A Stability Result for the Solutions of a Certain
Fourth-Order Vector Differential Equation
with Variable Delay

A. S. Motlaq 1, E. M. Elsayed 1,2 and F. Alzahrani 1

1 King Abdulaziz University, Faculty of Sciences
Mathematics Department, P. O. Box 80203
Jeddah 21589, Saudi Arabia

2 Department of Mathematics
Faculty of Science, Mansoura University, Egypt

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Abstract

By constructing an appropriate Lyapunov functional, we will give sufficient
conditions of the uniform stability of zero solution for nonlinear fourth-order
vector delay differential equation of the form

\[ X'''' + AX'' + \Psi(X')X'' + G(X'(t-r(t))) + H(X(t-r(t))) = 0 \]

and give example to illustrate truthfulness of our result.

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1 Introduction

In mathematical literature the stability of solutions receives broad attention from researchers, because it plays a fundamental role in the qualitative theory and applications of differential equations. Many methods have been improved to obtain information on the stability behaviour of differential equations when there are no analytical formulas for the solutions. One of the most interesting methods to determine the stability behaviour for the solutions of linear and non-linear differential equations is a method known as Lyapunov’s Second (or Direct) Method [5]. The main advantage of this method is that stability behaviour of solutions can be obtained without any previous knowledge of Exact solutions. That is, this method gives stability information directly, without solving the differential equation.

Today, this method is considered as an effective tool not only in the study of the stability of solutions for differential equations but also in the theory of control systems, analysis of energy system, dynamic systems, systems with a time lag and so on. It should be noted that any investigation on the stability of solutions for vector functional differential equations of fourth-order, using the Lyapunov functional method, first requires the construction of a suitable Lyapunov functional. In fact, the construction of an appropriate Lyapunov functional is in mostly a difficult work.

Over the past years, many new results have been obtained on the stability for solutions of ordinary and functional differential equations of higher-order without and with delay. For instance, we draw the attention of the interested reader to the book by Reissig et al [8] and the papers by Abou El-Ela et al [1,2,3]. Adesina et al.[4]. Omeike [6,7], Sadek[9], Tunc [10,11,12] and the references cited therein. As far as we know, researches that discussed the stability of solutions to vector differential equations can briefly be summarized as follows

First, in 2006 Tunc[11] gave sufficient conditions for the asymptotic stability of the zero solution $X = 0$ and boundedness of all solutions of equation

$$X^{(4)} + \Phi(X'')X''' + F(X,X')X'' + G(X') + H(X) = p(t,X,X',X'',X''')$$

Where $X \in \mathbb{R}^n$; $F$ and $\Phi$ are $n \times n$-symmetric matrices, $G$ and $H$ are $n$–vector continuous functions; $G(0) = H(0) = 0$
After that, in 2012 Abou-El-Ela et al. [2] established sufficient conditions for the uniform stability of the zero solution of the real fourth-order vector delay differential equation

\[ X^{(4)} + AX''' + \Phi(X'') + G(X') + H(X(t - r)) = 0. \]

Where \( X \in \mathbb{R}^n \); \( A \) is continuous \( n \times n \)-symmetric matrix; \( \Phi \), \( G \) and \( H \) are \( n \)-vector continuous functions; \( \Phi(0) = G(0) = H(0) = 0 \); \( r \) is a fixed delay and positive constant.

Lately, in 2015 Abou-El-Ela et al. [3] investigated sufficient conditions for the uniform stability of the zero solution \( X = 0 \) of real non-linear autonomous vector delay differential equation of the fourth-order

\[ X^{(4)} + F(X', X'')X'' + \Phi(X'') + G(X'(t - r)) + H(X(t - r)) = 0 \]

Where \( X \in \mathbb{R}^n \); \( F \) is an \( n \times n \)-symmetric matrix; \( \Phi \), \( G \) and \( H \) are \( n \)-vector continuous function; \( \Phi(0) = G(0) = H(0) = 0 \), and \( r \) is a bounded delay and positive constant.

The objective of this chapter is to study the uniform stability of the zero solution of vector delay differential equation of the form

\[ X^{(4)} + AX''' + \Psi(X')X'' + G(X'(t - r(t))) + H(X(t - r(t))) = 0 \] (1)

Where \( r(t) \) is continuously differentiable function; \( X \in \mathbb{R}^n \); \( A \) is a constant \( n \times n \) matrix such that \( 0 \leq r(t) \leq \delta \), \( \delta \) is a positive constant, \( r'(t) \leq \beta \), \( 0 < \beta < 1 \) in which \( t \in \mathbb{R}^+ \) and \( \Psi \) is \( n \times n \) continuous symmetric matrix function; \( G \) and \( H \) are \( n \)-vector continuous functions; \( G(0) = H(0) = 0 \).

It should be noted that the continuity of functions \( \Psi \), \( G \) and \( H \) is a sufficient condition for the existence of the solution of (1). In addition, we assume that the functions \( \Psi \), \( G \) and \( H \) satisfy a Lipschitz condition with respect to \( X \), \( X' \) and \( X'' \), this assumption is guaranteed the uniqueness of solution of (1).

Equation (1) can be represented as a system of real fourth-order delay differential equations:

\[ x_i^{(4)} + \sum_{k=1}^{n} a_{ik} x_k''' + \sum_{k=1}^{n} \psi_{ik}(x_1', \ldots, x_n') x_k'' + g_i(x_1'(t - r(t)), \ldots, x_n'(t - r(t))) \\
+ h_i(x_1(t - r(t)), \ldots, x_n(t - r(t))) = 0, \quad (i = 1, 2, \ldots, n) \]
Let \( J_G(Y) \), \( J_H(X) \) and \( J(\Psi(Y)Y/Y) \) denote the Jacobian matrices corresponding to the functions \( G(Y) \), \( H(X) \) and the matrix \( \Psi(Y) \) respectively, which given by the following relations;

\[
J_G(Y) = \left( \frac{\partial g_i}{\partial y_j} \right) , \quad J_H(X) = \left( \frac{\partial h_i}{\partial x_j} \right)
\]

and

\[
J(\Psi(Y)Y/Y) = \frac{\partial}{\partial y_j} \left( \sum_{k=1}^{n} \Psi_{ik}y_k \right) - \Psi(Y) + \left( \sum_{k=1}^{n} \frac{\partial \Psi_{ik}}{\partial y_j}y_k \right) . \quad (i, j = 1, 2, ..., n)
\]

Where \( x_i, y_i = x'_i, z_i = y'_i, w_i = z'_i, g_i, h_i, a_{ij} \) and \( \psi_{ij} \) represent \( X, Y, Z, W, G, H, A, \) and \( \Psi \) respectively.

In the following, we assume that the Jacobian matrices \( J_H(X), J_G(Y) \) and \( J(\Psi(Y)Y/Y) \) exist and are continuous. Besides, the symbol \( \langle X, Y \rangle \) corresponding to any pair \( X, Y \) in \( R^n \) denoted to the usual scalar product in \( R^n \), that is \( \langle X, Y \rangle = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n \); thus \( \langle X, X \rangle = \sum_{i=1}^{n} x_i^2 = x_1^2 + x_2^2 + \ldots + x_n^2 = \|X\|^2 \), \( \lambda_i(A)(i = 1, 2, ..., n) \) are the eigenvalues of the \( n \times n \) matrix \( A \).

