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Stability of a Predator-Prey Model with Beddington-DeAngelis Functional Response and Disease in the Prey [1]

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Abstract

In this paper, a predator-prey model with Beddington-DeAngelis type functional response and disease in prey is proposed. The existence of the equilibria are discussed. By means of characteristic equations, the conditions for the local stability of each equilibrium are obtained. By constructing new Lyapunov functions and using the LaSalle's invariance principle, we also establish the global stability of each equilibrium.

Keywords: Beddington-DeAngelis type functional response; LaSalle's invariance principle; Global stability

1 Introduction

In recent years, many scholars have applied mathematical modeling methods to study population dynamics and epidemiology, and have achieved fruitful results([1]-[5]). Due to the spread of diseases between populations, it is meaningful to consider the impact of diseases when studying population dynamics. There have been some results in this

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area. As in [6], the predator-prey model in which both groups are infected with SI type disease is studied. In [7], the predator-prey model in which both the predator and the prey are infected with SI type disease. In [8], the predator-prey model with Holling II type functional response and prey infection with SI type disease was studied. In [9], it also studied the predator-prey model with Holling II type functional response and prey infection with SI type disease. In [10], the predator-prey model of SIRS type disease was discussed. In [11], the predator-prey model with Holling II type functional response and disease factors was studied. In [12], the population model of SI type disease with Beddington-DeAngelis functional response function was analyzed. In [13], the predator-prey model of SIRS type disease is studied.

This article discusses the Beddington-DeAngelis functional response predator-prey model with SIRS type disease as the prey infection model:

$$\begin{aligned}
\frac{dS}{dt} &= \beta R + q(S+I+R) - \gamma SI - \frac{\alpha_1 SY}{a+bY+cS+eI+dR} - mS,\\
\frac{dI}{dt} &= \gamma SI - \rho I - \mu I - \frac{\alpha_2 IY}{a+bY+cS+eI+dR} - mI,\\
\frac{dR}{dt} &= \rho I - \beta R - \frac{\alpha_3 RY}{a+bY+cS+eI+dR} - mR,\\
\frac{dY}{dt} &= rY + \frac{f[\alpha_1 SY + \alpha_2 IY + \alpha_3 RY]}{a+bY+cS+eI+dR} - nY,
\end{aligned} \tag{1.1}$$

where, S(susceptible) + I(infective) + R(cured) indicates the prey population, Y represent the population density predator.

The parameters r and q stand for the intrinsic rate of growth for predator and prey. The parameters m and n means the natural mortality rate for predator and prey. γ represents the infection rate, β represents the cure rate, and μ represents the death rate. f represents the conversion factor denoting the newly born predators for each captured prey. $\alpha_1, \alpha_2, \alpha_3$ respectively represent the predator's capture rate of prey in different states. All the coefficients in the model (1.1) are positive, and will be regarded as the constants throughout we discussed.

S, I, R, Y meet the initial conditions

$$S(0) \geq 0, I(0) \geq 0, R(0) \geq 0, Y(0) \geq 0. \tag{1.2}$$

Infectious disease flow chart is as follows:

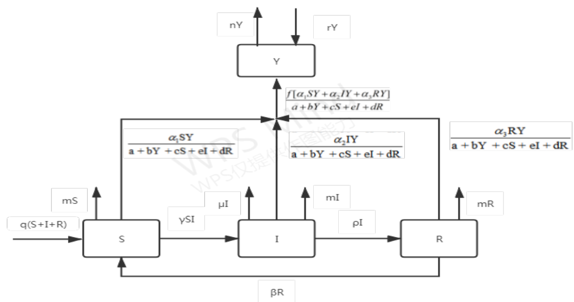

Figure 1: Model flow chart

This article is arranged as follows: in the second part, we prove the positivity and the boundedness of solution; in the third part, we analyze the equilibriums, including the existence and stability of equilibriums; in the last part, we give a brief summary.

