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Connected Components of a Network in Pretopology

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Abstract

In this paper, we present properties of connected components in the case of a network (which is defined as a family of pretopologies). The network can be analyse by the union or by the intersection or by the composition of the different pretopologies.

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1. Introduction

In Pretopology (see [1][2][5]), a network is defined as a family of pretopologies. Most often, it is studied by the union or by the intersection or by the composition of the different pretopologies constituting it (see [6]).

We have already studied the case of the strong connectivity (see [3][4]). So we highlight algorithms for searching the connected components (which is equivalent for searching the greatest connected subspaces (see [5])) of a network given the connected components of each pretopological space of the network.

2. Different Types of Pretopological Spaces (see [1][2][3])

Definition 1. Let X be a non empty set. P(X) denotes the family of subsets of X. We call pseudoclosure on X any mapping a from P(X) onto P(X) such as :

$$a(\emptyset) = \emptyset$$
$$\forall \text{ A} \subset \text{X, A} \subset a(\text{A})$$

(X, a) is then called pretopological space.

We can define 4 different types of pretopological spaces.

1- (X, a) is a V type pretopological space if and only if

$$\forall \text{ A} \subset \text{X}, \forall \text{ B} \subset \text{X, A} \subset \text{B} \Rightarrow a(\text{A}) \subset a(\text{B}).$$

2- (X, a) is a V_D type pretopological space if and only if

$$\forall \text{ A} \subset \text{X}, \forall \text{ B} \subset \text{X}, a(\text{A} \cup \text{B}) = a(\text{A}) \cup a(\text{B}).$$

3- (X, a) is a V_S type pretopological space if and only if

$$\forall \text{ A} \subset \text{X}, a(A) = \bigcup_{x \in A} a(\{x\}).$$

4- (X, a) a V_D type pretopological space, is a topological space if and only if

$$\forall \text{ A} \subset \text{X}, a(a(\text{A})) = a(\text{A}).$$

Property 2. *If (X, a) is a V_S space then (X, a) is a V_D space. If (X, a) is a V_D space then (X, a) is a V space.*

Example 3. Let X be a non empty set and R be a binary relationship defined on X.

The pretopology of descendants, noted a_d, is defined by :
$\forall$ A $\subset$ X, a_d(A) = { x$\in$ X/ R(x) $\cap$ A $\neq \emptyset$ } $\cup$ A with R(x)= { y $\in$ X/ x R y }.

The pretopology of ascendants, noted a_a, is defined by :
$\forall$ A $\subset$ X, a_a(A) = { x$\in$ X/ R^{-1}(x) $\cap$ A $\neq \emptyset$ } $\cup$ A with R^{-1}(x) = { y $\in$ X/ y R x}.

These pretopologies are V_S ones.

The pretopology of ascendant-descendants, noted a_{ad}, is defined by :
$\forall$ A $\subset$ X, a_{ad}(A) = { x $\in$ X/ R^{-1}(x) $\cap$ A $\neq \emptyset$ and R(x) $\cap$ A $\neq \emptyset$ } $\cup$ A.

This pretopology is only V one.

3. Different Pretopological Spaces Defined from a Space (X, *a*) and Closures (see [1][2][6])

Definition 4. Let (X, a) be a V pretopological space. Let A $\subset$ X. A is a closed subset if and only if a(A) = A.

We note $\forall$ A $\subset$ X, a^0(A) = A and $\forall$n, n$\geq$ 1, a^n(A) = $a(a^{n-1})$(A).

We name closure of A the subset of X, denoted F_a(A), which is the smallest closed subset which contains A.

F'_a , the inverse of the closure generated by a, is defined by : $\forall$ A $\subset$ X, $F'_a(A) = \{ y \in X / F_a(\{y\}) \cap A \neq \emptyset \}$.

We note $a'' = F'_a \odot F_a$ (a'' is the composed of the mapping F'_a and F_a) and F''_a the closure according to a''.

Remark 5. F_a(A) is the intersection of all closed subsets which contain A. In the case where (X, a) is a "general" pretopological space (i.e. is not a V space, nor a V_D space, nor a V_S space, nor a topological space), the closure may not exist.

Proposition 6. *Let (X, a) be a V space. Let A $\subset$ X. If one of the two following conditions is fulfilled :*

- X is a finite set

- a is of V_S type

then $F_a(A) = \bigcup_{n\geq 0} a^n(A)$.

Remark 7. If a is of V type then a^n, F_a , a'' and F''_a also are of V type and F'_a is of V_S type. If a is of V_S type then a^n, F_a , a'', F'_a and F''_a are also of V_S type.

Definition 8. Let (X, a) be a V pretopological space. Let A $\subset$ X. We define the induced pretopology on A by a, denoted a_A, by :

$\forall$ C $\subset$ A, $a_A(C) = a(C) \cap A$.

