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Minimal Closed Subsets and Strongly Connected Components in Pretopology

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Abstract

In this paper, we present the links between the minimal closed subsets, the elementary closed subsets and the strongly connected components in Pretopology.

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1. Introduction

In pretopology (see [3], [5]), some authors have developed a way of structuring the pretopological space using a particular method based on research known as minimal closed subsets (see [1], [3], [14], [15], [16], [18], [19]). We will show here that this method is very strongly linked to the search of the strongly connected components of the space.

2. Different Types of Pretopological Spaces (see [3], [5], [7])

Definition 1. Let X be a non empty set. P(X) denotes the family of subsets of X. We call pseudoclosure on X any mapping a from P(X) onto P(X) such as :

$$a(\emptyset) = \emptyset$$
$$\forall \text{ A} \subset \text{X, A} \subset a(\text{A})$$

(X, a) is then called pretopological space.

We can define 4 different types of pretopological spaces.

1- (X, a) is a V type pretopological space if and only if

$$\forall \text{ A} \subset \text{X}, \forall \text{ B} \subset \text{X, A} \subset \text{B} \Rightarrow a(\text{A}) \subset a(\text{B}).$$

2- (X, a) is a V_D type pretopological space if and only if

$$\forall \text{ A} \subset \text{X}, \forall \text{ B} \subset \text{X}, a(\text{A} \cup \text{B}) = a(\text{A}) \cup a(\text{B}).$$

3- (X, a) is a V_S type pretopological space if and only if

$$\forall \text{ A} \subset \text{X}, a(A) = \bigcup_{x \in A} a(\{x\}).$$

4- (X, a) a V_D type pretopological space, is a topological space if and only if

$$\forall \text{ A} \subset \text{X}, a(a(\text{A})) = a(\text{A}).$$

Property 2. *If (X, a) is a V_S space then (X, a) is a V_D space. If (X, a) is a V_D space then (X, a) is a V space.*

Example 3. Let X be a non empty set and R be a binary relationship defined on X.

The pretopology of descendants, noted a_d, is defined by :
$\forall$ A $\subset$ X, a_d(A) = { x $\in$ X/ R(x) $\cap$ A $\neq \emptyset$ } $\cup$ A with R(x)= { y $\in$ X/ x R y }.

The pretopology of ascendants, noted a_a, is defined by :
$\forall$ A $\subset$ X, a_a(A) = { x $\in$ X/ R^{-1}(x) $\cap$ A $\neq \emptyset$ } $\cup$ A with R^{-1}(x) = { y $\in$ X/ y R x}.

These pretopologies are V_S ones.

The pretopology of ascendant-descendants, noted a_{ad}, is defined by :
$\forall$ A $\subset$ X, a_{ad}(A) = { x $\in$ X/ R^{-1}(x) $\cap$ A $\neq \emptyset$ and R(x) $\cap$ A $\neq \emptyset$ } $\cup$ A.

This pretopology is only V one.

3. Different Pretopological Spaces Defined from a Space (X, *a*) and Closures (see [3], [5], [9])

Definition 4. Let (X, a) be a V pretopological space. Let A $\subset$ X. A is a closed subset if and only if a(A) = A.

We note $\forall$ A $\subset$ X, a^0(A) = A and $\forall$ n, n$\geq$ 1, a^n(A) = $a(a^{n-1})$(A).

We name closure of A the subset of X, denoted F_a(A), which is the smallest closed subset which contains A.

Remark 5. F_a(A) is the intersection of all closed subsets which contain A. In the case where (X, a) is a "general" pretopological space (i.e. is not a V space, nor a V_D space, nor a V_S space, nor a topological space), the closure may not exist.

Proposition 6. *Let (X, a) be a V pretopological space. Let A $\subset$ X. If one of the two following conditions is fulfilled :*
- X is a finite set
- a is of V_S type
then $F_a(A) = \bigcup_{n\geq 0} a^n(A)$.

Remark 7. If a is of V type then a^n and F_a also are of V type. If a is of V_S type then a^n and F_a are also of V_S type.

Definition 8. Let (X, a) be a V pretopological space. Let A $\subset$ X. We define the induced pretopology on A by a, denoted a_A, by :
$\forall$ C $\subset$ A, a_A(C) = a(C) $\cap$ A.
(A, a_A) (or more simply A) is said pretopological subspace of (X, a).
We note $(F_a)_A$ the closing obtained by restriction of closing F_a on A. $(F_a)_A$ is such as $\forall$ C $\subset$ A, $(F_a)_A$(C) = F_a(C) $\cap$A.

Definition 9 (see [3], [15]). Let (X, a) be a V pretopological space.
$\forall$ x $\in$ X, we note $F_x = F_a(\{ x \})$. F_x is called an elementary closed subset.
We note $\mathcal{F}$ the set of all closed subsets of X. We note $\mathcal{F}^* = \mathcal{F}$ - { ø } and we note $\mathcal{F}_e$ the set of all elementary closed subsets of X.

Remark 10. $\mathcal{F}_e \subset \mathcal{F}^* \subset \mathcal{F}$.