2 Positivity and boundedness of solution

2.1 Positivity of solution

Theorem 2.1.1 All solutions in system (1.1) with the initial condition (1.2) are positive.

Proof. For the system (1.1), under the initial conditions (1.2), for any $t > 0$, where

$$
\begin{aligned}
S(t) &= S_0\, e^{\int_0^t \left(\frac{\beta R+q(I+R)}{S}+q-\gamma I-\frac{\alpha_1 Y}{a+bY+cS+eI+dR}-m\right)dv_1} > 0,\\
I(t) &= I_0\, e^{\int_0^t \left(\gamma S-\rho-\mu-\frac{\alpha_2 Y}{a+bY+cS+eI+dR}-m\right)dv_2} > 0,\\
R(t) &= R_0\, e^{\int_0^t \left(\frac{\rho I}{R}-\beta-\frac{\alpha_3 Y}{a+bY+cS+eI+dR}-m\right)dv_3} > 0,\\
Y(t) &= Y_0\, e^{\int_0^t \left(r+\frac{f[\alpha_1 S+\alpha_2 I+\alpha_3 R]}{a+bY+cS+eI+dR}-n\right)dv_4} > 0.
\end{aligned}
$$

That is, the positive solution exists. □

2.2 Boundedness of solution

Theorem 2.2.1 All solutions in system (1.1) with the initial condition (1.2) are bounded for all $t \geq 0$.

Proof. Let $h(t) = S(t) + I(t) + R(t) + Y(t)$, where

$$\begin{aligned}&\tfrac{dh(t)}{dt} = \tfrac{dS}{dt} + \tfrac{dI}{dt} + \tfrac{dR}{dt} + \tfrac{dY}{dt}\\&= (S + I + R)(q - m) + (r - n)Y - \mu I - \tfrac{(1-f)[\alpha_1 SY + \alpha_2 IY + \alpha_3 RY]}{a + bY + cS + eI + dR}\\&\le (S + I + R)q + rY - mh.\end{aligned}$$

Here, $m = min\{1, n, \mu, S + I + R\}$. Hence,

$$\frac{dh(t)}{dt} + mh \le (S + I + R)q + rY.$$

From the constant variation method, we can get

$$h(t) \le (S + I + R)q + rY + \frac{h(0)}{e^{mt}}, m \ge 0.$$

where $t \to \infty$, $h \le (S + I + R)q + rY$.

So in $\Omega = \{(S, I, R, Y) \in \Re^4_+, h \le (S + I + R)q + rY + \varepsilon, \forall \varepsilon > 0\}$, the boundedness of the system is proved. □

3 Equalibrium analysis

3.1 Existence of equilibrium point

In system (1.1), we can obtained zero equilibrium point $E_0 = (0, 0, 0, 0)$. The following are the existence theorems for boundary equilibrium points $E_1 = (S_1, I_1, 0, 0)$, $E_2 = (S_2, I_2, R_2, 0)$ and positive equilibrium point $E^* = (S^*, I^*, R^*, Y^*)$.

Theorem 3.1.1 Where $m < q < \rho + \mu + m$, E_1 existed.

Proof. For the boundary equilibrium point E_1, by the system (1.1),

$$(S_1 + I_1) - \gamma S_1 I_1 - mS_1 = 0, \gamma S_1 I_1 - \rho I_1 - \mu I_1 - mI_1 = 0.$$

here

$$S_1 = \frac{\rho + \mu + m}{\gamma}, I_1 = \frac{(m - q)(\rho + \mu + m)}{\gamma(q - \rho - \mu - m)}.$$

Hence, where $m < q < \rho + \mu + m$, E_1 existed, $E_1 = (\frac{\rho+\mu+m}{\gamma}, \frac{(m-q)(\rho+\mu+m)}{\gamma(q-\rho-\mu-m)}, 0, 0)$. □

Theorem 3.1.2 Where $m > q$ and $(\beta + q)p + (\beta + m)(q - (p + \mu + m)) > 0$, E_2 existed.