**Definition 1.1** The function \( f \) satisfies the Lipschitz condition if

\[
\|f(x) - f(y)\| \leq L\|x - y\|
\]

where \( L \) is a positive constant.

**Theorem 1.1** Let \( V : C_H \rightarrow R \) be a continuous functional satisfying a Local Lipschitz condition, \( V(0) = 0 \) such that

(i) \( W_1(\|\Phi(0)\|) \leq V(\Phi) \leq W_2(\|\Phi\|) \), where \( W_1, W_2 \) are wedges, and

(ii) \( V'(\Phi) \leq 0 \), for \( \Phi \in C_H \).

Then the zero solution \( x = 0 \) of the following equation \( x' = f(x_t), x_t(s) = x(t + s), \) for \( -r \leq s \leq 0 \) is uniformly stable.

2 Main Result

The following will be our main stability result of (1).

**Theorem 2.1** Beside the basic assumptions which put on the functions \( A, \Psi, G \) and \( H \), we assume that there exist positive constants \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) and \( \alpha_4' \) such that for \( (i = 1, 2, ..., n) \) the following conditions are hold:
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(i) The matrix $A$ is symmetric and $\lambda_i(A) \geq \alpha_1$.

(ii) $\Psi(Y)$ is symmetric and $0 \leq \lambda_i(\Psi(Y) - \alpha_2 I) \leq \frac{1}{4} \alpha_1^3 \varepsilon_0$, for all $Y \in \mathbb{R}^n$.

(iii) $G(0) = 0$, $J_G(Y)$ is symmetric and $\lambda_i \left( \int_0^1 J_G(\sigma Y) d\sigma \right) \geq \alpha_3$, for all $Y \in \mathbb{R}^n$.

(iv) There is a finite constant $\triangle > 0$, such that

$$\frac{\alpha_1 \alpha_2 - \| J_G(Y) \| \alpha_3 - \alpha_1 \alpha_4 \| A \|}{\alpha_1^3} \geq \triangle,$$

for all $Y \in \mathbb{R}^n$.

(v) $0 \leq \lambda_i \left( J_G(Y) - \int_0^1 J_G(\sigma Y) d\sigma \right) \leq \eta < \frac{\alpha_1^4}{\alpha_3^2},$ for all $Y \in \mathbb{R}^n$.

(vi) $H(0) = 0$, $J_H(X)$ is symmetric and $\lambda_i \left( \int_0^1 J_H(\sigma X) d\sigma \right) \geq \alpha_4'$, for all $X \in \mathbb{R}^n$.

(vii) $J_H(X)$ commutes with $J_H(X')$, for all $X, X' \in \mathbb{R}^n$ and $0 \leq \lambda_i (\alpha_4 I - J_H(X)) \leq \varepsilon_0 D_0 \alpha_2^2,$ for all $X \in \mathbb{R}^n$. Where, $\varepsilon_0$ is a positive constant such that

$$\varepsilon_0 \leq \frac{1}{\alpha_1^2 \alpha_3 d_0^2 \alpha_4^2 \alpha_1^2} \left( \frac{2 \alpha_4}{\alpha_4} - \delta \right)$$

and,

$$D_0 = \alpha_1 \alpha_2 + \alpha_2 \alpha_3 \alpha_4^{-1}$$

(viii) $r(t) \leq \delta$ and $r'(t) \leq \beta, \ \beta \in (0, 1)$.

Then the zero solution of (1) is uniformly stable, provided that:

$$\delta < \min \left\{ \frac{2 \varepsilon \alpha_1^3 \alpha_2^2}{\alpha_1^2 \alpha_3 d_0^2 \alpha_4^2 \alpha_1^2}, \frac{2 \alpha_1 \left( \varepsilon - \frac{1}{4} \varepsilon_0 \right)}{\sqrt{n}(d_1 + d_2 + 1)}, \frac{2 \alpha_1 \left( \varepsilon - \frac{1}{4} \varepsilon_0 \right)}{\sqrt{n}(d_1^2 + d_1 d_2)}, \right\}$$

Where, $d_1 = \varepsilon + \frac{1}{\alpha_1}$ and $d_2 = \varepsilon + \frac{\alpha_4}{\alpha_3}$.

The following lemmas are required for proving Theorem 2.1.

**Lemma 2.1** Let $A$ be a real symmetric $n \times n$-matrix and

$$a' \geq \lambda_i(A) \geq a > 0 \ (i = 1, 2, ..., n),$$

where $a'$ and $a$ are constants. Then

$$a'(X, X) \geq \langle AX, X \rangle \geq a \langle X, X \rangle$$

and
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\[ a^2 \langle X, X \rangle \geq \langle AX, AX \rangle \geq a^2 \langle X, X \rangle \]

A proof is in Bellman.[13]

**Lemma 2.2** Suppose that \( X' = Y, Y' = Z, Z' = W \) Then the following relations are true:

1. \[ \frac{d}{dt} \int_0^1 \langle X(\sigma X), X \rangle d\sigma = \langle H(X), Y \rangle. \]
2. \[ \frac{d}{dt} \int_0^1 \langle G(\sigma Y), Y \rangle d\sigma = \langle G(Y), Z \rangle. \]
3. \[ \frac{d}{dt} \int_0^1 \langle \sigma \Psi(\sigma Y), Y \rangle d\sigma = \langle \Psi(Y)Z, Y \rangle. \]

### 3 Proof of Theorem 2.1.

For the proof of main stability Theorem 2.1, we rewrite equation (1) as the following equivalent system

\[ X' = Y, \quad Y' = Z, \quad Z' = W, \]

\[ W' = -AW - \Psi(Y)Z - G(Y) - H(X) + \int_{t-r(t)}^t J_G(Y(s)Z(s)) ds \]

\[ + \int_{t-r(t)}^t J_H(X(s))Y(s) ds \] (2)

The proof of Theorem 2.1. needs a Lyapunov function \( V = V(X_t, Y_t, Z_t, W_t) \), which is given by

\[ 2V(X_t, Y_t, Z_t, W_t) = 2d_2 \int_0^1 \langle H(\sigma X), X \rangle d\sigma - d_1 \langle \alpha_4 Y, Y \rangle + 2 \int_0^1 \langle G(\sigma Y), Y \rangle d\sigma \]

\[ + 2d_2 \int_0^1 \langle \sigma \Psi(\sigma Y)Y, Y \rangle d\sigma + d_1 \langle \alpha_2 Z, Z \rangle - d_2 \langle Z, Z \rangle + d_1 \langle W, W \rangle \]

\[ + \langle AZ, Z \rangle + 2d_2 \langle AZ, Y \rangle + 2 \langle H(X), Y \rangle + 2d_1 \langle H(X), Z \rangle \]

\[ + 2d_1 \langle G(Y), Z \rangle + 2d_2 \langle Y, W \rangle + 2 \langle Z, W \rangle + 2\mu \int_0^0 \int_{t-r(t)}^t \|Y(\theta)\|^2 d\theta ds \]

\[ + 2\lambda \int_{t-r(t)}^t \int_{t+s}^t \|Z(\theta)\|^2 d\theta ds, \] (3)

where, \( \mu \) and \( \lambda \) are positive constants, which will be determined later. First we shall prove that the Lyapunov function (3) satisfies the following relation:

\[ 2V(X_t, Y_t, Z_t, W_t) \geq D_1 (\|X^2\| + \|Y^2\| + \|Z^2\| + \|W^2\|), \]
where, $D_1$ is a positive constant. Let

$$
\Gamma(Y) = \int_0^1 J_G(\sigma Y) \, d\sigma.
$$

(4)

