On the Stability of Weierstrass Type Method with King’s Correction for Finding all Roots of Non-Linear Function with Engineering Application

Saima Akram 1, Mudassir Shams 2, Naila Rafiq 3, Nazir Ahmad Mir 2

1 Center for Advanced Studies in Pure and Applied Mathematics Bahauddin Zakariya University, Multan, Pakistan
2 Department of Mathematics and Statistics Riphah International University I-14, Islamabad 44000, Pakistan
3 Department of Mathematics, NUML, Islamabad, Pakistan

This article is distributed under the Creative Commons by-nc-nd Attribution License. Copyright © 2020 Hikari Ltd.

Abstract

In this article, we present a new family of simultaneous iterative method for determining all the roots of non-linear equation. Using King’s family of iterative methods as corrections, we accelerate the convergence order of basic weierstrass method from 2 to 5. Convergence analysis, basins of attraction, computational efficiency, numerical test examples and log of residual demonstrate the performance and efficiency of the newly constructed simultaneous method as compared to other existing methods in the literature.

Keywords: Distinct Roots; Non-Linear Equation; Simultaneous Iterative Methods; Basins of Attraction; Computational Efficiency

1. INTRODUCTION

The problem of solving a non-linear equation is one of the oldest problem of science in general and in mathematics in particular. These non-linear equations have a diverse application in many areas of science and engineering. The methods for the simultaneous determination of all roots of a non-linear equation are important roots solvers, as they overcome the difficulty of the successive removal of the linear factor and have a very fast convergence. Apart from these, such methods have a wider region of convergence, are more stable and
can be applied to parallel computing. For details on simultaneous methods, their convergence order, computational efficiency and parallel implementation can be seen in [1, 2, 3, 4, 5, 6, 7, 9, 10, 14, 15, 16, 20, 24, 26, 27, 31].

The main aim of this paper is to develop family of simultaneous methods with higher convergence order having more efficiency than the existing simultaneous methods in the literature. Further, a very high computational efficiency is achieved by using the suitable correction due to King’s family of iterative [18] method of low computational efficiency in weierstrass function. The correction enable us to achieve a very fast convergence (equal to five) with minimal number of function evaluations in each step. Basins of attraction for different values of parameter involve in our family of simultaneous methods shows the better convergence properties as compared to existing methods ZPH of same convergence order. Computational efficiency and local computational order ($\rho(\mathbf x)$) [30] shows the dominance behavior in efficiency of our simultaneous method as compared to ZPH.

Consider non-linear equation, say:

$$f(\mathbf x) = 0.$$  \tag{1}

2. CONSTRUCTIONS OF SIMULTANEOUS METHODS

Here, we construct a family of fifth order simultaneous method which is more efficient than the similar methods existing in literature.

2.1. Construction of Simultaneous Method for Distinct Roots. Considering fourth order King’s [18] family of iterative method:

$$z^{(k)} = y^{(k)} - \left( \frac{f(x^{(k)}) + \beta f(y^{(k)})}{f(x^{(k)}) + (\beta - 2)f(y^{(k)})} \right) f'(y^{(k)}), k = 0, 1, 2, \ldots \tag{2}$$

where $y^{(k)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}$.

Weierstrass [11] presents the following simultaneous method of convergence order 2 as:

$$y_i^{(k)} = x_i^{(k)} - \frac{n}{\prod_{j=1}^{n} (x_i^{(k)} - x_j^{(k)})}, (i,j = 1, 2, 3, \ldots, n) \tag{3}$$

Replacing $x_j^{(k)}$ by $z_j^{(k)}$ in (3), we have:

$$y_i^{(k)} = x_i^{(k)} - \frac{n}{\prod_{j=1}^{n} (x_i^{(k)} - z_j^{(k)})}, (i,j = 1, 2, 3, \ldots, n) \tag{4}$$

where

$$z_j^{(k)} = y_j^{(k)} - \left( \frac{f(x_j^{(k)}) + \beta f(y_j^{(k)})}{f(x_j^{(k)}) + (\beta - 2)f(y_j^{(k)})} \right) f'(y_j^{(k)}),$$
and \( y_j^{(k)} = x_j^{(k)} - \frac{f(x_j^{(k)})}{f'(x_j^{(k)})} \).