Proof. For the boundary equilibrium point E_2, by the system (1.1),

$$\beta R_2 + q(S_2 + I_2 + R_2) - \gamma S_2 I_2 - mS_2 = 0, \gamma S_2 I_2 - \rho I_2 - \mu I_2 - mI_2 = 0, \rho I_2 - \beta R_2 - mR_2 = 0,$$

here

$$S_2 = \frac{\rho+\mu+m}{\gamma}, I_2 = \frac{q(m-q)(\rho+\mu+m)(\beta+m)}{\rho\gamma[\beta p+(\beta+m)(q-(p+\mu+m))+pq]}, R_2 = \frac{q(m-q)(\rho+\mu+m)}{\gamma[\beta p+(\beta+m)(q-(p+\mu+m))+pq]}.$$

Hence, where $m > q$ and $(\beta + q)p + (\beta + m)(q - (p + \mu + m)) > 0$, E_2 existed, $E_2 = (\frac{\rho+\mu+m}{\gamma}, \frac{q(m-q)(\rho+\mu+m)(\beta+m)}{\rho\gamma[\beta p+(\beta+m)(q-(p+\mu+m))+pq]}, \frac{q(m-q)(\rho+\mu+m)}{\gamma[\beta p+(\beta+m)(q-(p+\mu+m))+pq]}, 0)$. □

Theorem 3.1.3 Where S^*, I^*, R^*, Y^* satisfied (3.3), E^* existed.

Proof. For the positive equilibrium point E^*, by the system (1.1),

$$\begin{aligned}
&\beta R^* + q(S^* + I^* + R^*) - \gamma S^* I^* - \tfrac{\alpha_1 S^* Y^*}{a+bY^*+cS^*+eI^*+dR^*} - mS^* = 0,\\
&\gamma S^* I^* - \rho I^* - \mu I^* - \tfrac{\alpha_2 I^* Y^*}{a+bY^*+cS^*+eI^*+dR^*} - mI^* = 0,\\
&\rho I^* - \beta R^* - \tfrac{\alpha_3 R^* Y^*}{a+bY^*+cS^*+eI^*+dR^*} - mR^* = 0,\\
&rY^* + \tfrac{f[\alpha_1 S^* Y^* + \alpha_2 I^* Y^* + \alpha_3 R^* Y^*]}{a+bY^*+cS^*+eI^*+dR^*} - nY^* = 0.
\end{aligned}$$

where S^*, I^*, R^*, Y^* is the solution of the following equation

$$f(q - m)(S^* + I^* + R^*) - \mu I^* f = (n - r)Y^*. \tag{3.3}$$

Hence, where S^*, I^*, R^*, Y^* satisfied (3.3), E^* existed, $E^* = (S^*, I^*, R^*, Y^*)$. □

3.2 Local stability of equilibrium point

(1)Stability of zero equilibrium point E_0

The Jacobi matrix of the system (1.1) at the point E_0 is

$$J(E_0) = \begin{bmatrix} q-m & q & \beta+q & 0 \\ 0 & -(\rho+\mu+m) & 0 & 0 \\ 0 & \rho & -(\beta+m) & 0 \\ 0 & 0 & 0 & r-n \end{bmatrix},$$

Hence

$$tr(E_0) = q + r - (\rho + \mu + \beta + n + 3m), \det(E_0) = (q - m)(\rho + \mu + m)(\beta + m)(r - n).$$

Theorem 3.2.1

When $q < m$, $r > n$ or $q > m$, $r < n$, the zero equilibrium point is saddle point;

When $q > m$, $r > n$ and $q+r < \rho+\mu+\beta+n+3m$ or $q < m$, $r < n$, the zero equilibrium point is asymptotic stability;

When $q > m$, $r > n$ and $q+r > \rho+\mu+\beta+n+3m$, the zero equilibrium point is unstable;
When $q > m$, $r > n$ and $q+r = \rho+\mu+\beta+n+3m$, the zero balance point as center.

