Applications of Modified Integral Transform to Solve Ordinary and Partial Differential Equations

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Abstract

In this paper, we characterize modified integral transform (called Alenezi transform in the sequel) and examine numerous properties and relations including altered Sumudu transforms of more functions. We examine this change with the other changes of Alenezi transform. We can exhibit that modified Alenezi transforms which are close to the state of the Laplace change. We can clarify the new Properties of changes utilizing the modified Alenezi transform. The modified Alenezi transform can applied to tackle differential, Partial and integral equation. We give modified Alenezi and examine a few relations and models.

Keywords: Ordinary differential equations, partial differential equations, Modified Alenezi transform

1. Introduction

Essential changes can be utilized to settle a few sorts of ordinary differential equations (ODEs), integral equations and partial differential equations (PDEs) [2–8]. These changes likewise can be combined with the homotopy perturbation and the Adomian decomposition methods to solve complicated types of ODEs and PDEs [9–14]. Aggarwal et al. [15–17] tackled a few issues utilizing the Laplace change. In [18], the creators introduced the utilization of Laplace change in cryptography. Fatoorehchi et al. proposed a nonlinear differential conditions arrangement dependent on a clever expansion of the Laplace change [19]. Higazy et al. [20] tackled the HIV-1 contaminations model by the Shehu change. The creators of [21] utilized the Sawi decay strategy for addressing the Volterra necessary
condition. An altered differential change strategy has been applied for addressing the vibration conditions of MDOF frameworks [22]. Alenezi drive an application of Annihilator Extension’s Method and Sumudu transform for nonlinear differential equations in [23-24]. This paper expects to find the arrangement of the arrangement of conventional differential conditions utilizing another change, we have called it modified Alenezi Transform. Modified Alenezi Transform can be utilized to portray some true issues, for example, the issue of the three-layer bar, electrical circuits, chain of substance responses, control of a flying contraption in enormous space, blending development of species, and mechanical vibration.

In this paper, we presented the modified Alenezi necessary change to get the specific arrangements of the respectful quotations. We can show a portion of these changes as displayed in tables [1, 2] that can exhibit the functions and those changes. The paper is facilitated as follows. To some section 2, we present the modified Alenezi fundamental change in the classification of Laplace change. Section 3, we match the changed modified Alenezi necessary change with the other fundamental changes in the class of Laplace change. The modified Alenezi basic change is used to the differential and essential conditions to get the specific arrangements to a limited Section 4. At long last, we summed up the finishes of my change to some Section 4.

**Table 2: Table of Transforms.**

<table>
<thead>
<tr>
<th>Transform Type</th>
<th>Transform Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alenezi Transforms</td>
<td>$A{j(s)} = m(s) \int_0^\infty h(t)e^{-\frac{t}{m(s)}}dt$</td>
</tr>
<tr>
<td>Elzaki transform</td>
<td>$E{z(t)} = s \int_0^\infty z(t)e^{-\frac{t}{s}}dt$</td>
</tr>
<tr>
<td>Sumudu transform</td>
<td>$S{z(t)} = \frac{1}{s} \int_0^\infty z(t)e^{-\frac{t}{s}}dt$</td>
</tr>
<tr>
<td>Natural transform</td>
<td>$n{z(t)} = R(s,u) = s \int_0^\infty z(ut)e^{-st}dt$</td>
</tr>
<tr>
<td>$\alpha$-Integral Laplace transform</td>
<td>$L_{\alpha}{z(t)} = \int_0^\infty z(t)e^{-\frac{1}{\alpha}t}dt, \ \alpha \in R_0^+$</td>
</tr>
<tr>
<td>Aboodh transform</td>
<td>$A{z(t)} = K(s) = \frac{1}{s} \int_0^\infty z(t)e^{-st}dt$</td>
</tr>
<tr>
<td>Mohand transform</td>
<td>$m{z(t)} = R(s) = s^2 \int_0^\infty z(t)e^{-st}dt$</td>
</tr>
</tbody>
</table>
Table 2: (Continued) Table of Transforms.

<table>
<thead>
<tr>
<th>Function</th>
<th>Alenezi integral transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pourreza transform</td>
<td>( H[J(z)] = s \int_0^\infty z(t)e^{-st}dt )</td>
</tr>
<tr>
<td>Kamal transform</td>
<td>( K[z(t)] = G(s) = \int_0^\infty z(t)e^{-\frac{t}{s}}dt )</td>
</tr>
<tr>
<td>Sawi transform</td>
<td>( Sa {z(t)} = \frac{1}{s^2} = \int_0^\infty z(t)e^{-\frac{t}{s}}dt )</td>
</tr>
<tr>
<td>Modified Alenezi transform</td>
<td>( M{T(s)} = \eta(s) \int_0^\infty z(t)e^{-\frac{\alpha t}{\delta(s)}}dt )</td>
</tr>
</tbody>
</table>

2. Definition of modified Alenezi transform

In this portion, we display modified Alenezi integral transform that envelope a widely integral transform in the group of Laplace transform.