Then, from (iii) and (v), we obtain

$$
\lambda_i(\Gamma(Y)) \geq \alpha_3, \quad \forall Y \in \mathbb{R}^n
$$

(5)

And,

$$
0 \leq \lambda_i(J_G(Y) - \Gamma(Y)) \leq \eta, \quad \forall Y \in \mathbb{R}^n.
$$

(6)

Since,

$$
2\mu \int_{-r(t)}^0 \int_{t+s}^t \|Y(\theta)\|^2 \, d\theta \, ds \quad \text{and} \quad 2\lambda \int_{-r(t)}^0 \int_{t+s}^t \|Z(\theta)\|^2 \, d\theta \, ds,
$$

are non-negative, then we get

$$
2V(X_t, Y_t, Z_t, W_t) \geq 2d_2 \int_0^1 \langle H(\sigma X), X \rangle \, d\sigma - d_1 \langle \alpha_1 Y, Y \rangle + 2 \int_0^1 \langle G(\sigma Y), Y \rangle \, d\sigma
$$

$$
+ 2d_2 \int_0^1 \langle \sigma \Psi(\sigma Y)Y, Y \rangle \, d\sigma + d_1 \langle \alpha_2 Z, Z \rangle - d_2 \langle Z, Z \rangle + d_1 \langle W, W \rangle
$$

$$
+ \langle AZ, Z \rangle + 2d_2 \langle AZ, Y \rangle + 2\langle H(X), Y \rangle + 2d_1 \langle H(X), Z \rangle
$$

$$
+ 2d_1 \langle G(Y), Z \rangle + 2d_2 \langle Y, W \rangle + 2\langle Z, W \rangle.
$$

Thus, we find

$$
2V(X_t, Y_t, Z_t, W_t) \geq 2d_2 \int_0^1 \langle H(\sigma X), X \rangle \, d\sigma - d_1 \langle \alpha_1 Y, Y \rangle + 2 \int_0^1 \langle G(\sigma Y), Y \rangle \, d\sigma
$$

$$
+ 2d_2 \int_0^1 \langle \sigma \Psi(\sigma Y)Y, Y \rangle \, d\sigma + d_1 \langle \alpha_2 Z, Z \rangle - d_2 \langle Z, Z \rangle + d_1 \langle W, W \rangle
$$

$$
- \|\Gamma^{-\frac{1}{2}} H(X)\|^2 - \|\Gamma^{\frac{1}{2}} Y\|^2 - d_1 \Gamma^{\frac{1}{2}} Z\|^2 - \|\frac{1}{2}A^{-\frac{1}{2}} W\|^2 - \|2d_2 A^{-\frac{1}{2}} Y\|^2
$$

$$
+ \|\Gamma^{-\frac{1}{2}} H(X) + \Gamma^{\frac{1}{2}} Y + d_1 \Gamma^{\frac{1}{2}} Z\|^2 + \|\frac{1}{2}A^{-\frac{1}{2}} W + A^{\frac{1}{2}} Z + d_2 A^{\frac{1}{2}} Y\|^2
$$

(7)

We notice that the matrix $\Gamma$ defined by (4) is symmetric because $J_G$ is symmetric. The eigenvalues of $A$ and $\Gamma$ are positive because (5). Accordingly, the square root $A^{\frac{1}{2}}$ and $\Gamma^{\frac{1}{2}}$ exist, which are symmetric and non-singular for all $Y \in \mathbb{R}^n$. Therefore from (7), we get

$$
2V(X_t, Y_t, Z_t, W_t) \geq 2d_2 \int_0^1 \langle H(\sigma X), X \rangle \, d\sigma - \langle \Gamma^{-1} H(X), H(X) \rangle + 2 \int_0^1 \langle G(\sigma Y), Y \rangle \, d\sigma
$$

$$
+ 2d_2 \int_0^1 \langle \sigma \Psi(\sigma Y)Y, Y \rangle \, d\sigma - (d_1 \alpha_4 + d_2 \|A\|)\|Y\|^2 - \langle \Gamma Y, Y \rangle
$$

$$
+ (d_1 \alpha_2 - d_2 - d_2 \|\Gamma\|)\|Z\|^2 + \left( d_1 - \frac{1}{\alpha_1} \right) \|W\|^2
$$

(8)
From (i) and lemma 1, we have
\[ 2d_2 \int_0^1 \langle \sigma \Psi(\sigma Y)Y,Y \rangle d\sigma \geq 2d_2 \alpha_2 \int_0^1 \langle Y,Y \rangle \sigma d\sigma = d_2 \alpha_2 \| Y \|^2 \]

Then, we can rewrite (8) as the following form
\[
2V(X_t, Y_t, Z_t, W_t) \geq 2d_2 \int_0^1 \langle H(\sigma X), X \rangle d\sigma - \langle \Gamma^{-1} H(X), H(X) \rangle + 2 \int_0^1 \langle G(\sigma Y), Y \rangle d\sigma \\
- \langle \Gamma Y, Y \rangle + (d_2 \alpha_2 - d_1 \alpha_4 - d_2^2 \| A \|) \| Y \|^2 \\
+ (d_1 \alpha_2 - d_2 - d_2^2 \| \Gamma \|) \| A \| Z \|^2 + \left( d_1 - \frac{1}{\alpha_1} \right) \| W \|^2
\]

It follows that
\[
2V(X_t, Y_t, Z_t, W_t) \geq V_1 + V_2 + V_3 + V_4, \quad (9)
\]

where,
\[
V_1 = 2d_2 \int_0^1 \langle H(\sigma X), X \rangle d\sigma - \langle \Gamma^{-1} H(X), H(X) \rangle \\
V_2 = 2 \int_0^1 \langle G(\sigma Y), Y \rangle d\sigma - \langle \Gamma Y, Y \rangle + (d_2 \alpha_2 - d_1 \alpha_4 - d_2^2 \| A \|) \| Y \|^2 \\
V_3 = (d_1 \alpha_2 - d_2 - d_2^2 \| \Gamma \|) \| Z \|^2 \\
V_4 = \left( d_1 - \frac{1}{\alpha_1} \right) \| W \|^2
\]

First, to estimate \( V_1 \), we know that
\[
\frac{\partial}{\partial \sigma_1} \langle H(\sigma_1 X), H(\sigma_1 X) \rangle = 2 \langle J_H(\sigma_1 X) X, H(\sigma_1 X) \rangle,
\]

by integrating both sides from \( \sigma_1 = 0 \) to \( \sigma_1 = 1 \) and because of \( H(0) = 0 \), we obtain
\[
\langle H(\sigma_1 X), H(\sigma_1 X) \rangle |^1_0 = 2 \int_0^1 \langle J_H(\sigma_1 X) X, H(\sigma_1 X) \rangle d\sigma_1 \\
\Rightarrow \langle H(X), H(X) \rangle = 2 \int_0^1 \langle J_H(\sigma_1 X) X, H(\sigma_1 X) \rangle d\sigma_1
\]

