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A Solution to the Langmuir Wave Equations Using Solitary Wave Methods

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Abstract

In this paper, we use solitary wave methods for the solution of the non-linear one-dimensional Langmuir wave equations, finding several families of solutions.

Keywords: Langmuir wave equations, Tanh method, Ricatti solutions

1 Introduction

The Langmuir waves is an important research field in plasma media. We can find studies in collapse waves [1], stability of plasma solitons [2]-[3], envelope solitons in solar bursts [4], among others. Moreover, the solitary wave methods, [5], have become very popular and widespread analytical tools, capable of finding solutions to nonlinear partial differential equations. In this work, we solve the Langmuir wave equations using the Tanh method [5] and the solutions of the Riccati equation [6].

2 Solitary wave solution

The Langmuir wave equations [1], in one dimension, are:

$$i\frac{\partial E}{\partial t} + \frac{\partial^2 E}{\partial x^2} - 2k_1 En = 0 \tag{1}$$

$$\frac{\partial^2 n}{\partial t^2} - \frac{\partial^2 n}{\partial x^2} - k_2 \frac{\partial^2 E^2}{\partial x^2} = 0 \tag{2}$$

Using

$$\xi = x - at \tag{3}$$

The derivatives are:

$$\frac{\partial}{\partial t} = -a\frac{\partial}{\partial \xi}; \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}; \quad \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \xi^2} \tag{4}$$

So, eqs. (1) and (2)

$$-ai\frac{\partial E}{\partial \xi} + \frac{\partial^2 E}{\partial \xi^2} + 2k_1 E n = 0 \tag{5}$$

$$a\frac{\partial^2 n}{\partial \xi^2} + \frac{\partial^2 n}{\partial \xi^2} + k_2 \frac{\partial^2 E^2}{\partial \xi^2} = 0 \tag{6}$$

Now, we apply the tanh method [5]. Then, we introduce an independent variable:

$$Y = \tanh(\mu\xi) \tag{7}$$

The derivatives of ξ in terms of Y, are:

$$\frac{d}{d\xi} = \mu(1 - Y^2)\frac{d}{dY}, \quad \frac{d^2}{d\xi^2} = -2\mu^2 Y(1 - Y^2)\frac{d}{dY} + \mu^2(1 - Y^2)^2\frac{d^2}{dY^2} \tag{8}$$

The solutions are postulated [5] as:

$$E = \sum_{i=1}^{n} a_i Y^i, \quad n = \sum_{i=1}^{m} b_i Y^i \tag{9}$$

Now, using eqs. (8) in eqs. (5) and (6), we get:

$$\begin{aligned} &-ai\mu(1 - Y^2)\frac{dE}{dY} - 2\mu^2 Y(1 - Y^2)\frac{dE}{dY} + \mu^2(1 - Y^2)^2\frac{d^2 E}{dY^2} \\ &+2k_1 E n = 0 \end{aligned} \tag{10}$$

$$(a+1)(-2\mu^2Y(1-Y^2)\frac{dn}{dY}+\mu^2(1-Y^2)^2\frac{d^2n}{dY^2}) \\ -2\mu^2Y(1-Y^2)\frac{d(E^2)}{dY}+k_2\mu^2(1-Y^2)^2\frac{d^2(E^2)}{dY^2}=0 \tag{11}$$

The parameters m and n, in eqs. (9), are obtained by applying the balance principle between the highest-order linear derivative with the highest nonlinear term in eqs. (10) and (11). So, we have:

$$Y^4\frac{d^2E}{dY^2}\rightarrow En\rightarrow n+2=m+n\rightarrow m=2 \tag{12}$$

$$Y^4\frac{d^2n}{dY^2}\rightarrow Y^4\frac{d^2(E^2)}{dY^2}\rightarrow m+2=2n+2\rightarrow n=1 \tag{13}$$