Definition 11 (see [3]). Let (X, a) be a V pretopological space. We name minimal closed subset of X for a each element of $\mathcal{F}^*$ which is minimal in terms of inclusion in $\mathcal{F}^*$.

We note $\mathcal{F}^*{}_m$ the set of all minimal closed subsets of X for a.

Remark 12 (see [3]). $\mathcal{F}^*{}_m$ can be empty. But $\mathcal{F}^*{}_m$ is not empty when X is a finite set.

Property 13. $\mathcal{F}^*{}_m \subset \mathcal{F}_e \subset \mathcal{F}^* \subset \mathcal{F}$.

Proof.

Let's show that each minimal closed subset of X for a is an elementary closed subset.

Let F $\in \mathcal{F}^*_m$. $\forall$ x $\in$ F, $F_x \subset$ F ((X, a) is a V pretopological space and F is a closed subset)

And then F_x = F (if $F_x \neq$ F then F is not a minimal closed subset of X for a because F_x is a closed subset).

Proposition 14. *Let (X, a) be a V pretopological space.*

F $\in \mathcal{F}^*_m$ if and only if F $\in \mathcal{F}_e$ and F is minimal in terms of inclusions in $\mathcal{F}_e$.

Proof.

Let F $\in \mathcal{F}^*_m$.

We know that F $\in \mathcal{F}_e$ (Property 13).

Let's show that F is minimal in terms of inclusions in $\mathcal{F}_e$.

F $\in \mathcal{F}^*_m$ so F is minimal in terms of inclusion in $\mathcal{F}^*$ (by definition)

Then $\forall$ G $\in \mathcal{F}^*$, G $\neq$ F, we have G $\not\subset$ F

And then $\forall$ G $\in \mathcal{F}_e$, G $\neq$ F, we have G $\not\subset$ F ($\mathcal{F}_e \subset \mathcal{F}^*$)

So F is minimal in terms of inclusions in $\mathcal{F}_e$.

Conversely, let F $\in \mathcal{F}_e$ and F minimal in terms of inclusions in $\mathcal{F}_e$. Let's show that F $\in \mathcal{F}^*_m$.

We know that F $\in \mathcal{F}^*$ (Property 13). It is enough to show that F is minimal in terms of inclusion in $\mathcal{F}^*$.

We hnow that there exists x $\in$ X, F = F_x (F $\in \mathcal{F}_e$).

If F is minimal in terms of inclusions in $\mathcal{F}_e$ and F is not minimal in terms of inclusion in $\mathcal{F}^*$ then $\forall$ y $\in$ X, y $\neq$ x, $F_y \not\subset F_x$ and there exists G $\in \mathcal{F}^*$, G $\neq F_x$, we have G $\subset F_x$ with x $\notin$ G (if x $\in$ G then F_x = F $\subset$ G and G $\subset F_x$ and then F_x = F = G)

So $\forall$ y $\in$ X, y $\neq$ x, $F_y \not\subset F_x$ and $\forall$ y' $\in$ G, y' $\neq$ x, $F_{y'} \subset$ G $\subset F_x$ and G $\neq$ F_x

But $F_{y'} \in \mathcal{F}_e$ so we have $\forall$ y' $\in$ G, y' $\neq$ x, $F_{y'} \not\subset F_x$ and $F_{y'} \subset$ G $\subset F_x$ with G $\neq F_x$.

And the contradiction. So if F is minimal in terms of inclusions in $\mathcal{F}_e$ then F is minimal in terms of inclusion in $\mathcal{F}^*$.

Remark 15. If X is a finite set, $\mathcal{F}_e$ is a hypergraph on X.

Proof.

If X is a finite set, $\mathcal{F}_e$ is a hypergraph on X (see [4]) if and only if :

1. $\forall$ x $\in$ X, $F_x \neq \emptyset$

2. $\bigcup_{x \in X} F_x = X$.

which is verified by definition.

Consequence. If X is a finite set, X can be structured by using the hypergraph $\mathcal{F}_e$ on X. Several examples use this method (see [1], [3], [14], [15], [16], [18], [19]).

Example 16 (see [17]**).** Let (X, a) a pretopological space with X a set of 32 children. Let a pretopology of descendants defined by the following graph 1. R(x) contains the children named by x (children close to him).

Child x (boys)	R(x)	Child x (girls)	R(x)
1	{2,6}	15	{21,25}
2	{1,4,8}	16	{19,21}
3	{5,6}	17	{19,21}
4	{2,5}	18	{20,26}
5	{6,22}	19	{20,21}
6	{3,4}	20	{18,21}
7	{3,4}	21	{19,20}
8	{2,11}	22	{20,30}
9	{3,4}	23	{20,27}
10	{4,12}	24	{9,30}
11	{6,8}	25	{15,31}
12	{8,10}	26	{18,21}
13	{3,8}	27	{21,23}
14	{8,11}	28	{26,29}
		29	{26,28}
		30	{24,32}
		31	{24,30}
		32	{30}

Graph 1

The set of all minimal closed subsets of X contains :

F_7 = {7}, F_{10} = F_{12} = {10,12}, F_{13} = {13}, F_{14} = {14}, F_{15} = F_{25} = {15,25}, F_{16} = {16}, F_{17} = {17}, F_{23} = F_{27} = {23,27}, F_{28} = F_{29} = {28,29}.