Hence,
\[
V_1 = 2d_2 \int_0^1 \langle H(\sigma X), X \rangle d\sigma - \langle \Gamma^{-1} H(X), H(X) \rangle \\
= 2d_2 \int_0^1 \langle H(\sigma_1 X), X \rangle d\sigma_1 - 2 \int_0^1 \langle \Gamma^{-1} J_H(\sigma_1 X) X, H(\sigma_1 X) \rangle d\sigma_1 \\
= 2 \int_0^1 \langle H(\sigma_1 X), \{ d_2 I - \Gamma^{-1} J_H(\sigma_1 X) \} X \rangle d\sigma_1 \quad (10)
\]
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But, from
\[
\frac{\partial}{\partial \sigma_2} \langle H(\sigma_1 \sigma_2 X), \{d_2 I - \Gamma^{-1} J_H(\sigma_1 X)\} X \rangle = \langle \sigma_1 J_H(\sigma_1 \sigma_2 X) X, \{d_2 I - \Gamma^{-1} J_H(\sigma_1 X)\} X \rangle,
\]
by integrating both sides from \( \sigma_2 = 0 \) to \( \sigma_2 = 1 \) and since \( H(0) = 0 \), we find
\[
\langle H(\sigma_1 X), \{d_2 I - \Gamma^{-1} J_H(\sigma_1 X)\} X \rangle = \int_0^1 \sigma_1 \langle J_H(\sigma_1 \sigma_2 X) X, \{d_2 I - \Gamma^{-1} J_H(\sigma_1 X)\} X \rangle d\sigma_2.
\]
Then, we can rewrite (10) as the following form
\[
V_1 = 2 \int_0^1 \int_0^1 \sigma_1 \langle J_H(\sigma_1 \sigma_2 X) X, \{d_2 I - \Gamma^{-1} J_H(\sigma_1 X)\} X \rangle d\sigma_2 d\sigma_1
= 2 \int_0^1 \int_0^1 \sigma_1 \langle J_H(\sigma_1 \sigma_2 X) \{d_2 I - \Gamma^{-1} J_H(\sigma_1 X)\} X, X \rangle d\sigma_2 d\sigma_1
\]
(11)
Therefore, from (vi) and (viii), we obtain
\[
V_1 \geq 2\varepsilon \int_0^1 \int_0^1 \langle J_H(\sigma_1 \sigma_2 X) \sigma_1 X, X \rangle d\sigma_2 d\sigma_1
+ \frac{2}{\alpha_3} \int_0^1 \int_0^1 \sigma_1 \langle J_H(\sigma_1 \sigma_2 X), \{\alpha_4 I - J_H(\sigma_1 X)\} X \rangle d\sigma_2 d\sigma_1
\geq 2\varepsilon \int_0^1 \left[ \int_0^1 \langle J_H(\sigma_2 \tilde{X}) \tilde{X}, X \rangle d\sigma_2 \right] d\sigma_1
\geq 2\varepsilon \int_0^1 \alpha_4^2 \langle \tilde{X}, X \rangle d\sigma_1 = 2\varepsilon \int_0^1 \alpha_4^2 \langle X, X \rangle \sigma_1 d\sigma_1 = \varepsilon \alpha_4 \|X\|^2
\]
(12)
Second, to estimate \( V_2 \), we need
\[
\alpha_2 d_2 - \alpha_4 d_1 - d_2^2 \|A\| = d_2 \{\alpha_2 - d_1 \|J_G(Y)\| - d_2 \|A\|\} + d_1 \{d_2 \|J_G(Y)\| - \alpha_4\}
\]
But, from (iii) and (v), we get
\[
d_2 \|J_G(Y)\| - \alpha_4 \geq \left( \varepsilon + \frac{\alpha_4}{\alpha_3} \right) \alpha_3 - \alpha_4 = \varepsilon \alpha_3 > 0
\]
Hence, we obtain
\[
\alpha_2 d_2 - \alpha_4 d_1 - d_2^2 \|A\| \geq d_2 \{\alpha_2 - d_1 \|J_G(Y)\| - d_2 \|A\|\}
\]
Therefore, from (iv), we have
\[
\alpha_2 - d_1 \|J_G(Y)\| - d_2 \|A\| = \alpha_2 - \frac{1}{\alpha_1} \|J_G(Y)\| - \frac{\alpha_4}{\alpha_3} \|A\| - \varepsilon (\|J_G(Y)\| + \|A\|)
\geq \frac{\alpha_1}{\alpha_1 \alpha_3} - \varepsilon (\alpha_1 \alpha_2 + \|A\|)
\geq \frac{\Delta}{\alpha_1 \alpha_3} - \varepsilon (\alpha_1 \alpha_2 + \alpha_2 \alpha_3 \alpha_4^{-1}) = \frac{\Delta}{\alpha_1 \alpha_3} - \varepsilon D_0
\]
(13)
From the identity
\[ \int_0^1 \sigma \langle J_G(\sigma Y)Y, Y \rangle \, d\sigma \equiv \langle G(Y), Y \rangle - \int_0^1 \langle G(\sigma Y), Y \rangle \, d\sigma. \]

it follows from (4) and lemma 2.1 that
\[ 2 \int_0^1 \langle G(\sigma Y), Y \rangle \, d\sigma - \langle G(Y), Y \rangle = \int_0^1 \sigma \langle J_G(\sigma Y)Y, Y \rangle \, d\sigma - \int_0^1 \sigma \langle J_G(\sigma Y)Y, Y \rangle \, d\sigma \]
\[ = - \int_0^1 \sigma \langle \{ J_G(\sigma Y) - \Gamma(\sigma Y) \} Y, Y \rangle \, d\sigma \]
\[ \geq - \int_0^1 \sigma \eta \langle Y, Y \rangle \, d\sigma = - \frac{1}{2} \eta \| Y \|^{2}; \]

Hence,
\[ V_2 \geq d_2 \left( \frac{\Delta}{\alpha_1 \alpha_3} - \varepsilon D_0 \right) \| Y \|^{2} - \frac{1}{2} \eta \| Y \|^{2} \]
\[ \geq \left\{ \frac{\alpha_4}{\alpha_3} \left( \frac{\Delta}{\alpha_1 \alpha_3} - \varepsilon D_0 \right) - \frac{1}{2} \eta \right\} \| Y \|^{2} \]

Since,
\[ \varepsilon < \frac{\alpha_3}{4 \alpha_4 D_0} \left( \frac{2 \Delta}{\alpha_1 \alpha_3} - \eta \right) \]

Then,
\[ V_2 \geq \left\{ \frac{\alpha_4}{\alpha_3} \left( \frac{\Delta}{\alpha_1 \alpha_3} - \frac{\alpha_3}{4 \alpha_4} \left( \frac{2 \Delta}{\alpha_1 \alpha_3} - \eta \right) \right) \right\} \| Y \|^{2} \]
\[ = \frac{1}{4} \left( \frac{2 \alpha_4 \Delta}{\alpha_1 \alpha_3} - \eta \right) \| Y \|^{2}, \quad (14) \]