Then, eqs. (10) and (11), are:

$$E=a_0+a_1Y\rightarrow E^2=a_0^2+2a_0a_1Y+a_1^2Y^2 \tag{14}$$

$$n=b_0+b_1Y+b_2Y^2 \tag{15}$$

Replacing

$$-ai\mu(1-Y^2)a_1-2\mu^2Y(1-Y^2)a_1 \\ +2k_1(a_0+a_1Y)(b_0+b_1Y+b_2Y^2)=0 \tag{16}$$

$$(a+1)(-2\mu^2Y(1-Y^2)(b_1+2b_2Y)+\mu^2(1-Y^2)^2 2b_2) \\ -2\mu^2Y(1-Y^2)(2a_0a_1+2a_1^2Y)+k_2\mu^2(1-Y^2)^2 2a_1^2=0 \tag{17}$$

Collecting the coefficients, in eqs. (16) and (17), in powers of Y^i and equating to zero, we get a set of nonlinear equations. Then, doing some algebra, we get:

$$\left\{a_0=0, a_1=0, a=-1, \mu=-i\sqrt{b_2}\sqrt{k_1}\right\}, \tag{18}$$

$$\left\{a_0=0, a_1=0, a=-1, \mu=i\sqrt{b_2}\sqrt{k_1}\right\}, \tag{19}$$

$$\left\{ b_1 = \frac{2a_0 b_2}{a_1}, a = \frac{-a_1^2 - b_2}{b_2}, \mu = 0, k_1 = 0, k_2 = 1 \right\}, \tag{20}$$

$$\left\{ a_1 = -i\sqrt{b_2}, b_1 = 2ia_0\sqrt{b_2}, a = 0, \mu = 0, k_1 = 0, k_2 = 1 \right\}, \tag{21}$$

$$\left\{ a_1 = i\sqrt{b_2}, b_1 = -2ia_0\sqrt{b_2}, a = 0, \mu = 0, k_1 = 0, k_2 = 1 \right\}, \tag{22}$$

$$\left\{ a_0 = 0, a_1 = -i\sqrt{b_2}, b_0 = -b_2, b_1 = 0, a = 0, \mu = -i\sqrt{b_2}\sqrt{k_1}, k_2 = 1 \right\}, \tag{23}$$

$$\left\{ a_0 = 0, a_1 = -i\sqrt{b_2}, b_0 = -b_2, b_1 = 0, a = 0, \mu = i\sqrt{b_2}\sqrt{k_1}, k_2 = 1 \right\}, \tag{24}$$

$$\left\{ a_0 = 0, a_1 = i\sqrt{b_2}, b_0 = -b_2, b_1 = 0, a = 0, \mu = -i\sqrt{b_2}\sqrt{k_1}, k_2 = 1 \right\}, \tag{25}$$

$$\left\{ a_0 = 0, a_1 = i\sqrt{b_2}, b_0 = -b_2, b_1 = 0, a = 0, \mu = i\sqrt{b_2}\sqrt{k_1}, k_2 = 1 \right\}, \tag{26}$$

$$\begin{aligned} &\{a_0 = a_1, b_0 = -3b_2, b_1 = 2b_2, a = \frac{-a_1^2 - b_2}{b_2}, \\ &\mu = -\frac{(a_1^2 + b_2)\, i}{6b_2}, k_1 = -\frac{(a_1^2 + b_2)^2\, i^2}{36b_2^3}, k_2 = 1\}, \end{aligned} \tag{27}$$

$$\begin{aligned} &\{a_0 = -a_1, b_0 = -3b_2, b_1 = -2b_2, a = \frac{-a_1^2 - b_2}{b_2}, \\ &\mu = \frac{(a_1^2 + b_2)\, i}{6b_2}, k_1 = -\frac{(a_1^2 + b_2)^2\, i^2}{36b_2^3}, k_2 = 1\} \end{aligned} \tag{28}$$

We find eleven families of solutions using Tanh method.

	A	C	F
1	1/2	-1/2	$coth(\xi) \pm cosh(\xi)$, $tanh(\xi) \pm isech(\xi)$
2	1/2	1/2	$sec(\xi) \pm itan(\xi)$
3	-1/2	- 1/2	$csc(\xi) \pm icot(\xi)$
4	1	- 1	$tanh(\xi)$, $coth(\xi)$
5	1	1	$tan(\xi)$
6	-1	-1	$cot(\xi)$

Table 1: Solutions for eqs. (**??**)-(**??**)[6] .