The set of all elementary closed subsets of X contains all minimal closed subsets of X and also :

F_{31} = {31,15,25},

$F_1 = F_2 = F_3 = F_4 = F_5 = F_6 = F_8 = F_9 = F_{11} = F_{22} = F_{24} = F_{30} = F_{32}$ $= \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,22,24,25,30,31,32\}$,

$F_{18} = F_{19} = F_{20} = F_{21} = F_{26} = X$.

The graphical representation of the corresponding hypergraph would allow these results to be interpreted. The minimum closed subsets are either single children (who do not receive friendships) or groups of two children naming each other without receiving other friendships. But these minimal closed subsets are contained in other elementary closed subsets, so we can see how the friendships are structured step by step. Children 18, 19, 20, 21 and 26 are indirectly at least chosen by all the other children (including themselves). The first mixed elementary closed subset contains 21 children.

4. Strong Connectivity in (X, *a*) (see [3], [5], [8], [10], [12])

Definition 17. Let (X, a) be a V pretopological space.

(X, a) is strongly connected if and only if $\forall$ C $\subset$ X, C $\neq \emptyset$, $F_a(C) = X$.

Definition 18. Let (X, a) be a V pretopological space. Let A $\subset$ X with A non empty.

A is a strongly connected subset of (X, a) if and only if A endowed with $(F_a)_A$ is strongly connected.

A is a strongly connected component of (X, a) if and only if A is a strongly connected subset of (X, a) and $\forall$ B, A $\subset$ B $\subset$ X with A$\neq$ B, B is not a strongly connected subset of (X, a).

A is a strongly connected subspace of (X, a) if and only if (A, a_A), as a pretopological space, is strongly connected.

A is a greatest strongly connected subspace of (X, a) if and only if (A, a_A) is a strongly connected subspace of (X, a) and $\forall$ B, A $\subset$ B $\subset$ X and A $\neq$ B, (B, a_B) is not a strongly connected subspace of (X, a).

Remark 19. Let (X, a) be a V pretopological space. Let A $\subset$ X with A non empty and A strongly connected subset of (X, a).

$\forall$ x $\in$ A, $F_x = F_a(A)$.

Proof.

$\forall$ x $\in$ A, $F_x \subset F_a(A)$.

Then $F_x \cap A = A$ (A is a strongly connected subset of (X, a)) so A $\subset F_x$ and $F_a(A) \subset F_x$.

Remark 20. Let (X, a) be a V pretopological space. Let A $\subset$ X with A non empty and A strongly connected component of (X, a). Let B $\subset$ X with B non empty and B strongly connected component of (X, a).

A $\neq$ B $\Leftrightarrow$ F_a(A) $\neq$ F_a(B).

Proof.

A and B are strongly connected components of (X, a) so A $\neq$ B $\Leftrightarrow$ A $\cap$ B = ø (because the set of the strongly connected components of (X, a) is a partition of X, see [3]).

So if A $\neq$ B and F_a(A) = F_a(B)

then A $\cap$ B = ø with A and B strongly connected subsets of (X, a) and A and B have the same closure

so A $\cap$ B = ø and A and B are in the same strongly connected component of (X, a) (see [8])

Which contradicts the fact that A is a strongly connected component of (X, a) and B is another one.

Conversely, if F_a(A) $\neq$ F_a(B) then A $\neq$ B.

Indeed if F_a(A) $\neq$ F_a(B) and A = B

Then F_a(A) $\neq$ F_a(B) and F_a(A) = F_a(B) which is a contradiction.

Proposition 21 (see [3]). *Let (X, a) be a V pretopological space. Let A $\subset$ X with A non empty.*

i- If A is a strongly connected subspace of (X, a) then A is a strongly connected subset of (X, a).

ii- If A is a closed subset of X for a then
A is a strongly connected subspace of (X, a) if and only if A is a strongly connected subset of (X, a).

Proposition 22. *Let (X, a) be a V pretopological space.* Let { A_i, $i \in I$ } *be a set of strongly connected subsets of (X, a) which is a partition of X.*
$\forall$ *i* $\in$ *I, there exists J* $\subset$ *I with* $F_a(A_i) = \bigcup_{j\in J} A_j$.

Proof.

Let's i $\in$ I, if x $\in$ $F_a(A_i)$ then there exists j $\in$ I, x $\in$ A_j ({ A_i, i $\in$ I } is a partition of X).

Let's show that $A_j \subset F_a(A_i)$.

We know that A_j is a strongly connected subsets of (X, a) so $F_x = F_a(A_j)$ (Remark 19).

But x $\in$ $F_a(A_i)$ implies $F_x \subset F_a(A_i)$ so $F_a(A_j) \subset F_a(A_i)$

And $A_j \subset F_a(A_i)$.

We denoted J = { j $\in$ I, $A_j \cap F_a(A_i) \neq$ ø }.
We have $\bigcup_{j\in J} A_j \subset F_a(A_i)$.