Third, to estimate \( V_3 \), by using (13), we get
\[ d_1 \alpha_2 - d_2 - d_1^2 \| \Gamma \| = \frac{d_1}{\alpha_2} \{ \alpha_2 - d_1 \| \Gamma \| - d_2 \| A \| \} + d_2 \{ d_1 \| A \| - 1 \} \]
\[ \geq d_1 \{ \alpha_2 - d_1 \| J_G(Y) \| - d_2 \| A \| \} \]
\[ \geq \frac{1}{\alpha_1} \left( \frac{\Delta}{\alpha_1 \alpha_3} - \varepsilon D_0 \right) \]

Since, \( \varepsilon < \frac{1}{4} \frac{\Delta}{\alpha_1 \alpha_3 D_0} \), then we have
\[ V_3 \geq \frac{3}{4} \left( \frac{\Delta}{\alpha_1 \alpha_3} \right) \| Z \|^{2} \quad (15) \]
Finally,

$$V_4 \geq \left(d_1 - \frac{1}{\alpha_1}\right)\|W\|^2 = \varepsilon\|W\|^2$$  \hspace{1cm} (16)

Therefore from (9), (12),(14), (15) and (16), we obtain:

$$2V(X_t, Y_t, Z_t, W_t) \geq \varepsilon\alpha_4'\|X\|^2 + \frac{1}{4}\left(\frac{2\alpha_4 \Delta}{\alpha_1 \alpha_3^2} - \eta\right)\|Y\|^2 + \frac{3\Delta_2}{4\alpha_1^2 \alpha_3}||Z||^2 + \varepsilon\|W\|^2.$$  \hspace{1cm} (17)

Since the coefficients are positive constants from (17), then there exists a positive constant $D_1$ such that

$$V(X_t, Y_t, Z_t, W_t) \geq D_1 (\|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2)$$  \hspace{1cm} (18)

Where,

$$D_1 = \frac{1}{2} \min \left\{ \varepsilon\alpha_4', \frac{1}{4}\left(\frac{2\alpha_4 \Delta}{\alpha_1 \alpha_3^2} - \eta\right), \frac{3\Delta_2}{4\alpha_1^2 \alpha_3}, \varepsilon \right\}$$

This derives that

$$V(X_t, Y_t, Z_t, W_t) \geq 0 \quad \text{if} \quad \|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2 \geq 0,$$

$$V(X_t, Y_t, Z_t, W_t) \to \infty \quad \text{if} \quad \|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2 \to \infty,$$

which satisfies the left hand side of the inequality in the condition (i) of Theorem 1.1.

Now we will prove the right hand side of the inequality in the condition (i) of Theorem 1.1. That is

$$V(X_t, Y_t, Z_t, W_t) \leq D_2 (\|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2),$$

for some positive constant $D_2$. By using the hypothesis of Theorem 2.1 we get

$$\|A\| \leq \alpha_2 \alpha_3 \alpha_4^{-1}$$  \hspace{1cm} (19)

From (ii), we find

$$\|
\Psi\| \leq \sqrt{n} \left(\alpha_2 + \frac{1}{4} \alpha_1^2 \varepsilon\right)$$  \hspace{1cm} (20)

Also, since $G(0) = 0$ and $\frac{\partial G(\sigma Y)}{\partial \sigma} = J_G(\sigma Y)Y$, then from (iv), we get

$$\|G(Y)\| = \| \int_0^1 J_G(\sigma Y)Y d\sigma \| \leq \int_0^1 \|J_G(\sigma Y)\| \|Y\| d\sigma$$

$$\leq \sqrt{n} \alpha_1 \alpha_2 \|Y\|$$  \hspace{1cm} (21)
Since \( H(0) = 0 \) and \( \frac{\partial H(\sigma X)}{\partial \sigma} = J_H(\sigma X)X \), then from (vii) we have

\[
\|H(X)\| = \| \int_0^1 J_H(\sigma X)X d\sigma \| \leq \int_0^1 \| J_H(\sigma X) \| \| X \| d\sigma \\
\leq 77\|X\|
\]

(22)

Also, from

\[
2\mu \int_{-r(t)}^0 \int_{t+s}^t \|Y(\theta)\|^2 d\theta ds = 2\mu \int_{t-r(t)}^t (\theta - t + r(t)) \|Y(\theta)\|^2 d\theta \\
\leq 2\mu \|Y\|^2 \int_{t-r(t)}^t (\theta - t + r(t)) d\theta \\
= \lambda r^2(t) \|Y\|^2
\]

(23)

And,

\[
2\lambda \int_{-r(t)}^0 \int_{t+s}^t \|Z(\theta)\|^2 d\theta ds = 2\lambda \int_{t-r(t)}^t (\theta - t + r(t)) \|Z(\theta)\|^2 d\theta \\
\leq 2\lambda \|Z\|^2 \int_{t-r(t)}^t (\theta - t + r(t)) d\theta \\
= \lambda r^2(t) \|Z\|^2
\]

(24)

Hence, by (19), (20), (21), (22), (23) and (24) there exists a positive constant \( D_2 \) satisfying

\[
V(X_t, Y_t, Z_t, W_t) \leq D_2 (\|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2)
\]

(25)

This complete the right hand side of the inequality in the condition (i) of Theorem 1.1.

Now, we will prove that \( V' \leq 0 \), by using (2), (3) and lemma 2.2, we obtain

\[
V'(X_t, Y_t, Z_t, W_t) = d_2 \langle H(X), Y \rangle - d_1 \langle \alpha_4 Y, Z \rangle + \langle G(Y), Z \rangle + d_2 \langle \Psi(Y)Z, Y \rangle \\
+ d_1 \langle \alpha_2 Z, W \rangle - d_2 \langle Z, W \rangle + d_1 \langle W, W' \rangle + \langle AZ, W \rangle \\
+ d_2 \langle AW, Y \rangle + d_2 \langle AZ, Z \rangle + \langle J_H(X)Y, Y \rangle + \langle H(X), Z \rangle \\
+ d_1 \langle J_H(X)Y, Z \rangle + d_1 \langle H(X), W \rangle + d_1 \langle J_G(Y)Z, Z \rangle \\
+ d_1 \langle G(Y), W \rangle + d_2 \langle Z, W \rangle + d_2 \langle Y, W' \rangle + \langle W, W \rangle + \langle Z, W' \rangle \\
+ \mu r(t) \|Y\|^2 - \mu (1 - r'(t)) \int_{t-r(t)}^t \|Y(\theta)\|^2 d\theta \\
+ \lambda r(t) \|Z\|^2 - \lambda (1 - r'(t)) \int_{t-r(t)}^t \|Z(\theta)\|^2 d\theta
\]