(A, a_A) (or more simply A) is said pretopological subspace of (X, a).

We note $(F_a)_A$ the closing obtained by restriction of closing F_a on A. $(F_a)_A$ is such as $\forall$ C $\subset$ A, $(F_a)_A(C) = F_a(C) \cap A$.

4. Connectivity in (X, *a*) (see [1][2][5][7][8][9][10][11])

Definition 9. Let (X, a) be a V pretopological space.

(X, a) is connected if and only if $\forall$ C $\subset$ X, C $\neq \emptyset$, F_a(C) = X or $F_a(X - F_a(C)) \cap F_a(C) \neq \emptyset$.

Definition 10. Let (X, a) be a V pretopological space. Let A a non empty subset of X. Let B a non empty subset of X. There exists a chain in (X,

a) from B to A if and only if B $\subset$ F"$_a$(A).

Proposition 11 (see [7]). *Let (X, a) be a V pretopological space.*

If $\forall$ $x \in X$ and $\forall$ $y \in X$, there exists a chain in (X, a) from $\{y\}$ to $\{x\}$ then (X, a) is connected.

Proposition 12 (see [7]). *Let (X, a) be a V_S pretopological space.*

i- (X, a) is connected $\Leftrightarrow$ $\forall$ $x \in X$ and $\forall$ $y \in X$, there exists a chain in (X, a) from $\{y\}$ to $\{x\}$.

ii- (X, a) is connected $\Leftrightarrow$ $\forall$ $x \in X$ and $\forall$ $y \in X$, there exists a sequence $x_0.........x_n$ of elements of X such as $x_0 = x$, $x_n = y$ with $\forall j = 0,......., n\text{-}1$, $x_{j+1} \in a(\{x_j\})$ or $x_j \in a(\{x_{j+1}\})$.

Definition 13. Let (X, a) be a V pretopological space. Let A $\subset$ X with A non empty.

A is a connected subset of (X, a) if and only if A endowed with $(F_a)_A$ is connected.

A is a connected component of (X, a) if and only if A is a connected subset of (X, a) and $\forall$ B, A $\subset$ B $\subset$ X with A $\neq$ B, B is not a connected subset of (X, a).

A is a connected subspace of (X, a) if and only if (A, a_A), as a pretopological space, is connected.

A is a greatest connected subspace of (X, a) if and only if (A, a_A) is a connected subspace of (X, a) and $\forall$ B, A $\subset$ B $\subset$ X and A $\neq$ B, (B, a_B) is not a connected subspace of (X, a).

Proposition 14 (see [5]). *Let (X, a) be a V pretopological space. Let A $\subset$ X with A non empty.*

A is a connected component of (X, a) if and only if A is a greatest connected subspace of (X, a).

Proposition 15. *Let (X, a) be a V pretopological space. Let A $\subset$ X with A non empty.*

i- If $\forall$ $x \in A$ and $\forall$ $y \in A$, there exists a chain in (A, a_A) from $\{y\}$ to $\{x\}$ then A is a connected subset of (X, a).

ii- If $\forall$ $x \in A$ and $\forall$ $y \in A$, there exists a sequence $x_0.........x_n$ of elements of A such as $x_0 = x$, $x_n = y$ with $\forall j = 0,......., n\text{-}1$, $x_{j+1} \in a(\{x_j\})$ or $x_j \in a(\{x_{j+1}\})$ then A is a connected subset of (X, a).

Proof.

i- If $\forall$ x $\in$ A and $\forall$ y $\in$ A, there exists a chain in (A, a_A) from {y} to {x}

Then (A, a_A) is a connected subspace of (X, a) (Proposition 11 and Definition 13)

and then A is a connected subset of (X, a) (see [5]).

ii- If ∀ x ∈ A and ∀ y ∈ A, there exists a sequence x_0.........x_n of elements of A such as x_0 = x, x_n= y with ∀j = 0,......., n-1, $x_{j+1}\in a(\{x_j\})$ or $x_j\in a(\{x_{j+1}\})$

then ∀ x ∈ A and ∀ y ∈ A, there exists a chain in (A, a_A) from {y} to {x} (see [7])

then A is a connected subset of (X, a) (Proposition 15-i).

Remark 16. Generally speaking, if ∀ x ∈ A and ∀ y ∈ A, there exists a chain in (X, a) from {y} to {x} then A is not a connected subset of (X, a).

Example 17. Let (X, a) a pretopological space with X = { a, b, c, d } and A = { a, d }. Let a pretopology of descendants defined by the following graph 1 :

x	R(x)
a	∅
b	{ a, c }
c	∅
d	{c}

Graph 1

∀ x ∈ A and ∀ y ∈ A, there exists a chain in (X, a) from {y} to {x} but A is not a connected subset of (X, a).