We have also $F_a(A_i) \subset \bigcup_{j\in J} A_j$.
Indeed, if $x \in F_a(A_i)$ then there exists $j \in I$, $x \in A_j$
So $x \in \bigcup_{j\in J} A_j$
And then $\forall x \in F_a(A_i)$, $x \in \bigcup_{j\in J} A_j$
And $F_a(A_i) \subset \bigcup_{j\in J} A_j$.

5. Minimal closed subsets and strongly connected components in (X, *a*)

In this part, we show the links between minimal closed subsets and strongly connected components. The previous authors didn't find and didn't explain them (see [1], [3], [14], [15], [16], [18], [19]).

Proposition 23. *Let (X, a) be a V pretopological space. Let $A \subset X$ with A non empty.*

A is a minimal closed subset of X for a $\Leftrightarrow$ A is a strongly connected component of (X, a) and A is a closed subset of X for a.

Proof.

Let's show that if A is a minimal closed subset of X for a then A is a strongly connected component of (X, a) and A is a closed subset of X for a.

A is a minimal closed subset of X for a then A is a closed subset of X for a.

Let's show that A is a strongly connected component of (X, a).

We have $\forall x \in A$, $F_x = A$ (A is a minimal closed subset of X for a)
So $\forall C \subset A$, $C \neq \emptyset$, $F_a(C) \cap A = A$
Then $\forall C \subset A$, $C \neq \emptyset$, $(F_a)_A(C) = A$
so A is strongly connected subset of (X, a) (by definition).

If A is not a strongly connected component of (X, a) then there exists B with $A \subset B$, $A \neq B$, B strongly connected component of (X, a)

But $\forall x \in A$, $F_x = A$ so $\forall x \in A$, $F_x \cap B = A$ with $A \subset B$, $A \neq B$
Which implies that B is not strongly connected subset of (X, a).
So A is a strongly connected component of (X, a).

Conversely, let's show that if A is a strongly connected component of (X, a) and if A is a closed subset of X for a then A is a minimal closed subset of X for a.

A is a closed subset of X for a and A is not empty so it is sufficient to show that A is minimal in terms of inclusion in $\mathcal{F}^*$.

If A is not minimal in terms of inclusion in $\mathcal{F}^*$ then there exists F a closed subset of X for a with F non empty and $F \neq A$ and $F \subset A$.

So $F_a(F) = F$ and $F_a(A) = A$ with $F \neq A$ and $F \subset A$ (F and A are closed subsets in X for a)

And then $F_a(F) \cap A = F \cap A = F$ with $F \neq A$ and $F \subset A$

Which contradicts the fact that A is a strongly connected subset of (X, a).

So A is a strongly connected component of (X, a) and A is a closed subset of X for a implies that A is a minimal closed subset of X for a.

Proposition 24. *Let (X, a) be a V pretopological space. Let $A \subset X$ with A non empty.*

A is a minimal closed subset of X for a $\Leftrightarrow$ A is a greatest strongly connected subspace of (X, a) and A is a closed subset of X for a.

Proof.

Obvious by Proposition 21-ii and Proposition 23.

Proposition 25. *Let (X, a) be a V pretopological space.*

$\mathcal{F}_e = \{ F_a(A) /$ *A strongly connected component of (X, a)*$\}$.

Proof.

Let's show that $\mathcal{F}_e \subset \{ F_a(A)/$ A strongly connected component of (X, a)$\}$

so that $\forall$ x $\in$ X, $F_x \in \{ F_a(A)/$ A strongly connected component of (X, a)$\}$.

We know that $\forall$ x $\in$ X, there exists $A \subset X$ with A strongly connected component of (X, a) and x $\in$ A (because the set of the strongly connected components of (X, a) is a partition of X, see [3])

So $\forall$ x $\in$ X, there exists $A \subset X$ with A strongly connected component of (X, a) and $F_x = F_a(A)$ (Remark 19).

So $\forall$ x $\in$ X, $F_x \in \{ F_a(A)/$ A strongly connected component of (X, a)$\}$.

Conversely, let's show that $\{ F_a(A)/$ A strongly connected component of (X, a)$\} \subset \mathcal{F}_e$

so that $\forall$ $A \subset X$ with A strongly connected component of (X, a), there exits x $\in$ X, $F_a(A) = F_x$.

We know that $\forall$ $A \subset X$ with A strongly connected component of (X, a), $\forall$ x $\in$ A, $F_a(A) = F_x$(Remark 19).

So $\forall$ $A \subset X$ with A strongly connected component of (X, a), there exits x $\in$ X, $F_a(A) = F_x$ (A is not empty).

Proposition 26. *Let (X, a) be a V pretopological space.*

Let $\{ A_i,\ i \in I \}$ be the set of strongly connected components of (X, a).

i- $\forall$ $x \in X$, there exits $J \subset I$, $F_x = \bigcup_{j \in J} A_j$.

ii- $\forall$ *x* $\in$ *X,* $\forall$ *y* $\in$ *X, if* $F_x \neq F_y$ *and* $F_x \subset F_y$ *then there exits* $J \subset I$, $J \neq \emptyset$, F_y - $F_x = \bigcup_{j\in J} A_j$.

