quad \beta(x) \in N_+, \quad \alpha(x) > \beta(x), \\ \frac{\Gamma(\beta(x)+1)}{\Gamma(\beta(x)-\alpha(x)+1)} (x-a)^{\beta(x)-\alpha(x)}, \end{cases} \tag{7} $$

$$ {}_b^C D_x^{\alpha(x)} (b-x)^{\beta(x)} = \begin{cases} 0, \quad \beta(x) \in N_+, \quad \alpha(x) > \beta(x), \\ \frac{\Gamma(\beta(x)+1)}{\Gamma(\beta(x)-\alpha(x)+1)} (b-x)^{\beta(x)-\alpha(x)}, \end{cases} \tag{8} $$

Definition 2.6 *The Mittag-Leffler function with single-function is defined by*

$$ E_{\alpha(x)}(x) := \sum_{k=0}^{\infty} \frac{x^{k\alpha(x)}}{\Gamma(k\alpha(x)+1)}, \ \alpha(x) > 0. \tag{9} $$

Lemma 2.7 *Suppose $f(x) \in [x_0, b]$ is an infinitely $\alpha(x)$-differentiable function. Then for $\alpha(x) > \alpha_0 = \alpha(0) > 0$, $f(x)$ has the variable-order power series expansion of the form*

$$ f(x) = \sum_{j=0}^{\infty} \frac{({}_{x_0}^C D_x^{j\alpha(x)} f)(x_0)}{\Gamma(j\alpha(x)+1)} (x-x_0)^{j\alpha(x)}, \tag{10} $$

and

$$ f(x) = \sum_{j=0}^{\infty} \frac{({}_x^C D_b^{j\alpha(x)} f)(b)}{\Gamma(j\alpha(x)+1)} (b-x)^{j\alpha(x)}. \tag{11} $$

Proof. Assume $f(x)$ has the expansion

$$f(x) = \sum_{j=0}^{\infty} C_i (x - x_0)^{j\alpha(x)+1}.$$

Apply the variable-order fractional derivative to the both sides of the above equation and let $x = x_0$, we have $C_0 = f(0)$, repeat this process n-times and until infinite, we have

$$C_1 = \frac{({}^{C}_{x_0}D_x^{\alpha(x)} f)(x_0)}{\Gamma(\alpha(x)+1)}, \ \cdots, \ C_n = \frac{({}^{C}_{x_0}D_x^{j\alpha(x)} f)(x_0)}{\Gamma(j\alpha(x)+1)}, \cdots$$

which yield the equation (11). In particular, when $x_0 = 0$, we have

$$f(x) = \sum_{j=0}^{\infty} \frac{({}^{C}_{0}D_x^{j\alpha(x)} f)(0)}{\Gamma(j\alpha(x)+1)} x^{j\alpha(x)}. \tag{12}$$

Example 2.8 *Suppose $\alpha(x) > \alpha_0 > 0$. Consider the variable-order power series expansion of the function $E_{\alpha(x)}(x)$ defined by (9). From the Definition 2.3 and equation (8), we derive*

$$E_{\alpha(0)}(0) = 1, \quad \left({}^{C}_{0}D_x^{j\alpha(x)} E_{\alpha(x)}(x)\right)(0) = 1, \quad j = 1, 2, \cdots$$

Then, by equation (12), we have

$$E_{\alpha(x)}(x) = \sum_{k=0}^{\infty} \frac{x^{k\alpha(x)}}{\Gamma(k\alpha(x)+1)}.$$

Similarly, we have

$$\frac{1}{1 - x^{\alpha(x)}} = 1 + x^{\alpha(x)} + x^{2\alpha(x)} + \cdots + x^{n\alpha(x)} + \cdots$$

3 Variable-order Legendre Polynomials

In this section, we introduce the definition of VOLPs and some useful formulas. The classic Legendre polynomials, denoted by $L_n(z)$, are orthogonal on the interval $[-1, 1]$, which is

$$\int_{-1}^{1} L_n(z) L_m(z) dz = \frac{2}{2n+1} \delta_{nm}, \tag{13}$$

where δ_{nm} is the Kronecker function.

we take the change variable $z = 2y - 1$ to Legendre polynomials, and denote $L_n(2y-1)$ by $SL_n(y)$, that is

$$\int_{-1}^{1} SL_n(y) SL_m(y) dy = \frac{1}{2n+1} \delta_{nm}. \tag{14}$$

We call $SL_n(y)$ the shifted Legendre polynomials, also can be obtained by the following three-term recurrence relations

$$(n+1)SL_{n+1}(y) = (2n+1)(2y+1)SL_n(y) - nSL_{n-1}(y),$$

$SL_0(y) = 1$ and $SL_1(y) = 2y - 1$. The analytic form of the shifted Legendre polynomials is given by