Remark 18. Generally speaking, the converse of ii- is not true.

Example 19. Let (X, a) a pretopological space with X = { a, b, c }. Let a pretopology of ascendant-descendants defined by the following graph 2 :

x	R(x)
a	{c}
b	{ a, c }
c	{b}

Graph 2

F_a({a}) = {a} and F_a({b}) = F_a({c}) = X with F_a({ b, c }) = X so F_a(X - F_a({a})) ∩ F_a({a}) = {a} ≠ ∅ then (X, a) is connected.

But there does not exist a sequence x_0.........x_n of elements of X such as x_0 = c, x_n= a with ∀j = 0,......., n-1, $x_{j+1}\in a(\{x_j\})$ or $x_j\in a(\{x_{j+1}\})$.

Proposition 20. *Let (X, a) be a V_S pretopological space. Let $A \subset X$ with A non empty.*

i- If A is a connected subset of (X, a) then $\forall$ $x \in A$ and $\forall$ $y \in A$, there exists a chain in (X, a) from $\{y\}$ to $\{x\}$.

ii- If A is a connected subset of (X, a) then $\forall$ $x \in A$ and $\forall$ $y \in A$, there exists a sequence x_0.........x_n of elements of X such as $x_0 = x$, $x_n = y$ with $\forall j$ = 0,......., n-1, $x_{j+1} \in a(\{x_j\})$ or $x_j \in a(\{x_{j+1}\})$.

Proof.

i- A is a connected subset of (X, *a*)

So $\forall$ C $\subset$ A, C $\neq \emptyset$, (1) $F_a(C) \cap A = A$ or (2) $F_a(A - (F_a(C) \cap A)) \cap F_a(C) \cap A \neq \emptyset$.

$\forall$ x $\in$ A, $F"_a(\{x\}) \cap A \subset A$ and $F"_a(\{x\}) \cap A \neq \emptyset$.

Let's show that $F"_a(\{x\}) \cap A$ does not verify (2) so that

$F_a(A - (F_a(F"_a(\{x\}) \cap A) \cap A)) \cap F_a(F"_a(\{x\}) \cap A) \cap A = \emptyset$.

We have $F_a(F"_a(\{x\}) \cap A) \cap A = F"_a(\{x\}) \cap A$.

Indeed, $F"_a(\{x\}) \cap A \subset F"_a(\{x\})$

So $F_a(F"_a(\{x\}) \cap A) \subset F"_a(\{x\})$ ($F"_a(\{x\})$ is closed for F_a)

And then $F_a(F"_a(\{x\}) \cap A) \cap A \subset F"_a(\{x\}) \cap A$.

We have also $F"_a(\{x\}) \cap A \subset F_a(F"_a(\{x\}) \cap A)$

Then $F"_a(\{x\}) \cap A \cap A \subset F_a(F"_a(\{x\}) \cap A) \cap A$

And then $F"_a(\{x\}) \cap A \subset F_a(F"_a(\{x\}) \cap A) \cap A$.

We must show that $F_a(A - (F"_a(\{x\}) \cap A)) \cap F"_a(\{x\}) \cap A = \emptyset$.

$F"_a(\{x\})$ is closed for F' so $\forall$ y $\in$ X - $F"_a(\{x\})$, y $\notin F'_a(F"_a(\{x\}))$

Then $\forall$ y $\in$ A - $(F"_a(\{x\}) \cap A)$, $F_a(\{y\}) \cap F"_a(\{x\}) = \emptyset$

So $F_a(A - (F"_a(\{x\}) \cap A)) \cap F"_a(\{x\}) \cap A = \emptyset$.

In result, (2) is not verified so (1) is verified and we have $F_a(F"_a(\{x\}) \cap A) \cap A = F"_a(\{x\}) \cap A = A$

So A $\subset F"_a(\{x\})$

And then $\forall$ x $\in$ A, $\forall$ y $\in$ A, y $\in F"_a(\{x\})$ and the result.

ii- A is a connected subset of (X, *a*)

then $\forall$ x $\in$ A and $\forall$ y $\in$ A, there exists a chain in (X, *a*) from {y} to {x} (see i-)

and then $\forall$ x $\in$ A and $\forall$ y $\in$ A, there exists a sequence x_0.........x_n of elements of X such as x_0 = x, x_n= y with $\forall$j = 0,......., n-1, $x_{j+1} \in a(\{x_j\})$ or $x_j \in a(\{x_{j+1}\})$ (see [7]).

Remarks 21.

1- Generally speaking, the converses of i- and ii- are not true (see previous example 17) .