Proof.

i- We know that $\forall$ x $\in$ X, there exists i $\in$ I with $F_x = F_a(A_i)$ (Proposition 25).

Also, $\forall$ i $\in$ I, there exists J $\subset$ I with $F_a(A_i) = \bigcup_{j\in J} A_j$ (Proposition 22).

Then $\forall$ x $\in$ X, there exists J $\subset$ I with $F_x = \bigcup_{j\in J} A_j$.

ii-$\forall$ x $\in$ X, $\forall$ y $\in$ X, if $F_x \neq F_y$ and $F_x \subset F_y$

then there exists i $\in$ I with $F_x = F_a(A_i)$ and there exists k $\in$ I with $F_y = F_a(A_k)$ with i $\neq$ k and $F_a(A_i) \neq F_a(A_k)$ and $F_a(A_i) \subset F_a(A_k)$ (Proposition 25, Remark 20).

Let's show that $A_k \subset F_a(A_k)$ - $F_a(A_i)$.

We know that $A_k \subset F_a(A_k)$. If $A_k \subset F_a(A_i)$ then $F_a(A_k) \subset F_a(A_i) \subset F_a(A_k)$ so $F_a(A_i) = F_a(A_k)$ which contradicts $F_a(A_i) \neq F_a(A_k)$.

Then there exits J $\subset$ I, J $\neq \emptyset$, F_y - $F_x = \bigcup_{j\in J} A_j$ (Proposition 26-i).

Proposition 27. *Let (X, a) be a V pretopological space with X a finite set. Let* $\{ A_i, i \in I \}$ *be the set of strongly connected components of (X, a).*

$\forall$ *i* $\in$ *I, there exits j* $\in$ *I,* A_j *strongly connected components of (X, a) with* A_j *closed subset of X for a and* $A_j \subset F_a(A_i)$*.*

Proof.

If A_i is a closed subset of X for *a* then the result is obvious.

If A_i is not a closed subset of X for *a* then there exists J $\subset$ I with $F_a(A_i)$ $= \bigcup_{j\in J} A_j$ (Proposition 22).

If $\forall$ j $\in$ J, A_j is not a closed subset of X for *a* then $\forall$ j $\in$ J, i $\neq$ j, $A_j \subset F_a(A_j) \subset F_a(A_i)$ with $A_j \neq F_a(A_j) \neq F_a(A_i)$ (Remark 20)

So $\forall$ j $\in$ J, i $\neq$ j, there exists k $\in$ J, k $\neq$ i, k $\neq$ j, $A_k \subset F_a(A_k) \subset F_a(A_j)$ $\subset F_a(A_i)$ with $A_k \neq F_a(A_k) \neq F_a(A_j) \neq F_a(A_i)$ and so on.

But X is a finite set so J is also a finite set and the contradiction.

So there exists j $\in$ J, A_j is a closed subset of X for *a* with $A_j \subset F_a(A_i)$.

Corollary 28. *Let (X, a) be a V pretopological space with X a finite set.* $\forall$ *x* $\in$ *X, there exists y* $\in$ *X,* F_y *minimal closed subset of X for a with* $F_y \subset F_x$*.*

Proof.

Obvious by Propositions 14, 23, 25 and 27.

Definition 29. Let (X, *a*) be a *V* pretopological space with X a finite set. Let $\{ A_i, i \in I \}$ be the set of strongly connected components of (X, *a*). Let

$\mathcal{R}$ be a binary relationship defined on $\{A_i, i \in I\}$ by :
$\forall$ i $\in$ I, $\forall$ j $\in$ I, i $\neq$ j, $A_i \mathcal{R} A_j$ if and only if $A_i \subset F_a(A_j)$ and there not exist k $\in$ I, k $\neq$ i, k $\neq$ j, $A_i \subset F_a(A_k) \subset F_a(A_j)$.

Proposition 30. *Let (X, a) be a V pretopological space with X a finite set. Let $\{ A_i, i \in I \}$ be the set of strongly connected components of (X, a). Let $\mathcal{R}$' be a binary relationship defined on $\{ F_a(A_i), i \in I \}$ by :*
$\forall i \in I, \forall j \in I, i \neq j, F_a(A_i)\ \mathcal{R}'\ F_a(A_j)$ if and only if $F_a(A_i) \subset F_a(A_j)$ and there not exist $k \in I, k \neq i, k \neq j, F_a(A_i) \subset F_a(A_k) \subset F_a(A_j)$.

$\forall i \in I, \forall j \in I, i \neq j, A_i \mathcal{R} A_j \Leftrightarrow F_a(A_i)\ \mathcal{R}'\ F_a(A_j)$.

Proof.

Let's show that $\forall$ i $\in$ I, $\forall$ j $\in$ I, i $\neq$ j, if $A_i \mathcal{R} A_j$ then $F_a(A_i)\ \mathcal{R}'\ F_a(A_j)$.