(2)Stability of boundary equilibrium point E_1

The Jacobi matrix of the system (1.1) at the point E_1 is

$$J(E_1)=\begin{bmatrix} q-\gamma I_1-m & q-\gamma S_1 & \beta+q & -\frac{\alpha_1 S_1}{a+cS_1+eI_1} \\ \gamma I_1 & 0 & 0 & -\frac{\alpha_2 I_1}{a+cS_1+eI_1} \\ 0 & \rho & -(\beta+m) & -\frac{\alpha_3 R_1}{a+cS_1+eI_1} \\ 0 & 0 & 0 & r-n+\frac{f(\alpha_1 S_1+\alpha_2 I_1)}{a+cS_1+eI_1} \end{bmatrix},$$

where

$$S_1=\frac{\rho+\mu+m}{\gamma}, I_1=\frac{(m-q)(\rho+\mu+m)}{\gamma(q-\rho-\mu-m)}.$$

$$tr(E_1)=q+r+\frac{f(\alpha_1 S_1+\alpha_2 I_1)}{a+cS_1+eI_1}-\gamma I_1-(\beta+2m+n),$$

$$\det(E_1)=\frac{\gamma I_1[(I_1 e+cS_1+a)(r-n)+f(\alpha_1 S_1+\alpha_2 I_1)](\gamma S_1-q-p)(\beta+m)}{a+cS_1+eI_1}.$$

Theorem 3.2.2
When $\det(E_1) < 0$, the boundary equilibrium point E_1 is saddle point;
When $tr(E_1) > 0$, $\det(E_1) > 0$ the boundary equilibrium point E_1 is unstable;
When $tr(E_1) = 0$, $\det(E_1) > 0$, the boundary equilibrium point E_1 as center;
When $tr(E_1) < 0$, $\det(E_1) > 0$, the boundary equilibrium point E_1 is asymptotic stability.

(3)Stability of boundary equilibrium point E_2

The Jacobi matrix of the system (1.1) at the point E_2 is

$$J(E_2)=\begin{bmatrix} q-\gamma I_2-m & q-\gamma S_2 & \beta+q & -\frac{\alpha_1 S_2}{a+cS_2+eI_2+dR_2} \\ \gamma I_2 & 0 & 0 & -\frac{\alpha_2 I_2}{a+cS_2+eI_2+dR_2} \\ 0 & \rho & -(\beta+m) & -\frac{\alpha_3 R_2}{a+cS_2+eI_2+dR_2} \\ 0 & 0 & 0 & r-n+\frac{f(\alpha_1 S_2+\alpha_2 I_2+\alpha_3 R_2)}{a+cS_2+eI_2+dR_2} \end{bmatrix},$$

where

$$S_2=\frac{\rho+\mu+m}{\gamma}, I_2=\frac{q(m-q)(\rho+\mu+m)(\beta+m)}{\rho\gamma[\beta p+(\beta+m)(q-(p+\mu+m))+pq]},$$

$$R_2=\frac{q(m-q)(\rho+\mu+m)}{\gamma[\beta p+(\beta+m)(q-(p+\mu+m))+pq]}.$$

$$trE_2=q+r+\frac{f(\alpha_1 S_2+\alpha_2 I_2+\alpha_3 R_2)}{a+cS_2+eI_2+dR_2}-(\gamma I_2+n+\beta+2m),$$

$$\det E_2=\frac{\gamma I_2(\beta+m)(\gamma S_2-q-p)[(cS_2+eI_2+dR_2+a)(n-r)-f(\alpha_1 S_2+\alpha_2 I_2+\alpha_3 R_2)]}{a+cS_2+eI_2+dR_2}.$$

Theorem 3.2.3
When $\det(E_2) < 0$, the boundary equilibrium point E_2 is saddle point;

When $tr(E_2) > 0$, $\det(E_2) > 0$ the boundary equilibrium point E_2 is unstable;
When $tr(E_2) = 0$, $\det(E_2) > 0$, the boundary equilibrium point E_2 as center;
When $tr(E_2) < 0$, $\det(E_2) > 0$, the boundary equilibrium point E_2 is asymptotic stability.