2- If A is a connected subset of (X, a) then, generally speaking, it is possible that there exists x ∈ A and y ∈ A such as there does not exist a chain in (A, a_A) from {y} to {x}.

3- If A is a connected subset of (X, a) then, generally speaking, it is possible that there exists x ∈ A and y ∈ A such as there does not exist a sequence x_0.........x_n of elements of A such as x_0 = x, x_n= y with ∀j = 0,......., n-1, $x_{j+1}\in a(\{x_j\})$ or $x_j\in a(\{x_{j+1}\})$.

Example 22. Let (X, a) a pretopological space with X = { a, b, c } and A = { a, b }. Let a pretopology of descendants defined by the following graph 3 :

x	R(x)
a	{c}
b	ø
c	{b}

Graph 3

A is a connected subset of (X, a) but there does not exist a chain in (A, a_A) from {a} to {b}. Morever, there does not exist a sequence x_0.........x_n of elements of A such as x_0 = a, x_n= b with ∀j = 0,......., n-1, $x_{j+1}\in a(\{x_j\})$ or $x_j\in a(\{x_{j+1}\})$.

5. Definition of a Network and Different Closures (see [6])

Definition 23. Let X a non empty set. Let I a countable family of indices. The family { (X, a_i), i∈I } of pretopological spaces is a network on X.

Definition 24. Let X a non empty set. For any pretopologies a_1 and a_2 defined on X, for any subset A of X, we define the three following mappings :
($a_1 \cup a_2$)(A) = a_1(A) ∪ a_2(A) [union of pretopologies]
($a_1 \cap a_2$)(A) = a_1(A) ∩ a_2(A) [intersection of pretopologies]
($a_1 \odot a_2$)(A) = a_1(a_2(A)) [composition of pretopologies]

More generally, in a network { (X, a_i), i∈I } such as for any i∈I, a_i is of V type, we note F_{ai} the closure according to a_i, $F_\cup$ (respectively $F_\cap$) the closure according to $\bigcup_{i\in I} a_i$ (respectively $\bigcap_{i\in I} a_i$), $F_{\cup F}$ (respectively $F_{\cap F}$) the closure according to $\bigcup_{i\in I} F_{ai}$ (respectively $\bigcap_{i\in I} F_{ai}$), F"$_\cup$ the closure according to ($\bigcup_{i\in I} a_i$)".
We define the mapping, denoted $\prod_{i\in I} a_i$, from P(X) onto P(X) by :

$\forall$ A $\subset$ X, $\prod_{i\in I} a_i(A)$= { x$\in$X/ there exists $n\in$I such as x $\in$ a_n(a_{n-1}(... (a_1(A))...)) }.
And we denote $F_{\prod}$ the closure according to $\prod_{i\in I} a_i$ and $F_{\prod F}$ the closure according to $\prod_{i\in I} F_{ai}$.

Proposition 25 (see [6]). *Let { (X, a_i), i$\in$I } a network such as for any i$\in$I, a_i is of V type.*
i- $F_{\cup} = F_{\cup F} = F_{\prod F} = F_{\prod}$.
ii- $\forall$ $A \subset X$, $F_{\cap}(A) \subset F_{\cap F}(A)$.

6. Connected Components in a Network

Definition 26 (see [1]). Let X a non empty set. Let a_1 and a_2 two pretopologies on X.
a_1 is thinner than a_2 if and only if $\forall$ A $\subset$ X, a_1(A) $\subset$ a_2(A).

Remark 27 (see [1]). Let X a non empty set. Let a_1 and a_2 two *V* type pretopologies on X.
If a_1 is thinner than a_2 then F_{a1} is thinner than F_{a2}.

Proposition 28. *Let X a non empty set. Let a_1 and a_2 two V type pretopologies on X such as a_1 thinner than a_2. Let A $\subset$ X with A non empty.*
If A is a connected subset of (X, a_1) then A is a connected subset of (X, a_2).