$$SL_n(y) = \sum_{j=0}^{n} (-1)^{n+j} \frac{(n+j)!y^j}{(n-j)!(j!)^2}.$$

Now, for $\alpha(x) > \alpha_0 > 0$, in order to apply Legendre polynomials on the interval $[0, 1]$, we define the VOLPs by introducing the change variable of $z = \left[2x^\beta + 1\right]_{\beta=\alpha(x)}$, which means that when doing the orthogonalization for VOLPs we treat the change variable as the function $z = 2x^{\alpha(x)} + 1$, otherwise z is treated as power function. We need such assumption due to the Definition 2.3. The VOLPs, denoted by $VL_n^{\alpha(x)}(x)$, are orthogonal polynomials with the weight function $w(x) = x^{\alpha(x)}\alpha'(x)\ln x + \alpha(x)x^{\alpha(x)-1}$, that is

$$\int_0^1 VL_n^{\alpha(x)}(x)VL_m^{\alpha(x)}(x)w(x)dx = \frac{1}{2n+1}\delta_{nm}.$$

In Figure **??**, we plot the first six terms of VOLPs. The following are some useful formulas about VOLPs, which are directly generalized from the classic Legendre polynomials.

1. The analytic form of VOLPs:

$$VL_n^{\alpha(x)}(x) = \sum_{j=0}^{n} b_{j,n} x^{j\alpha(x)},\ b_{j,n} = \sum_{j=0}^{n} (-1)^{n+j} \frac{(n+j)!}{(n-j)!(j!)^2}. \tag{15}$$

2. The three-term recurrence relations for VOLPs:

$$(n+1)VL_{n+1}^{\alpha(x)}(x) = (2n+1)(2x^{\alpha(x)}+1)VL_n^{\alpha(x)}(x) - nVL_{n-1}^{\alpha(x)}(x),$$

with $VL_0^{\alpha(x)}(x) = 1$ and $VL_1^{\alpha(x)}(x) = 2x^{\alpha(x)} - 1$.

3. Derivative recurrence relations for VOLPs:

$$(4n+2)VL_n^{\alpha(x)}(x) = x^{\alpha(x)-1}\left(VL_{n+1}^{\alpha(x)}(x) - VL_{n-1}^{\alpha(x)}(x)\right)'$$

4. The boundary value of VOLPs:

$$VL_n^{\alpha(0)}(0) = (-1)^n, VL_n^{\alpha(1)}(1) = 1, \quad n = 0, 1, 2, \cdots$$

5. Legendre's differential equation of variable-order:

$$\left[\left((x-x^{\beta+1})\left(VL_n^{\alpha(x)}(x)\right)'\right)' + \beta^2 n(n+1)x^{\beta-1}VL_n^{\alpha(x)}(x)\right]_{\beta=\alpha(x)} = 0.$$

Lemma 3.1 *Let* $\alpha(x) > \alpha_0 > 0$, *and*

$$d_{i,j}^{\alpha(x)} = \int_0^1 {}_0^C D_x^{\alpha(x)} VL_i^{\alpha(x)}(x) VL_j^{\alpha(x)}(x) w(x)dx,\ \ i,j = 0,1,2,\cdots$$

Then we have

$$d_{0,j} = 0;\ d_{i,j}^{\alpha(x)} = \sum_{n=1}^{i}\sum_{m=0}^{j} b_{n,i}b_{m,j}\frac{\Gamma(n\alpha(x)+1)}{\Gamma((n-1)\alpha(x)+1)(n+m)},\ \ i \geq 1.$$

Proof. By (6), we have $d_{0,j} = 0$. Then for $i \geq 1$, formulas (15), (6) and (8) lead to

$$\begin{aligned} d_{i,j}^{\alpha(x)} &= \int_0^1 \sum_{n=0}^{i} b_{n,i}\ {}_0^C D_x^{\alpha(x)} x^{n\alpha(x)} \sum_{m=0}^{j} b_{m,j} x^{m\alpha(x)} w(x)dx \\ &= \int_0^1 \sum_{n=1}^{i} b_{n,i} \left[\frac{\Gamma(n\beta+1)}{\Gamma((n-1)\beta+1}) \right]_{\beta=\alpha(x)} x^{(n-1)\alpha(x)} \sum_{m=0}^{j} b_{m,j} x^{m\alpha(x)} w(x)dx \\ &= \sum_{n=1}^{i}\sum_{m=0}^{j} b_{n,i}b_{m,j}\frac{\Gamma(n\alpha(x)+1)}{\Gamma((n-1)\alpha(x)+1)(n+m)}. \end{aligned} \tag{16}$$

From Lemma 3.1, we easily have

$${}_0^C D_x^{\alpha(x)} VL_i^{\alpha(x)}(x) = \sum_{j=0}^{i-1}(2j+1)d_{i,j}^{\alpha(x)} VL_j^{\alpha(x)}(x)$$

with $d_{i,j}^{\alpha(x)}$ is given by (16).