(4)Stability of positive equilibrium point E^*

The Jacobi matrix of the system (1.1) at the point E^* is

$$J(E^*) = \begin{bmatrix} a_{11}^* & a_{12}^* & a_{13}^* & a_{14}^* \\ a_{21}^* & a_{22}^* & a_{23}^* & a_{24}^* \\ a_{31}^* & a_{32}^* & a_{33}^* & a_{34}^* \\ a_{41}^* & a_{42}^* & a_{43}^* & a_{44}^* \end{bmatrix},$$

where

$$\begin{aligned}
&a_{11}^* = q - \gamma I^* - m - \tfrac{\alpha_1 Y^*(a+bY^*+eI^*+dR^*)}{(a+bY^*+cS^*+eI^*+dR^*)^2}, a_{12}^* = q - \gamma S^* + \tfrac{\alpha_1 e S^* Y^*}{(a+bY^*+cS^*+eI^*+dR^*)^2},\\
&a_{13}^* = \beta + q + \tfrac{\alpha_1 d S^* Y^*}{(a+bY^*+cS^*+eI^*+dR^*)^2}, a_{14}^* = -\tfrac{\alpha_1 S^*(a+cS^*+eI^*+dR^*)}{(a+bY^*+cS^*+eI^*+dR^*)^2},\\
&a_{21}^* = \gamma I^* + \tfrac{\alpha_2 c I^* Y^*}{(a+bY^*+cS^*+eI^*+dR^*)^2}, a_{22}^* = \gamma S^* - \rho - \mu - m - \tfrac{\alpha_2 Y^*(a+bY^*+cS^*+dR^*)}{(a+bY^*+cS^*+eI^*+dR^*)^2},\\
&a_{23}^* = \tfrac{\alpha_2 d I^* Y^*}{(a+bY^*+cS^*+eI^*+dR^*)^2}, a_{24}^* = -\tfrac{\alpha_2 I^*(a+cS^*+eI^*+dR^*)}{(a+bY^*+cS^*+eI^*+dR^*)^2}, a_{31}^* = \tfrac{\alpha_3 c R^* Y^*}{(a+bY^*+cS^*+eI^*+dR^*)^2},\\
&a_{32}^* = \rho + \tfrac{\alpha_3 e R^* Y^*}{(a+bY^*+cS^*+eI^*+dR^*)^2}, a_{33} = -\beta - m - \tfrac{\alpha_3 Y^*(a+bY^*+cS^*+eI^*)}{(a+bY^*+cS^*+eI^*+dR^*)^2},\\
&a_{34} = -\tfrac{\alpha_3 R^*(a+cS^*+eI^*+dR^*)}{(a+bY^*+cS^*+eI^*+dR^*)^2}, a_{41} = \tfrac{fY^*[\alpha_1(a+bY^*+eI^*+dR^*)-c(\alpha_2 I^*+\alpha_3 R^*)]}{(a+bY^*+cS^*+eI^*+dR^*)^2},\\
&a_{42} = \tfrac{fY^*[\alpha_2(a+bY^*+cS^*+dR^*)-e(\alpha_1 S^*+\alpha_3 R^*)]}{(a+bY^*+cS^*+eI^*+dR^*)^2}, a_{43} = \tfrac{fY^*[\alpha_3(a+bY^*+cS^*+eI^*)-d(\alpha_1 S^*+\alpha_2 I^*)]}{(a+bY^*+cS^*+eI^*+dR^*)^2},\\
&a_{44} = r - n + \tfrac{f(\alpha_1 S^*+\alpha_2 I^*+\alpha_3 R^*)(a+cS^*+eI^*+dR^*)}{(a+bY^*+cS^*+eI^*+dR^*)^2}.
\end{aligned}$$