Thus, we have constructed a new family of simultaneous method (4) for extracting all the distinct roots of non-linear equation (1).

2.2. Convergence Analysis. Here, we study the convergence analysis of family of simultaneous methods (4) in form of the following theorem:

**Theorem 1.** [1] Let \( \zeta_1, \zeta_2, ..., \zeta_n \) be \( n \) simple roots of non-linear equation (1). If \( x_1^{(0)}, x_2^{(0)}, ..., x_n^{(0)} \) be the initial approximations of the roots respectively and sufficiently close to actual roots, the convergence order of method (4) is five.

**Proof.** Let

\[
\begin{align*}
\epsilon_i & = x_i - \zeta_i, \\
\epsilon_i' & = y_i - \zeta_i,
\end{align*}
\]

be the errors in \( x_i \) and \( y_i \) approximations respectively. For simplification we omit iteration index. From (4), we have:

\[
\begin{align*}
y_i - \zeta_i & = x_i - \zeta_i - w(x_i), \\
\epsilon_i' & = i - \frac{w(x_i)}{i}, \\
\epsilon_i'' & = i(1 - G_i),
\end{align*}
\]

where \( w(x_i) = \frac{f(x_i)}{\prod_{j=1, j \neq i}^{n} (x_i - z_j)} \) and

\[
G_i = \frac{w(x_i)}{\prod_{j=1}^{n} (x_i - z_j)} = \frac{\prod_{j=1}^{n} (x_i - \zeta_j)}{\prod_{j=1, j \neq i}^{n} (x_i - z_j)} = \frac{(x_i - \zeta_i) \prod_{j=1}^{n} (x_i - \zeta_j)}{\prod_{j=1, j \neq i}^{n} (x_i - z_j)}.
\]

Now, if \( \zeta_i \) is a simple root, then for a small enough \( \epsilon \), \( |x_i - z_j| \) is bounded away from zero, and so

\[
\frac{x_i - \zeta_j}{x_i - z_j} = \frac{x_i - z_j + z_j - \zeta_j}{x_i - z_j} = 1 + \frac{z_j - \zeta_j}{x_i - z_j} = 1 + O(\epsilon),
\]
where \( z_j - \zeta_j = O (4) \) see [18]:

\[
\prod_{j \neq i}^{n} \left( \frac{x_i - \zeta_j}{x_i - z_j} \right) = (1 + O (4))^{n-1} = 1 + (n - 1) O (4) = 1 + O (4).
\]

Implies that:

\[
G_i = 1 + O (4),
\]

\[
G_i - 1 = O (4).
\]

Thus, (6) gives:

\[
\epsilon_i' = O (5).
\] (7)

which shows that convergence order of method (4) is five. Hence the theorem.