**Definition 1.** Let \( z(t) \) become an integrable function realized for \( t \geq 0 \), \( p(s) \) and \( n(s) \neq 0 \) are favorable real functions, we explain modified Alenezi transform \( T(s) \) of \( z(t) \) by the formula

\[
T(s) = \eta(s) \int_0^\infty z(t)e^{-\frac{\alpha t}{\delta(s)}}dt
\]
Theorem 1. Let \( z(t) \) is differentiable and \( \eta(s) \) and \( \delta(s) \) are positive real functions, then

(I) \[
T\{z'(t); s\} = \delta(s) \mathcal{T}(s) - \eta(s)z(0)
\]

(II) \[
T\{z''(t); s\} = \delta^2(s) \mathcal{T}(z(t); s) - \eta(s)\delta(s)z(0) - \eta(s)z'(0)
\]

(III) \[
T\{z^{(n)}(t); s\} = \delta^n(s) \mathcal{T}(z(t); s) - \eta(s) \sum_{k=0}^{n-1} \eta^k(s)z^k(0)
\]

Proof. (I). In view of (1) we have

\[
\mathcal{T}\{z'(t); s\} = \eta(s) \int_0^\infty z'(t)e^{-\frac{\eta(t)}{\delta(s)}}dt
\]

\[
= \eta(s) \left[ e^{-\frac{\eta(t)}{\delta(s)}}z(t) \right]_0^\infty + \delta(s) \int_0^\infty z(t)e^{-\frac{\eta(t)}{\delta(s)}}dt
\]

\[
= \delta(s) \mathcal{T}(z(t); s) - \eta(s)z(0),
\]

To prove (II), we assume \( z(t) = z'(t) \) so \( z''(t) = z'(t) \) now

\[
T\{z'(t); s\} = \eta(s) \int_0^\infty z'(t)e^{-\frac{\eta(t)}{\delta(s)}}dt = \delta(s) \mathcal{T}(z(t); s) - \eta(s)z(0)
\]

\[
= \delta(s) \mathcal{T}(z'(t); s) - \eta(s)z'(0) = \delta(s)[\delta(s)\mathcal{T}(z(t); s) - \eta(s)z(0)] - \eta(s)z'(0),
\]

Theorem 2. (Convolution) Let \( z_1(t) \) and \( z_2(t) \) have new integral transform \( F(s) \). Then the new integral transform of the Convolution of \( z_1 \) and \( z_2 \) is

\[
z_1 \ast z_2 = \int_0^\infty z_1(t) \ast z_2(t - \tau)d\tau = \frac{1}{\eta(s)}F_1(s) \ast F_2(s).
\]

Proof.

\[
T\{z_1 \ast z_2\} = \eta(s) \int_0^\infty e^{-\frac{\eta t}{\delta(s)}} \int_0^\infty z_1(t) \ast z_2(t - \tau)d\tau dt
\]

\[
= \eta(s) \int_0^\infty z_1(\tau)d\tau \int_0^\infty e^{-\frac{\eta t}{\delta(s)}} z_2(t - \tau)dt
\]

\[
= \eta(s) \int_0^\infty e^{-\frac{\eta t}{\delta(s)}} \int_0^\infty z_2(t) \ast z_2(t - \tau)d\tau dt
\]

\[
= \eta(s) \int_0^\infty e^{-\frac{\eta t}{\delta(s)}} z_1(\tau)d\tau \int_0^\infty e^{-\frac{\eta t}{\delta(s)}} z_2(t)dt
\]

\[
= \frac{1}{\eta(s)}F_1(s) \ast F_2(s)
\]
3. Solution of Ordinary differential equations using the modified Alenezi transform

In this part we present the modified Alenezi change for tackling Ordinary differential equations. Likewise, we applied it to get precise arrangement of few sorts of ODE and PDE.