here S^*, I^*, R^*, Y^* satisfied (3.3),

$$\begin{aligned}
tr(E^*) &= a_{11}^* + a_{22}^* + a_{33}^* + a_{44}^*\\
&= q + r + \gamma(S^* - I^*) - (3m - +\rho + \mu + \beta + n) + \tfrac{f(\alpha_1 S^*+\alpha_2 I^*+\alpha_3 R^*)(a+cS^*+eI^*+dR^*)}{(a+bY^*+cS^*+eI^*+dR^*)^2}\\
&\quad -\tfrac{\alpha_1 Y^*(a+bY^*+eI^*+dR^*)}{(a+bY^*+cS^*+eI^*+dR^*)^2} - \tfrac{\alpha_2 Y^*(a+bY^*+cS^*+dR^*)}{(a+bY^*+cS^*+eI^*+dR^*)^2} - \tfrac{\alpha_3 Y^*(a+bY^*+cS^*+eI^*)}{(a+bY^*+cS^*+eI^*+dR^*)^2},
\end{aligned}$$

$$\begin{aligned}
det(E^*) = {}& a_{11}^*a_{22}^*a_{33}^*a_{44}^* - a_{11}^*a_{22}^*a_{34}^*a_{43}^* - a_{11}^*a_{23}^*a_{32}^*a_{44}^* + a_{11}^*a_{23}^*a_{34}^*a_{42}^* + a_{11}^*a_{24}^*a_{32}^*a_{43}^* - a_{11}^*a_{24}^*a_{33}^*a_{42}^*\\
&-a_{12}^*a_{21}^*a_{33}^*a_{44}^* + a_{12}^*a_{21}^*a_{34}^*a_{43}^* + a_{12}^*a_{23}^*a_{31}^*a_{44}^* - a_{12}^*a_{23}^*a_{34}^*a_{41}^* - a_{12}^*a_{24}^*a_{31}^*a_{43}^* + a_{12}^*a_{24}^*a_{33}^*a_{41}^*\\
&+a_{13}^*a_{21}^*a_{32}^*a_{44}^* - a_{13}^*a_{21}^*a_{34}^*a_{42}^* - a_{13}^*a_{22}^*a_{31}^*a_{44}^* + a_{13}^*a_{22}^*a_{34}^*a_{41}^* + a_{13}^*a_{24}^*a_{31}^*a_{42}^* - a_{13}^*a_{24}^*a_{32}^*a_{41}^*\\
&-a_{21}^*a_{32}^*a_{41}^*a_{43}^* + a_{21}^*a_{33}^*a_{41}^*a_{42}^* + a_{22}^*a_{31}^*a_{41}^*a_{43}^* - a_{22}^*a_{33}^*{a_{41}^*}^2 - a_{23}^*a_{31}^*a_{41}^*a_{42}^* + a_{23}^*a_{32}^*{a_{41}^*}^2.
\end{aligned}$$

Theorem 3.2.4

When $\det(E^*) < 0$, the boundary equilibrium point E^* is saddle point;
When $tr(E^*) > 0$, $\det(E^*) > 0$ the boundary equilibrium point E^* is unstable;
When $tr(E^*) = 0$, $\det(E^*) > 0$, the boundary equilibrium point E^* as center;
When $tr(E^*) < 0$, $\det(E^*) > 0$, the boundary equilibrium point E^* is asymptotic stability.

3.3 Global stability of equilibrium point

Let T_1=$(q-\rho-\mu-m)(I-I_1)+\rho I-(\beta+m)R+\frac{(r-n)Y}{f}+Y\frac{\alpha_1 S_1+\alpha_2 I_1}{a+bY+cS_1+eI_1+dR}$.

Theorem 3.3.1 When $q<m$ and $T_1=0$, the boundary equilibrium point E_1 globally asymptotically stable.