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Also, we apply the method given in [6], in order to find solutions to $E(\xi)$ and $n(\xi)$, then:

$$E = \sum_{i=1}^{n} a_i F^i, \quad n = \sum_{i=1}^{m} b_i F^i \tag{29}$$

Here F satisfies, table (1), the Riccati equation, i.e.

$$\begin{aligned} &F^{'} = CF^2 + A \to F^{''} = C2FF^{'} = 2CF(CF^2 + A) \\ &= 2C^2F^3 + 2ACF \to F^{'''} = 6C^3F^4 + 8AC^2F^2 + 2A^2C \end{aligned} \tag{30}$$

Where A and C are constants, table (1). Replacing in eqs. (5) and (6), and taking $m = 2$ and $n = 1$, we get, $E = (a_0 + a_1F)$ and $n = (b_0 + b_1F + b_2F^2)$. So:

$$\begin{aligned} &E^{'} = b_1F^{'} = b_1CF^2 + Ab_1 \\ &E^{''} = 2b_1CF(CF^2 + A) = 2b_1C^2F^3 + 2Ab_1CF \\ &E^2 = (a_0^2 + 2a_0a_1F + a_1^2F^2) \\ &(E^2)^{'} = C2a_0a_1F^2 + A2a_0a_1 + 2Ca_1^2F^3 + 2Aa_1^2F \\ &(E^2)^{''} = (4Ca_0a_1 + 6Ca_1^2F^2 + 2Aa_1^2)(CF^2 + A) \end{aligned} \tag{31}$$

And

$$\begin{aligned} &n^{'} = b_1F^{'} + 2b_2FF^{'} = b_1A + 2Ab_2F + b_1CF^2 + 2Ca_2F^3 \\ &n^{''} = (2Ab_2 + 2b_1CF + 6Cb_2F^2)(CF^2 + A) \end{aligned} \tag{32}$$

Replacing ing eqs. (5) and eq. (6), and again doing some algebra

$$\{a_0 = 0, b_1 = 0, a = \frac{-b_2 - a_1^2 k_2}{b_2}, k_1 = 0\} \tag{33}$$

$$\{a_0 = -\frac{i\sqrt{A}a_1}{\sqrt{C}}, b_0 = \frac{Ab_1 i}{2AC - i\sqrt{A}\sqrt{C}i}, b_2 = \frac{2\left(-2i\sqrt{A}b_1 C^{3/2} + b_1 Ci\right)}{4AC + i^2}, \tag{34}$$
$$a = -1, k_1 = -\frac{i\left(2\sqrt{A}C^{3/2} - iCi\right)}{2a_1}, k_2 = 0\}$$

$$\{a_0 = \frac{i\sqrt{A}a_1}{\sqrt{C}}, b_0 = \frac{Ab_1 i}{2AC + i\sqrt{A}\sqrt{C}i}, b_2 = \frac{2\left(2i\sqrt{A}b_1 C^{3/2} + b_1 Ci\right)}{4AC + i^2}, \tag{35}$$
$$a = -1, k_1 = \frac{i\left(2\sqrt{A}C^{3/2} + iCi\right)}{2a_1}, k_2 = 0\}$$

Then, we get eighteen families of solutions using Ricatti method.

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Following the solution proposed in [7]:

$$E(x,t) = f(\xi)e^{i(p\xi - qt)}, \quad \xi = x - at \tag{36}$$

And replacing in eqs. (5)-(6), we get:

$$\frac{d^2 f}{d\xi^2} + i(2p - a)\frac{df}{d\xi} + (q - p^2 + 2k_1 n)f = 0 \tag{37}$$

$$(a^2 - 1)\frac{d^2 n}{d\xi^2} - k_2 \frac{d^2 f^2}{d\xi^2} = 0 \tag{38}$$

And integrating two times in eq. (38), assuming the integration constants as zero, and replacing n in eq. (37), yields

$$\frac{d^2 f}{d\xi^2} + i(2p - a)\frac{df}{d\xi} + (q - p^2)f + \frac{2k_1 k_2}{a^2 - 1}f^3 = 0 \tag{39}$$