4. Solving Ordinary differential equations with the initial condition

Consider the following initial value problem:

\[ Y^{(n)}(t) + a_1 Y^{(n)}(t) + \cdots + a_n Y(t) = g(x) \]  \hspace{1cm} (14)

\[ Y(0) = Y_0, \; Y'(0) = Y_1, \ldots, Y^{(n-1)}(0) = Y_{n-1}. \]  \hspace{1cm} (15)

Now we apply the modified Alenezi transform \( \mathcal{T} \), in view of Theorem (1) we have

\[ \mathcal{T}\{Y^{(n)}(t) + a_1 Y^{(n)}(t) + \cdots + a_n Y(t)\} = \mathcal{T}\{g(x)\} \]  \hspace{1cm} (16)

\[ \mathcal{T}\{Y^{(n)}(t)\} + a_1 \mathcal{T}\{Y^{(n)}(t)\} + \cdots + a_n \mathcal{T}\{Y(t)\} = \mathcal{T}\{g(x)\} \]  \hspace{1cm} (17)

Example 1. Consider the following third-order ODE

\[ \frac{d^3 y}{dx^3} + \frac{d^2 y}{dx^2} - 6 \frac{dy}{dx} = 0 \]  \hspace{1cm} (18)

\[ y(0) = 1, \; (0) = 1, \frac{dy}{dx}(0) = 0, \frac{d^2 y}{dx^2}(0) = 5. \]  \hspace{1cm} (19)

By applying the modified Alenezi transform \( \mathcal{T} \) on both sides, we have

\[ \mathcal{T}\left\{\frac{d^3 y}{dx^3} + \frac{d^2 y}{dx^2} - 6 \frac{dy}{dx}\right\} = \mathcal{T}\{0\}, \]  \hspace{1cm} (20)

\[ \mathcal{T}\{\frac{d^3 y}{dx^3}\} + \mathcal{T}\{\frac{d^2 y}{dx^2}\} - \mathcal{T}\{6 \frac{dy}{dx}\} = \mathcal{T}\{0\} \]  \hspace{1cm} (21)

\[ \delta^3 (s) \mathcal{T}(s) - \eta(s)(\delta^2(s) Y_0 + \delta(s)Y_1 + Y_2) + \delta^2(s) \mathcal{T}(s) - \eta(s)(\delta(s) Y_0 + Y_1 - 6 \mathcal{T}(s) = 0 \]

by replacing the initial conditions in above equation, we have

\[ \left[\delta^3(s) + \delta^2(s) - 6\delta(s)\right] \mathcal{T}(s) = \eta(s) \delta^2(s) - \eta(s) + \eta(s) \delta(s). \]  \hspace{1cm} (22)

\[ \mathcal{T}(s) = \frac{\eta(s)\delta^2(s) - \eta(s) + \eta(s)\delta(s)}{[\delta^3(s) + \delta^2(s) - 6\delta(s)]} - \frac{\eta(s)}{6\delta(s)} + \frac{\eta(s)}{3\delta(s)+9} + \frac{\eta(s)}{2\delta(s)-4} \]  \hspace{1cm} (23)
by applying $T^{-1}$ we find the exact solution as

$$Y(t) = \frac{1}{6} T^{-1}\left(\frac{\eta(s)}{\delta(s)}\right) + T^{-1}\left(\frac{\eta(s)}{3\delta(s)+9}\right) + T^{-1}\left(\frac{\eta(s)}{2\delta(s)-4}\right) = \frac{1}{6} + \frac{1}{3} e^{-3t} + \frac{1}{2} e^{2t}$$ (24)

Figure 1: Expansion of Example 1 based on modified Alenezi transform.

Example 2. Consider the following third-order ODE

$$Y''' + 2Y'' + 2Y' + 3Y = \sin t + \cos t$$ (25)

$$Y(0) = 1, Y'(0) = 1, Y''(0) = 0.$$ (26)

By applying $\mathcal{T}$ we have

$$\mathcal{T}\{Y''' + 2Y'' + 2Y' + 3Y\} = \mathcal{T}\{\sin t + \cos t\}$$ (27)

$$\mathcal{T}\{Y''''\} + 2\mathcal{T}\{Y''\} - \mathcal{T}\{6Y\} = \mathcal{T}\{(0)\}$$ (28)

$$\delta^3(s) \mathcal{T}(s) - \eta(s)(\delta^2(s)Y_0 + \delta(s)Y_1 + Y_2) + 2[\delta^2(s) \mathcal{T}(s) - \eta(s)(\delta(s) Y_0 + Y_1) + 2[\delta(s) \mathcal{T}(s) - \eta(s)Y_0] + 3 \mathcal{T}(s) = \frac{\eta(s)}{\delta^2(s)+1} + \frac{\delta(s)\eta(s)}{\delta^2(s)+1}$$ (29)

by replacing the initial conditions in above equation, we have

$$[\delta^3(s) + 2\delta^2(s) + 2\delta(s) + 3] \mathcal{T}(s)$$

$$= -\frac{\eta(s)}{\delta^2(s)+1} + \frac{\delta(s)\eta(s)}{\delta^2(s)+1} + \delta(s) \eta(s) + 2 \eta(s).$$

by simplification we got
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\[ \mathcal{T}(s) = \frac{\eta(s)}{\delta^2(s) + 1}. \] (30)

by applying \( \mathcal{T}^{-1} \), we find the exact solution as

\[ Y(t) = \mathcal{T}^{-1}\left(\frac{\eta(s)}{\delta^2(s) + 1}\right) = \sin(t) \]