Proof. For the boundary equilibrium point $E_1=(S_1,I_1,0,0)$, take the Lyapunov function,

$$V^1={V_1}^1+{V_2}^1,$$
$${V_1}^1=S-S_1-S_1\ln\frac{S}{S_1},{V_2}^1=\varphi_1^1(I-I_1-I_1\ln\frac{I}{I_1}),{V_3}^1={\varphi_2}^1R,{V_4}^1={\varphi_3}^1Y.$$

Take the derivative of V^1 along the trajectory of the system (1.1),

$$\begin{aligned}\frac{dV^1}{dt}&=\frac{dV_1^1}{dt}+\frac{dV_2^1}{dt}+\frac{dV_3^1}{dt}+\frac{dV_4^1}{dt}\\&=\frac{1}{S}(S-S_1)(\frac{dS}{dt}-\frac{dS_1}{dt})+\frac{\phi_1^1}{I}(I-I_1)(\frac{dI}{dt}-\frac{dI_1}{dt})+\phi_2^1\frac{dR}{dt}+\phi_3^1\frac{dY}{dt},\end{aligned}$$

let $\phi_1^1=\frac{(S-S_1)I}{S(I-I_1)}$, $\phi_2^1=\frac{(S-S_1)}{S}$, $\phi_3^1=\frac{(S-S_1)}{fS}$,

$$\begin{aligned}\frac{dV^1}{dt}&=\frac{(q-m)}{S}(S-S_1)^2+\frac{(S-S_1)}{S}[(q-\rho-\mu-m)(I-I_1)+\rho I-(\beta+m)R+\frac{(r-n)Y}{f}\\&+\frac{\alpha_1 S_1 Y}{a+bY+cS_1+eI_1+dR}+\frac{\alpha_2 I_1 Y}{a+bY+cS_1+eI_1+dR}]\\&=\frac{(q-m)}{S}(S-S_1)^2+\frac{(S-S_1)}{S}T_1.\end{aligned}$$

According to the Lyapunov-LaSalle invariance principle[14], when $q<m$ and $T_1=0$, $\frac{dV^1}{dt}<0$ the boundary equilibrium point E_1 is global stability. □

Let $T_2=(2\rho+\mu+m+q)(I-I_2)+(2\beta+m+q)(R-R_2)+\frac{(r-n)Y}{f}+Y\frac{\alpha_1 S_2+\alpha_2 I_2+\alpha_3 R_2}{a+bY+cS_2+eI_2+dR_2}$.

Theorem 3.3.2 When $q<m$ and $T_2=0$, the boundary equilibrium point E_2 globally asymptotically stable.

Proof. For the boundary equilibrium point $E_2=(S_2,I_2,R_2,0)$, take the Lyapunov function,

$$V^2=V_1^2+V_2^2+V_3^2+V_4^2,$$
$$V_1^2=S-S_2-S_2\ln\frac{S}{S_2},V_2^2=\phi_1^2(I-I_2-I_2\ln\frac{I}{I_2}),V_3^2=\phi_2^2(R-R_2-R_2\ln\frac{R}{R_2},V_4^2=\phi_3^2Y.$$

Take the derivative of V^2 along the trajectory of the system (1.1),

$$\begin{aligned}\frac{dV^2}{dt}&=\frac{dV_1^2}{dt}+\frac{dV_2^2}{dt}+\frac{dV_3^2}{dt}+\frac{dV_4^2}{dt}\\&=\frac{1}{S}(S-S_2)(\frac{dS}{dt}-\frac{dS_2}{dt})+\frac{\phi_1^2}{I}(I-I_2)(\frac{dI}{dt}-\frac{dI_2}{dt})+\frac{\phi_2^2}{R}(R-R_2)(\frac{dR}{dt}-\frac{dR_2}{dt})+\phi_3^2\frac{dY}{dt}.\end{aligned}$$