2.3. Complex Dynamical Study of Families of Simultaneous Iterative Methods. Here, we discuss the dynamical study of iterative methods NMS and ZPH. Let us recall some basic concepts of this study in the background context of complex dynamics. For more details on the dynamical behavior of the iterative methods one can consult [12, 25, 28]. Taking a rational function \( \mathbb{R}_f : \mathbb{C} \rightarrow \mathbb{C} \), where \( \mathbb{C} \) denotes the complex plane, the orbit \( x_0 \in \mathbb{C} \) is defined as a set such as \( \text{orb}(x) = \{ x_0, \mathbb{R}_f(x_0), \mathbb{R}_f^2(x_0), ..., \mathbb{R}_f^m(x_0), ... \} \). The convergence \( \text{orb}(s) \rightarrow x^* \) is understood in the sense if \( \lim_{k \rightarrow \infty} \mathbb{R}_f(x) = x^* \) exist. A point is \( x_0 \in \mathbb{C} \) known as attracting if \( |\mathbb{R}_f'(x)| < 1 \). An attracting point \( x^* \in \mathbb{C} \) defines basin of attraction \( \mathbb{R}(x^*) \), as the set of starting points whose orbit tends to \( x^* \). To generate basins of attraction, we take grid \( 2000 \times 2000 \) of square \( [-2.5 \times 2.5]^2 \in \mathbb{C} \). To each root of (1), we assign a color to which the corresponding orbit of the iterative methods starts and converges to a fixed point. Take color map as Jet. We use \( |f(x_i)| < 10^{-5} \) as a stopping criteria and maximum number of iteration is taken as 5 due to global convergence properties of simultaneous iterative methods as compared to single root find method see [13, 17, 21, 22, 23, 25]. Dark black point are assigns if the orbit of the simultaneous iterative methods does not converges to root after 5 iterations. we obtaine basins of attractions for the following three test function \( f_1(x) = x^5 - x^2 + x + 125 \), \( f_2(x) = e^x - x^2 + 25 \) and \( f_3(x) = \sin \left( \frac{x - 1}{2} \right) \sin \left( \frac{x - 2}{2} \right) \sin \left( \frac{x - 2.5}{2} \right) \). The exact root of \( f_1(x) \) is \(-5.00072\), exact root of \( f_2(x) \) are \(2.1 + 1.5i, 2.1 - 1.5i, -0.8 + 2.5i, -0.8 - 2.5i, -2.5\) and root of \( f_3(x) \) are \(1, 2, 2.5\). Brightness in color means less number of iterations steps. Finally, in Table 1, we present Elapsed time of basins of attraction correspond to iterative map NMS and ZPH using tic toc command in MATLAB (R2011b).

| Table 1: Elapsed Time in Seconds |
|-----------------|-----------------|-----------------|
| Methods | \( f_1(x) \) | \( f_2(x) \) | \( f_3(x) \) |
| ZPH     | 23.90345       | 2.110872       |
|         | Fig. 1(a)      | Fig. 1(c)       |
| NMS     | 2.884889       | 28.52985       |
|         | Fig. 1(b)      | Fig. 1(d)       |

|                     | 6.893102       | 9.757365       |
|                     | Fig. 1(e)      | Fig. 1(f)       |
Fig. 1(a)

Fig. 1(b)

Fig. 1(c)
Fig. 1(d)

Fig. 1(e)

Fig. 1(f)
Fig. 1(a,b), presents the basins of attraction of Methods NMS and ZPH for non-linear function \( f_1(s) \), Fig. 1(c,d), presents the basins of attraction of Methods NMS and ZPH for non-linear function \( f_2(s) \), Fig. 1(e,f), presents the basins of attraction of Methods NMS and ZPH for non-linear function \( f_3(s) \) respectively. Table 1, clearly shows that the elapsed time of NMS is less than ZPH, showing dominance efficiency and convergence behavior of our newly constructed method (NMS) over ZPH.

3. COMPUTATIONAL ASPECT

Here, we compare the computational efficiency and convergence behavior of the ZPH. [8] method and the new method (4). As presented in [21], the efficiency of an iterative method can be estimated using the efficiency index given by

\[
EF(m) = \frac{\log \bar{\alpha}}{D},
\]

where \( D \) is the computational cost and \( \bar{\alpha} \) is the order of convergence of the iterative method. Using arithmetic operation per iteration with certain weight depending on the execution time of operation to evaluate the computational cost \( D \). The weights used for division, multiplication and addition plus subtraction are \( \bar{w}_d, \bar{w}_s \) and \( \bar{w}_{\text{AS}} \) respectively. For a given polynomial of degree \( m \) and \( n \) roots, the number of division, multiplication addition and subtraction per iteration for all \( m \) roots are denoted by \( D_m, M_m \) and \( \bar{\alpha}S_m \). The cost of computation can be calculated as:

\[
D = D(m) = \bar{w}_{\text{AS}}\bar{\alpha}S_m + \bar{w}_mM_m + \bar{w}_dD_m,
\]

thus (8) becomes:

\[
EF(m) = \frac{\log \bar{\alpha}}{\bar{w}_{\text{AS}}\bar{\alpha}S_m + \bar{w}_mM_m + \bar{w}_dD_m}.
\]