Proof.
A is a connected subset of (X, a_1) $\Leftrightarrow$ $\forall$ C $\subset$ A, C $\neq \emptyset$, (1) F_{a1}(C) $\cap$ A = A or (2) F_{a1}(A - (F_{a1}(C) $\cap$ A)) $\cap$ F_{a1}(C) $\cap$ A $\neq \emptyset$ (by definition).
Let's show that $\forall$ C $\subset$ A, C $\neq \emptyset$, (1') F_{a2}(C) $\cap$ A = A or
(2') F_{a2}(A - (F_{a2}(C) $\cap$ A)) $\cap$ F_{a2}(C) $\cap$ A $\neq \emptyset$.
We note that F_{a1}(F_{a2}(C) $\cap$ A) $\cap$ A = F_{a2}(C) $\cap$ A.
Indeed, F_{a2}(C) $\cap$ A $\subset$ F_{a1}(F_{a2}(C) $\cap$ A) and F_{a2}(C) $\cap$ A $\subset$ A
Then F_{a2}(C) $\cap$ A $\subset$ F_{a1}(F_{a2}(C) $\cap$ A) $\cap$ A.
Conversely, F_{a1}(F_{a2}(C) $\cap$ A) $\subset$ F_{a2}(F_{a2}(C) $\cap$ A) (Remark 27)
Then F_{a1}(F_{a2}(C) $\cap$ A) $\subset$ F_{a2}(C)
And then F_{a1}(F_{a2}(C) $\cap$ A) $\cap$ A $\subset$ F_{a2}(C) $\cap$ A.
We have F_{a2}(C) $\cap$ A $\subset$ A with A connected subset of (X, a_1), so :
- if (1) F_{a1}(F_{a2}(C) $\cap$ A) $\cap$ A = A
Then F_{a2}(C) $\cap$ A = A (see previous remark)
And then (1') is verified.
- if (2) F_{a1}(A - (F_{a1}(F_{a2}(C) $\cap$ A) $\cap$ A)) $\cap$ F_{a1}(F_{a2}(C) $\cap$ A) $\cap$ A $\neq \emptyset$

Then $F_{a1}(A - (F_{a2}(C) \cap A)) \cap F_{a2}(C) \cap A \neq \emptyset$ (see previous remark)
And then $F_{a2}(A - (F_{a2}(C) \cap A)) \cap F_{a2}(C) \cap A \neq \emptyset$ (Remark 27)
So (2') is verified.

Proposition 29. *Let* $\{ (X, a_i), i \in I \}$ *a network such as for any* $i \in I$, a_i *is of V type. Let* $A \subset X$ *with A non empty.*

i- The following assertions are equivalent :
(1) A connected subset of $(X, \bigcup_{i\in I} a_i)$
(2) A connected subset of $(X, \bigcup_{i\in I} F_{ai})$
(3) A connected subset of $(X, \prod_{i\in I} a_i)$
(4) A connected subset of $(X, \prod_{i\in I} F_{ai})$.
ii- If A is connected subset of $(X, \bigcap_{i\in I} a_i)$ *then A is connected subset of* $(X, \bigcap_{i\in I} F_{ai})$.

Proof.
i- Let's show that (1) is équivalent to (2) :
A is a connected subset of $(X, \bigcup_{i\in I} a_i)$
$\Leftrightarrow \forall C \subset A, C \neq \emptyset$, (1) $F_\cup(C) \cap A = A$ or
(2) $F_\cup(A - (F_\cup(C) \cap A)) \cap F_\cup(C) \cap A \neq \emptyset$ (by definition)
$\Leftrightarrow \forall C \subset A, C \neq \emptyset$, (1) $F_{\cup F}(C) \cap A = A$ or
(2) $F_{\cup F}(A - (F_{\cup F}(C) \cap A)) \cap F_{\cup F}(C) \cap A \neq \emptyset$ (Proposition 25-i)
$\Leftrightarrow$ A is a connected subset of $(X, \bigcup_{i\in I} F_{ai})$.

Let's show that (1) is équivalent to (4) :
A is a connected subset of $(X, \bigcup_{i\in I} a_i)$
$\Leftrightarrow \forall C \subset A, C \neq \emptyset$, (1) $F_\cup(C) \cap A = A$ or
(2) $F_\cup(A - (F_\cup(C) \cap A)) \cap F_\cup(C) \cap A \neq \emptyset$ (by definition)
$\Leftrightarrow \forall C \subset A, C \neq \emptyset$, (1) $F_{\prod F}(C) \cap A = A$ or
(2) $F_{\prod F}(A - (F_{\prod F}(C) \cap A)) \cap F_{\prod F}(C) \cap A \neq \emptyset$ (Proposition 25-i)
$\Leftrightarrow$ A is a connected subset of $(X, \prod_{i\in I} F_{ai})$.

Let's show that (3) is équivalent to (4) :
A is a connected subset of $(X, \prod_{i\in I} a_i)$
$\Leftrightarrow \forall C \subset A, C \neq \emptyset$, (1) $F_{\prod}(C) \cap A = A$ or
(2) $F_{\prod}(A - (F_{\prod}(C) \cap A)) \cap F_{\prod}(C) \cap A \neq \emptyset$
$\Leftrightarrow \forall C \subset A, C \neq \emptyset$, (1) $F_{\prod F}(C) \cap A = A$ or
(2) $F_{\prod F}(A - (F_{\prod F}(C) \cap A)) \cap F_{\prod F}(C) \cap A \neq \emptyset$ (Proposition 25-i)
$\Leftrightarrow$ A is a connected subset of $(X, \prod_{i\in I} F_{ai})$.
ii- $\bigcap_{i\in I} a_i$ is thinner than $\bigcap_{i\in I} F_{ai}$.
Indeed, $\forall A \subset X, \forall i \in I$, $a_i(A) \subset F_{ai}(A)$
Then $\bigcap_{i\in I} a_i(A) \subset \bigcap_{i\in I} F_{ai}(A)$.
and the result according to Proposition 28.