$\forall$ i $\in$ I, $\forall$ j $\in$ I, i $\neq$ j, if $A_i \mathcal{R} A_j$ then $A_i \subset F_a(A_j)$ and there not exist k $\in$ I, k $\neq$ i, k $\neq$ j, $A_i \subset F_a(A_k) \subset F_a(A_j)$

So $F_a(A_i) \subset F_a(A_j)$ and there not exist k $\in$ I, k $\neq$ i, k $\neq$ j, $F_a(A_i) \subset F_a(A_k) \subset F_a(A_j)$

then $F_a(A_i)\ \mathcal{R}'\ F_a(A_j)$.

Conversely, let's show that $\forall$ i $\in$ I, $\forall$ j $\in$ I, i $\neq$ j, if $F_a(A_i)\ \mathcal{R}'\ F_a(A_j)$ then $A_i \mathcal{R} A_j$.

$\forall$ i $\in$ I, $\forall$ j $\in$ I, i $\neq$ j, if $F_a(A_i)\ \mathcal{R}'\ F_a(A_j)$ then $F_a(A_i) \subset F_a(A_j)$ and there not exist k $\in$ I, k $\neq$ i, k $\neq$ j, $F_a(A_i) \subset F_a(A_k) \subset F_a(A_j)$

So $A_i \subset F_a(A_j)$ and there not exist k $\in$ I, k $\neq$ i, k $\neq$ j, $A_i \subset F_a(A_k) \subset F_a(A_j)$

Then $A_i \mathcal{R} A_j$.

Proposition 31. *Let (X, a) be a V pretopological space with X a finite set. Let $\{ A_i, i \in I \}$ be the set of strongly connected components of (X, a). Let $\mathcal{R}$ the previous binary relationship defined on $\{A_i, i \in I\}$.*

$\forall i \in I, \forall j \in I, i \neq j, A_i \mathcal{R} A_j \Leftrightarrow \forall x \in A_i, \forall y \in A_j, F_x \neq F_y, F_x \subset F_y$ and there not exist $z \in A_k, F_x \neq F_z, F_y \neq F_z, F_x \subset F_z \subset F_y$.

Proof.

If $\forall$ i $\in$ I, $\forall$ j $\in$ I, i $\neq$ j, $A_i \mathcal{R} A_j$

Then $\forall$ i $\in$ I, $\forall$ j $\in$ I, i $\neq$ j, $F_a(A_i)\ \mathcal{R}'\ F_a(A_j)$ (Proposition 30)

So $\forall$ i $\in$ I, $\forall$ j $\in$ I, i $\neq$ j, $F_a(A_i) \subset F_a(A_j)$ and there not exist k $\in$ I, k $\neq$ i, k $\neq$ j, $F_a(A_i) \subset F_a(A_k) \subset F_a(A_j)$ (by definition of $\mathcal{R}$')

Then $\forall$ i $\in$ I, $\forall$ j $\in$ I, i $\neq$ j, $\forall$ x $\in$ A_i, $\forall$ y $\in$ A_j, $F_x \subset F_y$ and there not exist k $\in$ I, k $\neq$ i, k $\neq$ j, $\forall$ z $\in$ A_k, $F_x \subset F_z \subset F_y$ (Remark 19)

And then $\forall$ x $\in$ A_i, $\forall$ y $\in$ A_j, $F_x \neq F_y$, $F_x \subset F_y$ and there not exist z $\in$ A_k, $F_x \neq F_z$, $F_y \neq F_z$, $F_x \subset F_z \subset F_y$(Remark 20).

Conversely, if $\forall$ i $\in$ I, $\forall$ j $\in$ I, i $\neq$ j, $\forall$ x $\in$ A_i, $\forall$ y $\in$ A_j, $F_x \neq F_y$, $F_x \subset F_y$ and there not exist z $\in$ A_k, $F_x \neq F_z$, $F_y \neq F_z$, $F_x \subset F_z \subset F_y$

so $\forall$ i $\in$ I, $\forall$ j $\in$ I, i $\neq$ j, $F_a(A_i) \subset F_a(A_j)$ and there not exist k $\in$ I, k $\neq$ i, k $\neq$ j, $F_a(A_i) \subset F_a(A_k) \subset F_a(A_j)$ (Remarks 19 and 20)

then $\forall$ i $\in$ I, $\forall$ j $\in$ I, i $\neq$ j, $F_a(A_i)$ $\mathcal{R}$' $F_a(A_j)$

and $\forall$ i $\in$ I, $\forall$ j $\in$ I, i $\neq$ j, A_i $\mathcal{R}$ A_j(Proposition 30).

Proposition 32. *Let (X, a) be a V pretopological space with X a finite set. Let* $\{ A_i,\ i \in I \}$ *be the set of strongly connected components of (X, a). Let* $\mathcal{R}$ *the previous binary relationship defined on* $\{A_i,\ i \in I\}$.

The relation $\mathcal{R}$ *is anti-symmetrical, anti-transitive and without circuit.*

Proof.

Obvious by definition and Remark 20.

Consequence. We can apply to $\mathcal{R}$ the level function in Graph Theory which gives, for any vertex A_i, the length of the longest path arriving at A_i (see [13]).