(26)
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Then, from (2), we can rewrite (26) as following

\[
V'(X_t, Y_t, Z_t, W_t) = d_2\langle H(X), Y \rangle - d_1\langle \alpha_4 Y, Z \rangle + \langle G(Y), Z \rangle + d_2\langle \Psi(Y) Z, Y \rangle \\
+ d_1\langle \alpha_2 Z, W \rangle - d_2\langle Z, W \rangle + d_1\langle W, -AW - \Psi(Y)Z - G(Y) \rangle \\
- H(X) + \int_{t-r(t)}^{t} J_G(Y(s)Z(s)) \, ds + \int_{t-r(t)}^{t} J_H(X(s))Y(s) \, ds \\
+ \langle AZ, W \rangle + d_2\langle AW, Y \rangle + d_2\langle AZ, Z \rangle + \langle J_H(X)Y, Y \rangle + d_2\langle Z, W \rangle \\
+ \langle H(X), Z \rangle + d_1\langle J_H(X)Y, Z \rangle + d_1\langle H(X), W \rangle + d_1\langle J_G(Y)Z, Z \rangle \\
+ d_1\langle G(Y), W \rangle + d_2\langle Y, -AW - \Psi(Y)Z - G(Y) - H(X) \rangle \\
+ \int_{t-r(t)}^{t} J_G(Y(s)Z(s)) \, ds + \int_{t-r(t)}^{t} J_H(X(s))Y(s) \, ds + \langle W, W \rangle \\
+ \langle Z, -AW - \Psi(Y)Z - G(Y) - H(X) + \int_{t-r(t)}^{t} J_G(Y(s)Z(s)) \, ds \\
+ \int_{t-r(t)}^{t} J_H(X(s))Y(s) \, ds + \mu r(t)\|Y\|^2 + \lambda r(t)\|Z\|^2 \\
- \mu(1-r'(t)) \int_{t-r(t)}^{t} \|Y(\theta)\|^2 \, d\theta - \lambda(1-r'(t)) \int_{t-r(t)}^{t} \|Z(\theta)\|^2 \, d\theta
\]

(27)

It follows that

\[
V'(X_t, Y_t, Z_t, W_t) = -d_1\langle \alpha_4 Y, Z \rangle + d_1\langle \alpha_2 Z, W \rangle - d_1\langle W, \Psi(Y)Z \rangle + d_2\langle AZ, Z \rangle \\
+ d_1\langle W, \int_{t-r(t)}^{t} J_G(Y(s)Z(s)) \, ds \rangle + d_1\langle W, \int_{t-r(t)}^{t} J_H(X(s))Y(s) \, ds \rangle \\
+ \langle J_H(X)Y, Y \rangle + d_1\langle J_H(X)Y, Z \rangle + d_1\langle J_G(Y)Z, Z \rangle - d_1\langle W, AW \rangle \\
+ d_2\langle Y, \int_{t-r(t)}^{t} J_G(Y(s)Z(s)) \, ds \rangle + d_2\langle Y, \int_{t-r(t)}^{t} J_H(X(s))Y(s) \, ds \rangle \\
- d_2\langle Y, G(Y) \rangle + \langle W, W \rangle - \langle Z, \Psi(Y)Z \rangle + \langle Z, \int_{t-r(t)}^{t} J_G(Y(s)Z(s)) \, ds \rangle \\
+ \langle Z, \int_{t-r(t)}^{t} J_H(X(s))Y(s) \, ds \rangle + \mu r(t)\|Y\|^2 + \lambda r(t)\|Z\|^2 \\
- \mu(1-r'(t)) \int_{t-r(t)}^{t} \|Y(\theta)\|^2 \, d\theta - \lambda(1-r'(t)) \int_{t-r(t)}^{t} \|Z(\theta)\|^2 \, d\theta
\]
By done some calculations, we get

\[ V'(X_t, Y_t, Z_t, W_t) = - \left\{ d_2 \langle Y, G(Y) \rangle - \alpha_4 \|Y\|^2 \right\} - \left\{ \alpha_2 - d_1 \|J_G(Y)\| - d_2 \|A\| \right\} \|Z\|^2 \]

\[ - \left\{ d_1 \|A\| - 1 \right\} \|W\|^2 + \langle d_1 W + d_2 Y + Z, \int_{t-r(t)}^t J_G(Y(s)Z(s)) \, ds \rangle \]

\[ + \langle d_1 W + d_2 Y + Z, \int_{t-r(t)}^t J_H(X(s))Y(s) \, ds \rangle + \mu r(t) \|Y\|^2 \]

\[ + \lambda r(t) \|Z\|^2 - \mu (1 - r'(t)) \int_{t-r(t)}^t \|Y(\theta)\|^2 \, d\theta \]

\[ - \lambda (1 - r'(t)) \int_{t-r(t)}^t \|Z(\theta)\|^2 \, d\theta + V_5 + V_6 \]  

(28)

Where,

\[ V_5 = \langle \alpha_2 Z, Z \rangle - d_1 \langle W, \Psi(Y) \rangle Z - \langle Z, \Psi(Y) \rangle Z + d_1 \langle \alpha_2 Z, W \rangle \]

\[ V_6 = \langle J_H(X)Y, Y \rangle - d_1 \langle \alpha_4 Y, Z \rangle + d_1 \langle J_H(X)Y, Z \rangle - \langle \alpha_4 Y, Y \rangle \]

But,

\[ V_5 = -d_1 \langle \{ \Psi(Y) - \alpha_2 I \} Z, W \rangle - \langle \{ \Psi(Y) - \alpha_2 I \} Z, Z \rangle \]

Since, \( \lambda_i(\Psi(Y) - \alpha_2 I) \) is non-negative by (ii), and by using (viii), we obtain

\[ V_5 \leq \frac{d_1^2}{4} \langle \{ \Psi(Y) - \alpha_2 I \} W, W \rangle \]

\[ \leq \frac{1}{16} \left( \varepsilon + \frac{1}{\alpha_1} \right)^2 \left( \alpha_1^2 \varepsilon_0 \|W\|^2 \right) \]

\[ \leq \frac{1}{16} \left( \frac{2}{\alpha_1} \right)^2 \left( \alpha_1^2 \varepsilon_0 \|W\|^2 \right) \]

\[ = \frac{1}{4} \alpha_1 \varepsilon_0 \|W\|^2 \]  

(29)

Also,

\[ V_6 = -d_1 \langle \{ \alpha_4 I - J_H(X) \} Y, Z \rangle - \langle \{ \alpha_4 I - J_H(X) \} Y, Y \rangle \]

Since, \( \lambda_i(\alpha_4 I - J_H(X)) \) is non-negative by (vii), then we get

\[ V_6 \leq \frac{1}{4} d_1^2 \langle \{ \alpha_4 I - J_H(X) \} Z, Z \rangle \]

\[ \leq \frac{1}{4} \left( \varepsilon + \frac{1}{\alpha_1} \right)^2 \varepsilon_0 D_0 \alpha_1^2 \|Z\|^2 \]

Since \( \varepsilon < \frac{1}{\alpha_1} \), then we have

\[ V_6 \leq \frac{1}{4} \left( \frac{2}{\alpha_1} \right)^2 \varepsilon_0 D_0 \alpha_1^2 \|Z\|^2 \]

\[ = \varepsilon_0 D_0 \|Z\|^2 \]  