Considering the number of operations of a complex polynomial with real and complex roots reduce to operation of real arithmetic, given in Table 2 as a polynomial of degree \( m \) taking the dominion term of order \( (m^2) \). Applying (10) and data given in Table 2, we calculate the percentage ratio \( \rho((4), (ZPH)) \) [21] given by:

\[
\rho((4), (ZPH)) = \left( \frac{EF(4)}{EF(ZPH)} - 1 \right) \times 100
\]

\[
\rho((ZPH), (4)) = \left( \frac{EF(ZPH)}{EF(4)} - 1 \right) \times 100
\]

Fig. 2(a), shows computational efficiency of methods (NMS) w.r.t method ZPH, and Fig. 2(b), shows computational efficiency of method ZPH w.r.t
Fig. 2(a)

Fig. 2(b)

(NMS). Figure 2(a,b) clearly shows that the newly constructed simultaneous method (4) is more efficient as compared to ZPH method.

Table 2: The number of basic arithmetic operations

<table>
<thead>
<tr>
<th>Methods</th>
<th>$\tilde{r}$</th>
<th>$\tilde{A}_{SM}$</th>
<th>$M_m$</th>
<th>$D_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NMS</td>
<td>5</td>
<td>$7m^2 + O(m)$</td>
<td>$7m^2 + O(m)$</td>
<td>$2m^2 + O(m)$</td>
</tr>
<tr>
<td>ZPH</td>
<td>5</td>
<td>$11m^2 + O(m)$</td>
<td>$3m^2 + O(m)$</td>
<td>$2m^2 + O(m)$</td>
</tr>
</tbody>
</table>

4. NUMERICAL RESULTS

Here, some numerical examples are considered in order to demonstrate the performance of our newly constructed simultaneous methods, namely NMS. We compare our method with ZPH [8] method of convergence order five (abbreviated as ZPH). All the computations are performed using Maple 18 with 64 digits floating point arithmetic. We take $\varepsilon = 10^{-30}$ as tolerance and use the following stopping criteria:

$$
\xi^{(k)} = \left| f \left( x_i^{(k+1)} \right) \right| < \varepsilon,
$$

where $\xi^{(k)}$ represents the absolute error of function values. Numerical tests examples from [19, 20, 25] are provided in Tables 4.1-4.5 and 4.6. In all Tables,
CO represents the convergence order, n represents the number of iterations, \( \rho^{(k)} \) local computational order of convergence and CPU represents execution time in seconds for approximating roots of (1). The figures 3, show that residue fall of the methods NMS and ZPH for the examples 1–3, shows that methods NMS is more efficient as compared to ZPH. We observe that numerical results of the method NMS are better than ZPH method on same number of iteration.

Example 1: Fractional Conversion [25] (An Engineering Application)

As expression describe in [28, 29]

\[
f_4(x) = x^4 - 7.79075x^3 + 14.7445x^2 + 2.511x - 1.674
\]

is the fractional conversion of nitrogen, hydrogen feed at 250 atm. and 227k. The exact roots of (1) is \( \zeta_1 = 3.9485 + 0.3161i \), \( \zeta_2 = 3.9485 - 0.3161i \), \( \zeta_3 = -0.3841 \), \( \zeta_4 = 0.2778 \). The initial estimates for \( f_4(x) \) are taken as:

\[
(x_1^{(0)} = 3.5 + 0.3i, \quad x_2^{(0)} = 3.5 - 0.3i, \quad x_3^{(0)} = -0.3 + 0.01i, \quad x_4^{(0)} = 1.8 + 0.01i)
\]