Example 33. We can build the relation $\mathcal{R}$ applied to the previous example 16. We have the following strongly connected components which are closed subsets (Proposition 23) : $F_7 = \{7\}$, $F_{10} = \{10,12\}$, $F_{13} = \{13\}$, $F_{14} = \{14\}$, $F_{15} = \{15,25\}$, $F_{16} = \{16\}$, $F_{17} = \{17\}$, $F_{23} = \{23,27\}$, $F_{28} = \{28,29\}$.

Then, we know ([5], [8]) that the union of all x of X which have the same closure is a strongly connected components of (X, a). So we can obtain the following strongly connected components which are not closed subsets : $SCC_{31} = \{31\}$, $SCC_1 = \{1,2,3,4,5,6,8,9,11,22,24,30,32\}$, $SCC_{18} = \{18, 19, 20, 21, 26\}$.

The graph of $\mathcal{R}$ is the following graph 2.

A (component)	Level of A	B component / B ∈ $\mathcal{R}$(A)
F_7 = {7}	0	SCC_1
F_{10} = {10,12}	0	SCC_1
F_{13} = {13}	0	SCC_1
F_{14} = {14}	0	SCC_1
F_{15} = {15,25}	0	SCC_{31}
F_{16} = {16}	0	SCC_{18}
F_{17} = {17}	0	SCC_{18}
F_{23} = {23,27}	0	SCC_{18}
F_{28} = {28,29}	0	SCC_{18}
SCC_{31} = {31}	1	SCC_1
SCC_1= {1,2,3,4,5,6,8,9,11,22,24,30,32}	2	SCC_{18}
SCC_{18} = {18, 19, 20, 21, 26}	3	

Graph 2

We read that 9 groups of isolated children (singletons) or in groups of two children receive no link and remain marginal. A mixed group of 13 children (SCC_1) who exchange directly or indirectly with each other receive links from 6 of these groups (child 31 acting as intermediary for the group containing 15 and 25). At the top of the hierarchy, there is a group of 5 girls (SCC_{18}) who centralize the links sent without return to all the other groups, these girls form a group apparently united since they have direct or indirect links between them. The class therefore has a hierarchical structure that can easely be represented using $\mathcal{R}$ and the different levels.

Definition 34. Let (X, a) be a V pretopological space with X a finite set. Let $\mathcal{R}$" the binary relationship defined on $\mathcal{F}_e$ by :

$\forall$ x $\in$ X, $\forall$ y $\in$ X, F_x $\mathcal{R}$" F_y $\Leftrightarrow$ $F_x \neq F_y$, $F_x \subset F_y$ and there not exist z $\in$ X , $F_x \neq F_z$, $F_y \neq F_z$, $F_x \subset F_z \subset F_y$.

Remark 35. $\forall$ x $\in$ X, $\forall$ y $\in$ X, if F_x $\mathcal{R}$" F_y then it exits i $\in$ I, it exits j $\in$ I, i $\neq$ j, x $\in$ A_i and y $\in$ A_j with A_i and A_j strongly connected components of (X, a).

Proof.
Obvious by Definition 34, Remarks 19 and 20.

Proposition 36. *Let (X, a) be a V pretopological space with X a finite set. Let { A_i, $i \in I$ } be the set of strongly connected components of (X, a).*

Let $\mathcal{R}$ the previous binary relationship defined on $\{A_i,\ i \in I\}$. Let $\mathcal{R}$" the previous binary relationship defined on $\mathcal{F}_e$.

$\forall\ i \in I, \forall\ j \in I,\ i \neq j,\ A_i\ \mathcal{R}\ A_j \Leftrightarrow \forall\ x \in A_i, \forall\ y \in A_j$, F_x $\mathcal{R}$" F_y.

Proof.

Obvious by Proposition 31, Definition 34, Remarks 19, 20.

Remark 37. The definition of the relation $\mathcal{R}$ and its properties make it possible to clarify the approach of the previous authors who give an algorithm to find the different inclusions, step by step, between the elementary closed subsets using $\mathcal{R}$" (see [1], [3], [14], [15], [16], [18], [19]) and propose the representation of the relation $\mathcal{R}$ and its levels but without explaining it.

The links with strongly connected components can also allow to use more directly an algorithm to search for strongly connected components rather than a specific algorithm (see [14], [15]) : we know (see [5], [8]) that the union of all x of X which have the same closure is a strongly connected components of (X, a). Also, the search for minimal closed subsets is equivalent to search the strongly connected components which are closed.

This new more familiar approach also allows for a more direct and easier interpretation of the results.

Definition 38. Let (X, a) be a V_S pretopological space with X a finite set. Let { A_i, i ∈ I } be the set of strongly connected components of (X, a).

Let $\mathcal{R}^*$ be a binary relationship defined on { A_i, i ∈ I } by :

∀ i ∈ I, ∀ j ∈ I, i ≠ j, A_i $\mathcal{R}^*$ A_j ⇔ it exits x ∈ A_i, it exits y ∈ A_j, x ∈ $a(\{y\})$.