Corollary 30. *Let { (X, a_i), i∈I } a network such as for any i∈I, a_i is of V type. Let A ⊂ X with A non empty.*

The following assertions are equivalent :

(1) A connected component of (X, $\bigcup_{i\in I} a_i$)

(2) A connected component of (X, $\bigcup_{i\in I} F_{ai}$)

(3) A connected component of (X, $\prod_{i\in I} a_i$)

(4) A connected component of (X, $\prod_{i\in I} F_{ai}$).

Proof.

Obvious from Proposition 29-i and Definition 13.

Consequence. Decomposing (X, $\bigcup_{i\in I} a_i$) into connected components is equivalent to decomposing (X, $\prod_{i\in I} a_i$) into connected components. So we will propose algorithms to look for connected components in a network in the case of the union and in the case of the intersection of the different pretopologies.

Proposition 31. *Let { (X, a_i), i∈I } a network such as for any i∈I, a_i is of V type. Let A ⊂ X with A non empty.*

i- If there exists i ∈ I such as A connected subset of (X, a_i) then A is connected subset of (X, $\bigcup_{i\in I} a_i$).

ii- If A is connected subset of (X, $\bigcap_{i\in I} a_i$) then for any i∈I, A is connected subset of de (X, a_i).

Proof.

i- ∀ *i*∈I, a_i is thinner than $\bigcup_{i\in I} a_i$. We get the result according to Proposition 28.

ii- ∀ *i*∈I, $\bigcap_{i\in I} a_i$ is thinner than a_i. We get the result according to Proposition 28.

Remark 32. The converses of i- and ii- are not true generally speaking.

Examples 33.

i- Let { (X, a_i), *i*∈I } a network with X = { a, b, c }, I = { 1, 2 }, a_1 and a_2 pretopologies of descendants defined respectively by the following graphs 4 and 5 :

x	R(x)
a	{b}
b	ø
c	ø

Graph 4

x	R(x)
a	∅
b	∅
c	{b}

Graph 5

X is connected subset of (X, $\bigcup_{i\in I} a_i$) but X is not connected subset of (X, a_1) and X is not connected subset of (X, a_2).

ii- Let { (X, a_i), i∈I } a network with X = { a, b, c }, I = { 1, 2 }, a_1 and a_2 respectively pretopology of ascendants and pretopology of descendants defined by the following graph 6 :

x	R(x)
a	{c}
b	{c}
c	∅

Graph 6

X is connected subset of (X, a_1) and connected subset of (X, a_2) but X is not connected subset of (X, $\bigcap_{i\in I} a_i$).
Indeed, $(a_1 \cap a_2)(\{a\}) = a_1(\{a\}) \cap a_2(\{a\}) = \{a\} = F_{a1\cap a2}(\{a\})$ with $F_{a1\cap a2}(\{a\}) \neq X$ and $F_{a1\cap a2}(X - \{a\}) \cap \{a\} = \{ b, c \} \cap \{a\} = \emptyset$.

Consequence. It does not seem possible to find a more judicious algorithm from the study of each a_i for the study of $\bigcap_{i\in I} a_i$. We will take into account only the case of $\bigcup_{i\in I} a_i$.

Proposition 34. *Let { (X, a_i), i∈I } a network such as for any i∈I, a_i is of V type. Let { S_k, k∈K } a family of subsets non empty of X such as :*

1- $\bigcup_{k\in K} S_k = X$

2- ∀ k∈K, there exists { A_j, j∈J } a family of subsets non empty of X such as:

2-1- $S_k = \bigcup_{j\in J} A_j$

2-2- ∀ j∈J, there exists i∈I, A_j connected component of (X, a_i)

2-3- ∀ j∈J, ∀ j'∈J, there exists a sequence $j_0 \ldots j_r$ of elements of J such as $j_0 = j$, $j_r = j'$ and ∀ l = 0,...,r-1, $A_{jl} \cap A_{jl+1} \neq \emptyset$

2-4- ∀ A' ⊂ X, A' ∉ { A_j, j∈J }, if there exists i∈I such as A' connected component of (X, a_i) then A' ∩ S_k = ∅.