(30)
Therefore, from (28), (29) and (30), we find

\[
V' \leq -\alpha_3 \|Y\|^2 - \{\alpha_2 - d_1 \|J_G(Y)\| - d_2 \|A\| - \varepsilon_0 D_0\} \|Z\|^2
\]
\[
- \left\{d_1 \|A\| - \frac{1}{4} \alpha_1 \varepsilon_0 - 1\right\} \|W\|^2
\]
\[
+ \langle d_1 W + d_2 Y + Z, \int_{t-r(t)}^{t} \ J_H(X(s))Y(s) \ ds \rangle
\]
\[
+ \langle d_1 W + d_2 Y + Z, \int_{t-r(t)}^{t} \ J_G(Y(s))Z(s) \ ds \rangle
\]
\[
+ \mu r(t) \|Y\|^2 + \lambda r(t) \|Z\|^2 - \mu (1 - r'(t)) \int_{t-r(t)}^{t} \|Y(\theta)\|^2 \ d\theta
\]
\[
- \lambda (1 - r'(t)) \int_{t-r(t)}^{t} \|Z(\theta)\|^2 \ d\theta
\]

(31)

Here, since \(\|J_H(X)\| \leq \alpha_4 \sqrt{n}\) (by (vii)) and by using Cauchy-Schwartz inequality \(|\langle U, V \rangle| \leq \frac{1}{2} (\|U\|^2 + \|V\|^2)\), we get

\[
|\langle d_1 W + d_2 Y + Z, \int_{t-r(t)}^{t} \ J_H(X(s))Y(s) \ ds \rangle| \leq \|d_1 W + d_2 Y + Z\| \| \int_{t-r(t)}^{t} \ J_H(X(s))Y(s) \ ds \|
\]
\[
\leq \|d_1 W + d_2 Y + Z\| \int_{t-r(t)}^{t} \alpha_4 \sqrt{n} \|Y(s)\| \ ds
\]
\[
\leq (d_1 \|W\| + d_2 \|Y\| + \|Z\|) \int_{t-r(t)}^{t} \alpha_4 \sqrt{n} \|Y(s)\| \ ds
\]
\[
\leq \frac{d_1 \alpha_4 \sqrt{n}}{2} \left( \|W\|^2 r(t) + \int_{t-r(t)}^{t} \|Y(s)\|^2 \ ds \right)
\]
\[
+ \frac{d_2 \alpha_4 \sqrt{n}}{2} \left( \|Y\|^2 r(t) + \int_{t-r(t)}^{t} \|Y(s)\|^2 \ ds \right)
\]
\[
+ \frac{\alpha_4 \sqrt{n}}{2} \left( \|Z\|^2 r(t) + \int_{t-r(t)}^{t} \|Y(s)\|^2 \ ds \right)
\]
Also, since \( \|J_G(Y)\| \leq \alpha_1\alpha_2\sqrt{n} \) (by (iv)) and by using Cauchy-Schwartz inequality, we find

\[
|\langle d_1W + d_2Y + Z, \int_{t-r(t)}^t J_G(Y(s))Z(s) \, ds \rangle| \leq \|d_1W + d_2Y + Z\| \int_{t-r(t)}^t \|J_G(Y(s))Z(s)\| \, ds \\
\leq \|d_1W + d_2Y + Z\| \int_{t-r(t)}^t \alpha_1\alpha_2\sqrt{n} \|Z(s)\| \, ds \\
\leq (d_1\|W\| + d_2\|Y\| + \|Z\|) \int_{t-r(t)}^t \alpha_1\alpha_2\sqrt{n} \|Z(s)\| \, ds \\
\leq \frac{d_1\alpha_1\alpha_2\sqrt{n}}{2} \left( \|W\|^2 r(t) + \int_{t-r(t)}^t \|Z(s)\|^2 \, ds \right) \\
+ \frac{d_2\alpha_1\alpha_2\sqrt{n}}{2} \left( \|Y\|^2 r(t) + \int_{t-r(t)}^t \|Z(s)\|^2 \, ds \right) \\
+ \frac{\alpha_1\alpha_2\sqrt{n}}{2} \left( \|Z\|^2 r(t) + \int_{t-r(t)}^t \|Z(s)\|^2 \, ds \right)
\]

Therefore, from the above inequalities and by (vii), we get

\[
V' \leq - \left\{ \varepsilon\alpha_3 - \frac{d_2\alpha_1\alpha_2\sqrt{n}}{2} - \frac{d_2\alpha_1\alpha_2\delta\sqrt{n}}{2} - \mu\delta \right\} \|Y\|^2 \\
- \left\{ \left( \frac{\Delta}{\alpha_1\alpha_3} - 2\varepsilon D_0 \right) - \frac{\alpha_4\delta\sqrt{n}}{2} - \frac{\alpha_1\alpha_2\delta\sqrt{n}}{2} - \lambda\delta \right\} \|Z\|^2 \\
- \left\{ \alpha_1 \left( \varepsilon - \frac{1}{4} \varepsilon_0 \right) - \frac{d_1\alpha_4\delta\sqrt{n}}{2} - \frac{d_1\alpha_1\alpha_2\delta\sqrt{n}}{2} \right\} \|W\|^2 \\
+ \left\{ \frac{d_1\alpha_4\sqrt{n}}{2} + \frac{d_2\alpha_1\alpha_2\sqrt{n}}{2} + \frac{\alpha_4\sqrt{n}}{2} - \mu(1 - \beta) \right\} \int_{t-r(t)}^t \|Y(s)\|^2 \, ds \\
+ \left\{ \frac{d_1\alpha_1\alpha_2\sqrt{n}}{2} + \frac{d_2\alpha_1\alpha_2\sqrt{n}}{2} + \frac{\alpha_1\alpha_2\sqrt{n}}{2} - \lambda(1 - \beta) \right\} \int_{t-r(t)}^t \|Z(s)\|^2 \, ds \tag{32}
\]

If, we take

\[
\mu = \frac{\alpha_4\sqrt{n}(d_1 + d_2 + 1)}{2(1 - \beta)} \quad \text{and} \quad \lambda = \frac{\alpha_1\alpha_2\sqrt{n}(d_1 + d_2 + 1)}{2(1 - \beta)}
\]

Then, we have

\[
V' \leq - \left\{ \varepsilon\alpha_3 - \frac{d_2\alpha_1\alpha_2\sqrt{n}}{2} - \frac{d_2\alpha_1\alpha_2\delta\sqrt{n}}{2} - \frac{\alpha_4\sqrt{n}(d_1 + d_2 + 1)}{2(1 - \beta)} \right\} \|Y\|^2 \\
- \left\{ \left( \frac{\Delta}{\alpha_1\alpha_3} - 2\varepsilon D_0 \right) - \frac{\alpha_4\delta\sqrt{n}}{2} - \frac{\alpha_1\alpha_2\delta\sqrt{n}}{2} - \frac{\alpha_1\alpha_2\sqrt{n}(d_1 + d_2 + 1)}{2(1 - \beta)} \right\} \|Z\|^2 \\
- \left\{ \alpha_1 \left( \varepsilon - \frac{1}{4} \varepsilon_0 \right) - \frac{d_1\alpha_4\delta\sqrt{n}}{2} - \frac{d_1\alpha_1\alpha_2\delta\sqrt{n}}{2} \right\} \|W\|^2 \tag{33}
\]
Therefore, if

\[ \delta < \min \left[ \frac{2\varepsilon\alpha_3(1 - \beta)}{\sqrt{n}\{d_2(1 - \beta)(\alpha_4 + \alpha_1\alpha_2) + \alpha_4(d_1 + d_2 + 1)\}}, \frac{2\alpha_1(\varepsilon - \frac{1}{4}\varepsilon_0)}{\sqrt{n}(d_1\alpha_4 + d_1\alpha_1\alpha_2)}, \frac{2\alpha_1(\varepsilon - \frac{1}{4}\varepsilon_0)}{\sqrt{n}(1 - \beta)} \right] \]

Then, we obtain

\[ V' \leq -D_3(||Y||^2 + ||Z||^2 + ||W||^2), \text{ for some } D_3 > 0 \] (34)

Therefore, from (18), (25) and (34), we conclude that the functional \( V(X_t, Y_t, Z_t, W_t) \) satisfies all conditions of the Theorem 1.1. So that, the zero solution of (1) is uniformly stable. Thus the proof of Theorem 2.1 is completed.