<table>
<thead>
<tr>
<th>Method</th>
<th>CO</th>
<th>CPU</th>
<th>n</th>
<th>( e_1^{(k)} )</th>
<th>( e_2^{(k)} )</th>
<th>( e_3^{(k)} )</th>
<th>( e_4^{(k)} )</th>
<th>( \rho_1^{(k)} )</th>
<th>( \rho_2^{(k)} )</th>
<th>( \rho_3^{(k)} )</th>
<th>( \rho_4^{(k)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>ZPH</td>
<td>5</td>
<td>0.015</td>
<td>1</td>
<td>3.1</td>
<td>3.2</td>
<td>0.1</td>
<td>18.5</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2</td>
<td>0.5</td>
<td>0.5</td>
<td>4.5e-3</td>
<td>10.9</td>
<td>-0.59</td>
<td>-0.59</td>
<td>2.32</td>
<td>0.81</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>3</td>
<td>0.019</td>
<td>0.016</td>
<td>1.7e-6</td>
<td>1.3</td>
<td>5.71</td>
<td>5.92</td>
<td>2.45</td>
<td>0.10</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>4</td>
<td>5.5e-6</td>
<td>3.5e-6</td>
<td>6.4e-14</td>
<td>2.6e-4</td>
<td>3.05</td>
<td>3.03</td>
<td>2.23</td>
<td>0.01</td>
</tr>
<tr>
<td>NMS</td>
<td>5</td>
<td>0.012</td>
<td>1</td>
<td>2.3</td>
<td>2.3</td>
<td>1.5</td>
<td>0.4</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2</td>
<td>0.3</td>
<td>0.2</td>
<td>0.02</td>
<td>0.05</td>
<td>-1.30</td>
<td>-1.90</td>
<td>-9.61</td>
<td>3.23</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>3</td>
<td>2.1e-1</td>
<td>2.6e-4</td>
<td>1.1e-6</td>
<td>3.1e-6</td>
<td>7.61</td>
<td>5.129</td>
<td>3.52</td>
<td>4.22</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>4</td>
<td>7.1e-17</td>
<td>4.5e-17</td>
<td>2.8e-20</td>
<td>9.0e-20</td>
<td>4.39</td>
<td>4.51</td>
<td>3.21</td>
<td>3.45</td>
</tr>
</tbody>
</table>

Example 2 [19]: Consider

\[
f_5(x) = (x + 1)(x - 2)(x - 1 - i)(x - 1 + i)
\]

with exact roots:

\[
\zeta_1 = -1, \quad \zeta_2 = 2, \quad \zeta_3 = 1 + i, \quad \zeta_4 = 1 - i.
\]

The initial approximations have been taken as:

\[
(x_1^{(0)} = -1.1 + 0.2i, \quad x_2^{(0)} = 2.1 - 0.2i, \quad x_3^{(0)} = 0.8 + 1.2i, \quad x_4^{(0)} = 0.9 - 1.2i)
\]
Example 3 [20]: Consider

\[ f_6(x) = (e^{x-1}(x-2)(x-3) - 1), \]

with exact roots:

\[ \zeta_1 = 0, \quad \zeta_2 = 1, \quad \zeta_3 = 2, \quad \zeta_4 = 3. \]

The initial approximations have been taken as:

\[ (x_1^{(0)} = 0.1, \quad x_2^{(0)} = 0.9, \quad x_3^{(0)} = 1.8, \quad x_4^{(0)} = 2.9. \]
Fig. 3(a-c), shows residual fall of iterative methods (NMS) and ZPH for non-linear function $f_4(s)$, $f_5(s)$ and $f_6(s)$ respectively.

5. CONCLUSION

We have developed here a family of simultaneous method of order five for determination of all the distinct roots of non-linear equation, namely NMS. From Tables 3 – 6 and Figures 1-3, we observe that our method is superior in term of efficiency, CPU time and residual error as compared to the existing method ZPH.

REFERENCES


Received: March 30, 2020; Published: July 4, 2020