We have :

i- $\forall$ $k\in K$, S_k is a connected subset of (X, $\bigcup_{i\in I} a_i$).

ii- If for any $i\in I$, a_i is of V_S type, { S_k, $k\in K$ } is the family of connected component of (X, $\bigcup_{i\in I} a_i$) .

iii- { S_k, $k\in K$ } is a partition of X.

Proof.

i- $\forall$ $j\in$J, there exists $i\in$I, A_j connected component of (X, a_i) then $\forall$ $j\in$J, A_j is connected subset of (X, $\bigcup_{i\in I} a_i$) (Proposition 31-i).

Morever, the union of two connected subsets with a non empty intersection is a connected subset (see [1]) hence the result.

ii- $\forall$ $k\in$K, S_k is a connected subset of (X, $\bigcup_{i\in I} a_i$) (see i-).
Let's show that S_k is connected component of (X, $\bigcup_{i\in I} a_i$).

If S_k is not connected component of (X, $\bigcup_{i\in I} a_i$) then there exists B such as $S_k \subset B \subset X$ with $S_k \neq B$ and such as B is a connected subset of (X, $\bigcup_{i\in I} a_i$).

But $\bigcup_{i\in I} a_i$ is of V_S type (see [1]), so there exists B such as $S_k \subset B \subset X$ with $S_k \neq B$ and $\forall$ x $\in$ B and $\forall$ y $\in$ B, there exists a sequence x_0.........x_n of elements of X such as x_0 = x, x_n= y with $\forall$ l = 0,......., n-1, $x_{l+1} \in \bigcup_{i\in I} a_i(\{x_l\})$ or $x_l \in \bigcup_{i\in I} a_i(\{x_{l+1}\})$ (Proposition 20-ii).

Then there exists B such as $S_k \subset B \subset X$ with $S_k \neq B$ and $\forall$ x $\in S_k$ and $\forall$ y $\in$ B - S_k, there exists a sequence x_0.........x_n of elements of X such as x_0 = x, x_n= y with $\forall$ l = 0,......., n-1, $x_{l+1} \in \bigcup_{i\in I} a_i(\{x_l\})$ or $x_l \in \bigcup_{i\in I} a_i(\{x_{l+1}\})$

Then there exists B such as $S_k \subset B \subset X$ with $S_k \neq B$ and there exists x $\in S_k$ and there exists y $\in$ B - S_k such as x $\in \bigcup_{i\in I} a_i(\{y\})$ or y $\in \bigcup_{i\in I} a_i(\{x\})$

So there exists B such as $S_k \subset B \subset X$ with $S_k \neq B$ and there exists x $\in S_k$, there exists y $\in$ B - S_k and there exists $i\in$I such as x $\in a_i(\{y\})$ or y $\in a_i(\{x\})$

And then there exists B such as $S_k \subset B \subset X$ with $S_k \neq B$ and there exists x $\in S_k$, there exists y $\in$ B - S_k and there exists $i\in$I such as { x, y } is connexted subset of (X, a_i) (Proposition 15-ii)

And there exists A' such as { x, y } $\subset$ A' $\subset$ X and A' connected component of (X, a_i) with x $\in$ A' $\cap S_k$ and A' $\notin$ { A_j, $j\in$J }

So there exists A' such as { x, y } $\subset$ A' $\subset$ X and A' connected component of (X, a_i) with A' $\cap S_k \neq \emptyset$ and A' $\notin$ { A_j, $j\in$J }

Which is in contradiction with 2-4.

In result, S_k is a connected component of (X, $\bigcup_{i\in I} a_i$).

iii- It is sufficient to show that $\forall$ $k\in$K, $\forall$ $k'\in$K with $k \neq k'$, $S_k \cap S_{k'} = \emptyset$ which is ensured by 2.

Remark 35. Generally speaking, if there exists $i\in$I, a_i is of V type but not of V_S type then S_k is not connected component of (X, $\bigcup_{i\in I} a_i$).

Example 36. Let { (X, a_i), $i\in$I } a network with X = { a, b, c, d, e }, I = { 1, 2 }, a_1 and a_2 prétopologies of ascendant-descendants defined respectively by the following graphs 7 and 8 :

x	R(x)
a	{ b, c }
b	{a}
c	{b}
d	{e}
e	{d}

Graph 7

x	R(x)
a	∅
b	∅
c	{e}
d	∅
e	{b}

Graph 8

Let A_1 = { a, b, c }. A_1 is connected component of (X, a_1). Let A_2 = {a}, A_3 = {b}, A_4= {c}. A_2, A_3, A_4 are connected component of (X, a_2). Let S = $A_1 \cup A_2 \cup A_3 \cup A_4$. S checks the conditions of the Proposition so S is connected subset of (X, $\bigcup_{i\in I} a_i$) but S is not connected component of (X, $\bigcup_{i\in I} a_i$). Indeed, X is connected component of (X, $\bigcup_{i\in I} a_i$).