let $\phi_1^2=\frac{I(S-S_2)}{S(I-I_2)}$, $\phi_2^2=\frac{R(S-S_2)}{S(R-R_2)}$, $\phi_3^2=\frac{(S-S_2)}{fS}$,

$$\begin{aligned}\frac{dV^2}{dt}&=\frac{1}{S}(q-m)(S-S_2)^2-\frac{(S-S_2)}{S}[(2\rho+\mu+m+q)(I-I_2)+(2\beta+m+q)(R-R_2)\\&+\frac{(r-n)Y}{f}+Y\frac{\alpha_1 S_2}{a+bY+cS_2+eI_2+dR_2}+Y\frac{\alpha_2 I_2}{a+bY+cS_2+eI_2+dR_2}+Y\frac{\alpha_3 R_2}{a+bY+cS_2+eI_2+dR_2}]\\&=\frac{1}{S}(q-m)(S-S_2)^2-\frac{(S-S_2)}{S}T_2.\end{aligned}$$

According to the Lyapunov-LaSalle invariance principle[14], when $q < m$ and $T_2 = 0$, $\frac{dV^2}{dt} < 0$, the boundary equilibrium point E_2 is global stability. □

Theorem 3.3.3 When $q < m$, $\rho + m = 1$, $\rho + q = \mu + m$, $Y = Y^*$, the positive equilibrium point E^* globally asymptotically stable.

Proof. For the positive equilibrium point $E^* = (S^*, I^*, R^*, Y^*)$, take the Lyapunov function,

$$V^2 = V_1^2 + V_2^2 + V_3^2 + V_4^2,$$
$$V_1^2 = S - S_2 - S_2 \ln \frac{S}{S_2}, V_2^2 = \phi_1^2(I - I_2 - I_2 \ln \frac{I}{I_2}), V_3^2 = \phi_2^2(R - R_2 - R_2 \ln \frac{R}{R_2}, V_4^2 = \phi_3^2 Y.$$

Take the derivative of V^* along the trajectory of the system (1.1),

$$\frac{dV^*}{dt} = \frac{dV_1}{dt}^* + \frac{dV_2}{dt}^* + \frac{dV_3}{dt}^* + \frac{dV_4}{dt}^*$$
$$= \frac{1}{S}(S - S^*)(\frac{dS}{dt} - \frac{dS^*}{dt}) + \frac{\phi_1}{I}(I - I^*)(\frac{dI}{dt} - \frac{dI^*}{dt}) + \frac{\phi_2}{R}(R - R^*)(\frac{dR}{dt} - \frac{dR^*}{dt}) + \frac{\phi_3}{Y}(Y - Y^*)(\frac{dY}{dt} - \frac{dY^*}{dt}),$$

let $\phi_1^* = \frac{I(S-S^*)}{S(I-I^*)}$, $\phi_2^* = \frac{R(S-S^*)}{S(R-R^*)}$, $\phi_3^* = \frac{Y(S-S^*)}{fS(Y-Y^*)}$,

$$\frac{dV^*}{dt} = \frac{(S-S^*)}{S}[(\rho + m - 1)(R - R^*) + (\rho + q - \mu - m)(I - I^*) + \frac{(r+n)(Y-Y^*)}{f}] + \frac{(q-m)(S-S^*)^2}{S}.$$

According to the Lyapunov-LaSalle invariance principle[14], when $q < m$, $\rho + m = 1$, $\rho + q = \mu + m$, $Y = Y^*$, $\frac{dV^*}{dt} < 0$, the positive equilibrium point E^* is global stability. □

4 Summary

In this paper, we mainly discuss the predator-prey model in which the disease spread in the prey population. the existence of positive solutions is proved. the boundedness of the solution is proved by using the constant variation method. the conditions for the existence of the equilibrium point and the conditions for the local asymptotic stability of the boundary equilibrium point are obtained by analyzing the model. by constructing a new Lyapunov function, the conditions for the global asymptotic stability of the positive equilibrium point are obtained by using LaSalle invariance principle.

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