![Figure 2: Expansion of Example 2 based on modified Alenezi transform.](image)

**Example 3. Solve the Partial differential equation**

\[ 2x \frac{\partial y}{\partial t} + \frac{\partial y}{\partial x} = 2x \] (31)

Given that

\[ Y(x, 0) = 1, \quad Y(0, t) = 1. \] (32)

We have

\[ 2xY_y(x, t) + Y_x(x, t) = 2x \] (33)

\[ 2x[sy(x, s) - Y(x, 0)] + y_x(x, s) = \frac{2x}{s} \] (34)

\[ 2x[sy(x, s) - 1] + y_x(x, s) = \frac{2x}{s} \] (35)

\[ \frac{dy}{dx} + 2x sy = 2x + \frac{2x}{s} \] (36)
\[ = 2x\left(1 + \frac{1}{s}\right) \] (37)

This is linear differential equation of the first order. The integrating factor is

\[ e^{\int 2sx dx} = e^{sx^2} \] (38)

\[ e^{sx^2} = \int 2x\left(1 + \frac{1}{s}\right)e^{sx^2}dx + c \] (39)

\[ = \frac{1}{s}\left(1 + \frac{1}{s}\right)e^{sx^2} + c \] (40)

\[ y(x, s) = \frac{1}{s}\left(1 + \frac{1}{s}\right) + c e^{-sx^2} \] (41)

\[ \frac{1}{s} = \frac{1}{s}\left(1 + \frac{1}{s}\right) + c \]

\[ c = -\frac{1}{s^2} \]

\[ y(x, s) = \frac{1}{s}\left(1 + \frac{1}{s}\right) - \frac{1}{s^2} e^{-sx^2} \] (42)

\[ Y(x, t) = \begin{cases} 1 + t & 0 \leq t \leq x^2 \\ 1 + x^2 & t \geq x^2 \end{cases} \] (43)

**Figure 3:** Expansion of Example 3 based on modified Alenezi transform.
Example 4. Solve the Linear difference equations:

\[ 4X(\tau) - 5X(\tau - 1) + X(\tau - 2) = \tau^2 \]
given that \( X(\tau) = \lambda \)

Solution using Modified Alenezi Transform: we have

\[ 4\mathcal{T}\{X(\tau)\} - 5\mathcal{T}\{X(\tau - 1)\} + \mathcal{T}\{X(\tau - 2)\} = \mathcal{T}\{\tau^2\} \]  \hspace{1cm} (45)

i.e.,

\[ 4\mathcal{T}\tilde{Y}_L - 5\mathcal{T}\{X(\tau - 1)\} + \mathcal{T}\{X(\tau - 2)\} = 2 S^2 \]  \hspace{1cm} (46)

where \( \{\mathcal{T}\{X(\tau)\}\} = \tilde{Y}_L = \tilde{Y}_L(s) \).

Using the definition

Following the procedure as described in case of Modified Alenezi Transform or using the duality between Alenezi Transform and Modified Alenezi Transform, we have

\[ K\{X(\tau - n)\} = e^{-n \frac{s}{S}} \tilde{Y}_L + Ks(1 - e^{-n \frac{s}{S}}) \]  \hspace{1cm} (47)

Hence the above equation becomes

\[ X(\tau) = \lambda S + \frac{1}{2} S^3 + \frac{2}{3} \sum_{n=1}^{\infty} (1 - 2^{-2n-2}) e^{-n S^3} \]  \hspace{1cm} (48)

Taking inverse Modified Alenezi Transform and using \( X(\tau) = \begin{cases} 0, & \text{if } \tau \leq n \\ 1, & \text{if } \tau > n \end{cases} \) we have

\[ X(\tau) = \lambda + \frac{1}{4} \tau^2 + \frac{1}{3} \sum_{n=1}^{\infty} (1 - 2^{-2n-2})(\tau - 2)^2 \]  \hspace{1cm} (49)

which is the same as obtained by Alenezi Transform.

6. Results and Discussions

In this paper, we made a comparative discussion on Alenezi Transform and Modified Alenezi Transform through some of their properties and transforms of some frequently used functions, derivatives and integrals. Further, in the application section we solved a difference equation using both the transforms. Study shows that both the transform techniques are parallel and closely connected to each other by the duality relation mentioned in the above section.

References


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