Proposition 39. *Let (X, a) be a V_S pretopological space with X a finite set. Let $\{A_i,\ i \in I\}$ be the set of strongly connected components of (X, a). Let $\mathcal{R}$ and $\mathcal{R}^*$ be the previous relationships defined on $\{A_i,\ i \in I\}$.*
$\forall\ i \in I, \forall\ j \in I,\ i \neq j,\ A_i\ \mathcal{R}\ A_j \Leftrightarrow A_i\ \mathcal{R}^*\ A_j$ and there not exist a sequence A_0......... A_n of $\{A_i,\ i \in I\}$ with n≥2 such as $A_0 = A_i$, $A_n = A_j$ and ∀k = 0,......., n-1, $A_k\ \mathcal{R}^*\ A_{k+1}$.

Proof.

If A_i $\mathcal{R}$ A_j then $A_i \subset F_a(A_j)$ and there not exist k ∈ I, k ≠ i, k ≠ j, $A_i \subset F_a(A_k) \subset F_a(A_j)$ (by Definition)

So ∀ x ∈ A_i, ∀ y ∈ A_j, x ∈ F_y and there not exist k ∈ I with z ∈ A_k, k ≠ i, k ≠ j, x ∈ $F_z \subset F_y$(Remark 19, Proposition 26-i)

Then ∀ x ∈ A_i, ∀ y ∈ A_j, there exists a path in (X, a) from { x } to { y } and ∀ k ∈ I with z ∈ A_k, k ≠ i, k ≠ j, there not exist a path in (X, a) from { x } to { z } or there not exist a path in (X, a) from { z } to { y } (see [10])

And then ∀ x ∈ A_i, ∀ y ∈ A_j, there exists a sequence x_0.........x_n of elements of X such as x_0 = y, x_n = x with ∀l = 0,......., n-1, $x_{l+1} \in a(\{ x_l\})$ and ∀ k ∈ I with z ∈ A_k, k ≠ i, k ≠ j, there not exist a sequence x_0.........x_n of elements of X such as x_0 = z, x_n = x with ∀l = 0,......., n-1, $x_{l+1} \in a(\{ x_l\})$ or there not exist a sequence x_0.........x_n of elements of X such as x_0 = y, x_n = z with ∀l = 0,......., n-1, $x_{l+1} \in a(\{ x_l\})$ (see [10])

And then A_i $\mathcal{R}^*$ A_j and there not exist a sequence A_0......... A_n of { A_i, i ∈ I } with n≥2 such as A_0 = A_i, A_n = A_j and ∀k = 0,......., n-1, A_k $\mathcal{R}^*$ A_{k+1}(see [10]).

Conversely, if A_i $\mathcal{R}^*$ A_j and there not exist a sequence A_0......... A_n of { A_i, i ∈ I } with n≥2 such as A_0 = A_i, A_n = A_j and ∀k = 0,......., n-1, A_k $\mathcal{R}^*$ A_{k+1}

Then there exits x ∈ A_i, there exits y ∈ A_j, x ∈ $a(\{ y\})$ and ∀ k ∈ I with z ∈ A_k, k ≠ i, k ≠ j, there not exist a sequence x_0.........x_n of elements of X such as x_0 = z, x_n = x with ∀l = 0,......., n-1, $x_{l+1} \in a(\{x_l\})$ or there not exist a sequence x_0.........x_n of elements of X such as x_0 = y, x_n = z with ∀l = 0,......., n-1, $x_{l+1} \in a(\{ x_l\})$

So there exits x ∈ A_i, there exits y ∈ A_j, x ∈ F_y and there not exist k ∈ I with z ∈ A_k, k ≠ i, k ≠ j, there exists a path in (X, a) from { x } to { z } and there exists a path in (X, a) from { z } to { y } (see [10])

And then there exits x ∈ A_i, there exits y ∈ A_j, x ∈ F_y and there not exist k ∈ I with z ∈ A_k, k ≠ i, k ≠ j, x ∈ F_z and z ∈ F_y (see [10])

Then $A_i \subset F_a(A_j)$ and there not exist k ∈ I, k ≠ i, k ≠ j, $A_i \subset F_a(A_k) \subset F_a(A_j)$ (Remark 19, Proposition 26-i)

And the result A_i $\mathcal{R}$ A_j.

Remark 40. The definition of the relation $\mathcal{R}$ in the case of a V_S pretopological space, and more particularly in the case of a graph (which is based on the pretopology of the descendants), simply amounts to representing the strongly connected components and their links as in a reduced graph (in Graph Theory) without any form of transitivity. In this case, we could therefore adapt a reduced graph construction algorithm in Graph Theory. The previous example shows this.

6. Conclusion

Finally, the method based on the representation of minimal closed subsets is a generalization in pretopology of the notion of reduced graph in Graph Theory. But we note that in some cases, the pretopological space admits only one (or a few) strongly connected component(s) (or greatest strongly connected subspace(s)) and that the aim is then to propose a decomposition of the component(s) (or greatest strongly connected subspace(s)). For this, we have proposed new concepts in pretopology and therefore in graph theory ([5], [6], [11]). The sofware Réso implements these concepts ([2]).

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