Remark 37. Generally speaking, $S_k \subset F"_{\cup}(S_k)$ with $S_k \neq F"_{\cup}(S_k)$.

Example 38. See the previous example 36. Indeed, the connected component of (X, a_1) are A_1 = { a, b, c }, A_5 = { d, e }. The connected component of (X, a_2) are A_2 = {a}, A_3 = {b}, A_4 = {c}, A_6 = {d} and A_7 = {e}. So we have S_1 = { a, b, c } and S_2 = { d, e } and then $F_{\cup}(S_1)$ = X. The result is $S_1 \subset F"_{\cup}(S_1)$ with $S_1 \neq F"_{\cup}(S_1)$.

Remark 39. $\forall\ k\in$K, $F"_{\cup}(S_k)$ is a connected subset of (X, $\bigcup_{i\in I} a_i$) (Proposition 34-i and Propositin 5.5 of [8]).

Consequence. The Proposition 5.13 of [8] can be applied iteratively to { S_k, $k\in$K } because it is a family of connected subset of (X, $\bigcup_{i\in I} a_i$) which is a partition of X. So we can build a greatest partition noted { F^*_k, $k\in K^*$ } such as $\forall\ k\in K^*$, $F"_{\cup}(F^*_k) = F^*_k$. Indeed, we can apply the next Proposition (see Propoition 5.19 of [8]).

Proposition 40. *Let $\{ (X, a_i), i \in I \}$ a network such as for any $i \in I$, a_i is of V type. The same conditions apply as in Proposition 34.*

Let $\{ F^*_k, k \in K^* \}$ the family which is iteratively built from $\{ S_k, k \in K \}$ such as $\forall k \in K^*$, $F''_\cup(F^*_k) = F^*_k$. (see Proposition 5.13 of [8]).

Let $A \subset X$ with A non empty.

A is connected component of $(X, \bigcup_{i \in I} a_i)$

$$\Leftrightarrow$$

1- It exists $J_A \subset K^*$ such as $A = \bigcup_{j \in J_A} F*_j$

And 2 - $\forall\ C \subset A$, $C \neq \emptyset$, $C = \bigcup_{j \in J} F*_j$ with $J \subset J_A$, we have $F_\cup(C) \cap A = A$ or $F_\cup(A - (F_\cup(C) \cap A)) \cap F_\cup(C) \cap A \neq \emptyset$

And 3- $\forall\ B$, $A \subset B \subset X$ with $A \neq B$, $B = \bigcup_{j \in J_B} F*_j$ with $J_A \subset J_B \subset K^*$ and $J_A \neq J_B$, there exists $C \subset B$, $C \neq \emptyset$, $C = \bigcup_{j \in J} F*_j$ with $J \subset J_B$, such as $F_\cup(C) \cap B \neq B$ and $F_\cup(B - (F_\cup(C) \cap B)) \cap F_\cup(C) \cap B = \emptyset$.

Proof.

See Proposition 34, Propositions 5.13 and 5.19 of [8].

7. Conclusion

Finally, if X is a finite set, we can give two algorithms to find connected components of $(X, \bigcup_{i \in I} a_i)$ (i.e. of $(X, \prod_{i \in I} a_i)$).

In the case where for any i, a_i is of V type, we can use the first algorithm :

- For any $i \in$I, find the connected components of (X, a_i)
- Build $\{ S_k, k \in K \}$ by joining step by step all connected components with a non empty intersection (Proposition 34)
- For any $k \in$K, compute $F''_\cup(S_k)$
- while there exists $k \in$K such as $S_k \subset F''_\cup(S_k)$ with $S_k \neq F''_\cup(S_k)$, do :
    - build $\{ F_{k'}, k' \in K' \}$ by joining step by step all $F''_\cup(S_k)$ with a non empty intersection (see [8])
    - K = K'
    - For any $k' \in$K', $k = k'$ and $S_k = F_{k'}$
    - For any $k \in$K, compute $F''_\cup(S_k)$
- End while
- K* = K
- For any $k \in$K, $k' = k$ and $F^*_{k'} = S_k$
- build the subset defined as in Proposition 40. There are the connected components of $(X, \bigcup_{i \in I} a_i)$ (i.e. $(X, \prod_{i \in I} a_i)$).

In the case where for any $i \in$I, a_i is of V_S type, we can use the second algorithm :

- For any i∈I, find the connected components of (X, a_i)

- Build { S_k, k∈K } by joining step by step all connected components with a non empty intersection. There are the connected components of (X,$\bigcup_{i\in I} a_i$) (i.e. (X, $\prod_{i\in I} a_i$)) (Proposition 34-ii).

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