Also, applying the tanh method [5], used in section (2), we look for solutions as $f = a_0 + a_1 Y$, we obtain:

$$\left\{a_0 = 0, a_1 = -\frac{\sqrt{\mu^2 - a^2\mu^2}}{\sqrt{k_1}\sqrt{k_2}}, q = \frac{1}{4}\left(a^2 + 8\mu^2\right), p = \frac{a}{2}\right\}, \tag{40}$$

$$\left\{a_0 = 0, a_1 = \frac{\sqrt{\mu^2 - a^2\mu^2}}{\sqrt{k_1}\sqrt{k_2}}, q = \frac{1}{4}\left(a^2 + 8\mu^2\right), p = \frac{a}{2}\right\}, \tag{41}$$

defining $l_1 = a^2 i^2 - 12ai\mu + 36\mu^2 + 32i^2\mu^2$ and $l_2 = a^2 i^2 + 12ai\mu + 36\mu^2 + 32i^2\mu^2$

$$\left\{a_0 = -\frac{\sqrt{\mu^2 - a^2\mu^2}}{\sqrt{k_1}\sqrt{k_2}}, a_1 = -\frac{\sqrt{-\left(-1 + a^2\right)\mu^2}}{\sqrt{k_1}\sqrt{k_2}}, q = \frac{l_1}{4i^2}, p = \frac{ai - 6\mu}{2i}\right\}, \tag{42}$$

$$\left\{a_0 = \frac{\sqrt{\mu^2 - a^2\mu^2}}{\sqrt{k_1}\sqrt{k_2}}, a_1 = \frac{\sqrt{-\left(-1 + a^2\right)\mu^2}}{\sqrt{k_1}\sqrt{k_2}}, q = \frac{l_1}{4i^2}, p = \frac{ai - 6\mu}{2i}\right\}, \tag{43}$$

$$\left\{a_0 = -\frac{\sqrt{\mu^2 - a^2\mu^2}}{\sqrt{k_1}\sqrt{k_2}}, a_1 = \frac{\sqrt{-\left(-1 + a^2\right)\mu^2}}{\sqrt{k_1}\sqrt{k_2}}, q = \frac{l_2}{4i^2}, p = \frac{ai + 6\mu}{2i}\right\}, \tag{44}$$

$$\left\{a_0 = \frac{\sqrt{\mu^2 - a^2\mu^2}}{\sqrt{k_1}\sqrt{k_2}}, a_1 = -\frac{\sqrt{-\left(-1 + a^2\right)\mu^2}}{\sqrt{k_1}\sqrt{k_2}}, q = \frac{l_2}{4i^2}, p = \frac{ai + 6\mu}{2i}\right\}, \tag{45}$$

So, we obtain six families of solutions.

5 Conclusions

We solve the Langmuir wave equations using solitary wave methods. We get eleven families of solutions using Tanh method, and eighteen families of solutions using Ricatti solutions. Also, using the postulated solution, we obtain six families of solutions, again appliying the tanh method. As a future work, we can look for solutions of the model extended to higher dimensions.

$$\begin{aligned} E(x,t) &= a_0 + a_1 \tanh\left(\mu(x - at)\right), \\ n(x,t) &= b_0 + b_1 \tanh\left(\mu(x - at)\right) + b_2 \tanh\left(\mu(x - at)\right)^2 \end{aligned} \tag{46}$$

$$E(x,t) = a_0 + a_1 F(x - at),\ n(x,t) = b_0 + b_1 F(x - at) + b_2 F^2(x - at) \quad (47)$$

$$\begin{aligned} E(x,t) &= (a_0 + a_1 \tanh(\mu(x - at)))e^{i(p\xi - qt)}, \\ n(x,t) &= \frac{2k_1k_2}{a^2 - 1}(a_0 + a_1 \tanh(\mu(x - at)))^2 \end{aligned} \quad (48)$$

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