4 Illustrative Example

We display an example to illustrate the sufficient conditions, which are given in Theorem 2.1.

**Example 4.1** In a special case of (1) for \( n = 2 \), we choose

\[
A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \quad \Psi(W) = \begin{pmatrix} 6 + y_1^2(t) \\ 0 \end{pmatrix}, \quad H(X(t - r(t))) = \begin{pmatrix} \frac{1}{6}x_1(t - r(t)) + \frac{1}{4}\arctan(x_1(t - r(t))) \\ \frac{1}{6}x_2(t - r(t)) + \frac{1}{4}\arctan(x_2(t - r(t))) \end{pmatrix}
\]

\[ G(Y(t - r(t))) = \begin{pmatrix} y_1(t - r(t)) \\ y_2(t - r(t)) \end{pmatrix}, \quad H(X(t - r(t))) = \begin{pmatrix} \frac{1}{6}x_1(t - r(t)) + \frac{1}{4}\arctan(x_1(t - r(t))) \\ \frac{1}{6}x_2(t - r(t)) + \frac{1}{4}\arctan(x_2(t - r(t))) \end{pmatrix}
\]

From the above matrices, we find that

\[
(A - \lambda I) = \begin{pmatrix} 3 - \lambda & 0 \\ 0 & 3 - \lambda \end{pmatrix} = (3 - \lambda)(3 - \lambda) = 0
\]

Then, we find

\[ \lambda_1(A) = 3 \quad \text{and} \quad \alpha_1 = 3 \]

Also, we can see that the matrix \( \Psi(Y) \) is symmetric and

\[ \lambda_1(\Psi(Y)) = 6 + y_1^2(t), \quad \lambda_2(\Psi(Y)) = 6 + y_2^2(t) \]
Then, we obtain

\[ \lambda_i(\Psi(Y)) \geq 6 \quad \text{and} \quad \alpha_2 = 6 \]

We find that \( G(0) = 0 \) and

\[ J_G(Y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

is symmetric, then

\[ \int_0^1 J_G(\sigma Y) \, d\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

So, we get

\[ \lambda_1 \left( \int_0^1 J_G(\sigma Y) \, d\sigma \right) = 1 \quad \text{and} \quad \lambda_2 \left( \int_0^1 J_G(\sigma Y) \, d\sigma \right) = 1 \]

Thus, we obtain

\[ \lambda_i \left( \int_0^1 J_G(\sigma Y) \, d\sigma \right) = 1 \quad \text{and} \quad \alpha_3 = 1 \]

We have \( H(0) = 0 \) and

\[ J_H(X) = \begin{pmatrix} \frac{1}{6} + \frac{1}{4(1 + x_1^2(t - r(t)))} & 0 \\ 0 & \frac{1}{6} + \frac{1}{4(1 + x_2^2(t - r(t)))} \end{pmatrix} \]

is symmetric, then

\[ \int_0^1 J_H(\sigma X) \, d\sigma = \begin{pmatrix} \int_0^1 \left( \frac{1}{6} + \frac{1}{4(1 + x_1^2(t - r(t)))} \right) \, d\sigma & 0 \\ 0 & \int_0^1 \left( \frac{1}{6} + \frac{1}{4(1 + x_2^2(t - r(t)))} \right) \, d\sigma \end{pmatrix} \]

\[ \int_0^1 J_H(\sigma X) \, d\sigma = \begin{pmatrix} \frac{1}{6} + \frac{1}{4x_1(t - r(t))} \arctan x_1(t - r(t)) & 0 \\ 0 & \frac{1}{6} + \frac{1}{4x_2(t - r(t))} \arctan x_2(t - r(t)) \end{pmatrix} \]
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So, we obtain

\[
\lambda_1 \left( \int_0^1 J_H(\sigma X) \, d\sigma \right) = \frac{1}{6} + \frac{1}{4x_1(t-r(t))} \arctan x_1(t-r(t)).
\]

\[
\lambda_2 \left( \int_0^1 J_H(\sigma X) \, d\sigma \right) = \frac{1}{6} + \frac{1}{4x_2(t-r(t))} \arctan x_2(t-r(t)).
\]

Therefore, we get

\[
\lambda_i \left( \int_0^1 J_H(\sigma X) \, d\sigma \right) \geq \frac{1}{6} \quad \text{and} \quad \alpha_4 = \frac{1}{6}.
\]

Clearly, \( J_H(X) \) commutes with \( J_H(X') \) for all \( X, X' \in \mathbb{R}^n \) and

\[
\lambda_1(J_H(X)) = \frac{1}{6} + \frac{1}{4(1 + x_1^2(t-r(t)))}, \quad \lambda_2(J_H(X)) = \frac{1}{6} + \frac{1}{4(1 + x_2^2(t-r(t)))}
\]

Then, we get

\[
\lambda_i(J_H(X)) \leq \frac{5}{12} \quad \text{and} \quad \alpha_4 = \frac{5}{12}
\]

Now, since \( \|J_G(Y)\| = \sqrt{\lambda_{\text{max}}(J_G^T(Y)J_G(Y))} = 1 \) and \( \|A\| = \sqrt{\lambda_{\text{max}}(A^T A)} = 3 \) where \( J_G^T(Y) \) and \( A^T \) are transpose of matrix \( J_G(Y) \) and \( A \) respectively, then there is a finite positive constant \( \triangle \) such that

\[
\triangle \leq \{\alpha_1 \alpha_2 - \|J_G(Y)\|\} \alpha_3 - \alpha_1 \alpha_4 \|A\| = \frac{53}{4}
\]

Finally we have

\[
0 \leq \lambda_i \left( J_G(Y) - \int_0^1 J_G(\sigma Y) \, d\sigma \right) \leq \eta < \frac{265}{72}
\]

and if we choose \( \eta = 3 \), we get

\[
\varepsilon_0 < \varepsilon = \min \left\{ \frac{1}{3}, \frac{5}{12}, \frac{265}{7776}, \frac{49}{3888} \right\} \approx 0.01260288066
\]

If we take \( \varepsilon_0 = 0.012 \) and \( \beta = 0.5 \), then all conditions of the main Theorem are hold provided that

\[
\delta < \min \{0.0018991, 0.06184, 0.006394\} \approx 0.0018991.
\]
References


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