4 Application

In this section, we apply the method of VOLPs tau to solve the variable-order fractional differential equation

$$\begin{cases} {}_0^C D_x^{\alpha(x)} u(x) + u(x) = f(x), \\ u(0) = 0. \end{cases} \tag{17}$$

Suppose $f(x) = \left[\frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)}\right]_{\beta=\alpha(x)} \cdot x^{\alpha(x)} + x^{2\alpha(x)}$. Then the exact solution for (17) is $u(x) = x^{2\alpha(x)}$. Now we use the method of VOLPs tau to obtain it. Let

$$u(x) \simeq \sum_{i=0}^{n-1} c_i VL_i^{\alpha(x)}(x) = C^T\phi(x), \tag{18}$$

with $C^T = [c_0, c_1, \cdots, c_{n-1}]$ and $\phi(x)^T = [VL_0^{\alpha(x)}(x), VL_1^{\alpha(x)}(x), \cdots, VL_{n-1}^{\alpha(x)}(x)]$. From Lamma 3.1, we have

$$ {}_0^C D_x^{\alpha(x)} u(x) \simeq \sum_{i=0}^{n-1} c_i \, {}_0^C D_x^{\alpha(x)} VL_i^{\alpha(x)}(x) = C^T D_n^{\alpha(x)} \phi(x), \tag{19}$$

with the matrix $D_n^{\alpha(x)} = \{d_{i,j}\}_{n\times n}$, where $d_{i,j}$ is given by (16). Assume

$$ f(x) \simeq \sum_{i=0}^{n-1} f_i VL_i^{\alpha(x)}(x) = F^T \phi(x), \tag{20}$$

with $F^T = [f_0, f_1, \cdots, f_{n-1}]$, where $f_i = (2i+1)\int_0^1 f(x) VL_i^{\alpha(x)}(x) w(x) dx$, $i = 0, 1, \cdots, n-1$. Set $n = 3$. Then

$$ VL_0^{\alpha(x)}(x) = 1,\ \ VL_1^{\alpha(x)}(x) = 2x^{\alpha(x)} - 1,\ \ VL_2^{\alpha(x)}(x) = 6x^{2\alpha(x)} - 6x^{\alpha(x)} + 1, \tag{21}$$

$$ f_0 = \frac{\Gamma(2\alpha(x)+1)}{2\Gamma(\alpha(x)+1)} + \frac{1}{3},\ \ f_1 = \frac{\Gamma(2\alpha(x)+1)}{2\Gamma(\alpha(x)+1)} + \frac{1}{2},\ \ f_2 = \frac{1}{6}.$$

By (18), (19) and (20), we have

$$ C^T D_3^{\alpha(x)} + C^T = F^T $$

with three order matrix

$$ D_3^{\alpha(x)} = \begin{pmatrix} 0 & 0 & 0 & ; & d_{1,0} & d_{1,1} & d_{1,2} & ; & d_{2,0} & d_{2,1} & d_{2,2} \end{pmatrix}, $$

which in accordance with $u(0) = \sum_{i=0}^{2} c_i VL_i^{\alpha(0)}(0) = c_0 - c_1 + c_2 = 0$ yields $C^T = [\frac{1}{3}, \frac{1}{2}, \frac{1}{6}]$. Finally, (21) and (18) lead to the exact solution to (17).

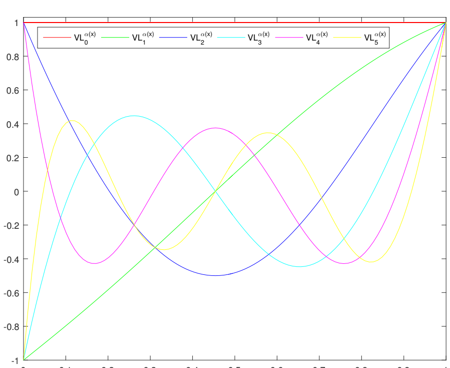

Figure 1: The first six terms of VOLPs for $\alpha(x) = 0.5 + 0.5\cos(0.5\pi x)$.

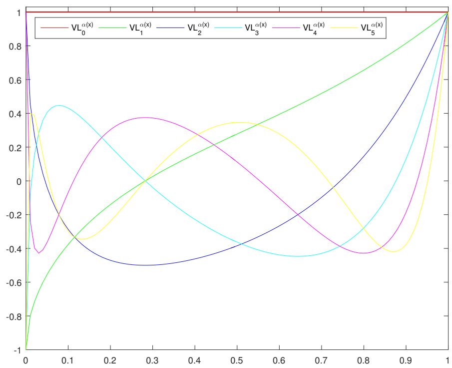

Figure 2: The first six terms of VOLPs for $\alpha(x) = 1 - 0.5\cos(0.5\pi x)$.

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