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# Inverse Problems for Sturm-Liouville-Type Operators with Delay: Symmetric Case

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**Abstract**

We study the inverse problem of recovering the potential $q(x)$ from the spectrum of the operator $-y''(x)+q(x)y(x-a)$, $y(0)=y^{(k)}(\pi)=0$, where $k\in\{0,1\}$ and $a\in[\frac{\pi}{3},\frac{2\pi}{5})$ is a delay. It is proven that a potential is an uniquely determined from two spectra, when a potential is symmetric on a subinterval.

**Mathematics Subject Classification:** 34A55, 34K29, 45D05

**Keywords:** Inverse problem, differential operators with constant delay, Volterra integral operator

# 1 Introduction

In this paper we study two boundary spectral problems $L_k=L_k(q,a)$, $k=0,1$ generated by

$$-y''(x)+q(x)y(x-a)=\lambda y(x),\ \ x\in(0,\pi) \tag{1}$$

$$y(0)=0 \tag{2}$$

$$y^{(k)}(\pi)=0 \tag{3}$$

where $\lambda$ is a spectral parametar, $a\in[\frac{\pi}{3},\frac{2\pi}{5})$ is a delay, $q(x)$ is a complex-valued function such that $q\in L^2(0,\pi)$ and $q(x)\equiv 0$ for $x\in[0,a]$. The boundary value problem $L_k$, $k=0,1$ has a countable set of eigenvalues $(\lambda_{nk})_{n=1}^{\infty}$ and for $n\to\infty$ (see [2])

$$\sqrt{\lambda_{nk}} = (n - \tfrac{k}{2}) + \tfrac{\cos(a(n-\frac{k}{2}))}{2\pi n} \int\limits_a^\pi q(t)dt + o(\tfrac{1}{n}), (n \to \infty).$$

We are interested to recover the potential $q(x)$ from those spectra. More about differential equations with deviating argument can be found in [5]. The case $a \in (\frac{\pi}{2}, \pi)$ has been processed in [3] and [7]. Inverse problem for delay $a \in (\frac{2\pi}{5}, \frac{\pi}{3})$ can be found in [2] and [6]. In [1] and [4] authors studied incomplete inverse problems differetial equation with a deviating argument. Inverse problems for differential operators with two delays can be found in [8].

## 2 Main results

Let the function $Y(x, \lambda)$ be a solution of the differential equation (1) which satisfying intial conditions $Y(0, \lambda) = 0$, $Y'(0, \lambda) = 1$. Let $\lambda = z^2$, the eigenvalues $(\lambda_{nk})_{n=1}^\infty$ of the boundary value problem $L_k, k = 0, 1$ coincide with the zeros of its characteristic function $\Delta_k(\lambda) = Y^{(k)}(\pi, \lambda)$, where (see [4] and [5])

$$Y(\pi, \lambda) = \frac{\sin(z\pi)}{z} + \frac{1}{z^2} \int\limits_a^\pi q(t) \sin(z(t-a)) \sin(z(\pi - t))dt +$$

$$\frac{1}{z^3} \int\limits_{2a}^\pi \int\limits_a^{t-a} q(t)q(t_1) \sin(z(\pi - t)) \sin(z(t_1 - a)) \sin(z(t - a - t_1))dt_1 dt, \quad (4)$$

$$Y'(\pi, \lambda) = \cos(z\pi) + \frac{1}{z} \int\limits_a^\pi q(t) \sin(z(t-a)) \cos(z(\pi - t))dt +$$

$$\frac{1}{z^2} \int\limits_{2a}^\pi \int\limits_a^{t-a} q(t)q(t_1) \cos(z(\pi - t)) \sin(z(t_1 - a)) \sin(z(t - a - t_1))dt_1 dt. \quad (5)$$

The delay $a$ and the integral $I_1 = \int\limits_a^\pi q(t)dt$, are uniquely determined by the spectrum $(\lambda_{n0})_{n=1}^\infty$ of $L_0$ (see [5]). The functions $\Delta_k(\lambda)$, given by (4-5), are uniquely determined by the spectra $(\lambda_{nk})_{n=1}^\infty$ of $L_k$, $k = 0, 1$ (see[2]).
We introduce a notation

$$I_2 = \int\limits_{2a}^\pi q(t) \int\limits_a^{t-a} q(s)dsdt \ , \ F_0(z) = z\Delta_0(\lambda), F_1(z) = z\Delta_1(\lambda).$$

Using the well-known method transformation of characteristic functions (4-5) we obtain (see [4])

$$F_0(z) = \sin(z\pi) + \frac{1}{2z} \left(\widetilde{a_c}(z) - \cos(z(\pi - a))I_1\right) - \frac{1}{2z} K_c^*(z) - \frac{1}{2z^2} I_2 \sin(z(\pi - 2a)) \quad (6)$$

$$F_1(z) = z\cos(z\pi) + \frac{1}{2}\left(-\widetilde{a_s}(z) + \sin(z(\pi - a))I_1\right) + \frac{1}{2}K_s^*(z) \tag{7}$$

where

$$\widetilde{a_c}(z) = \int\limits_{\frac{a}{2}}^{\pi-\frac{a}{2}} \widetilde{q}(t)\cos(z(\pi - 2t))dt, \widetilde{a_s}(z) = \int\limits_{\frac{a}{2}}^{\pi-\frac{a}{2}} \widetilde{q}(t)\sin(z(\pi - 2t))dt.$$

$$\widetilde{q}(t) = \left\{ \begin{array}{c} q\left(t + \frac{a}{2}\right); t \in \left[\frac{a}{2}, \pi - \frac{a}{2}\right] \\ 0; t \in \left(0, \frac{a}{2}\right) \cup \left(\pi - \frac{a}{2}, \pi\right) \end{array} \right\}$$

$$K_c^*(z) = \int\limits_a^{\pi-a}\int\limits_a^t K(s)\cos(z(\pi - 2t))dsdt,\ K_s^*(z) = \int\limits_a^{\pi-a}\int\limits_a^t K(s)\sin(z(\pi - 2t))dsdt,$$

$$K(t) = \left\{ \begin{array}{c} q(t + a)\int\limits_a^t q(s)ds - q(t)\int\limits_{t+a}^{\pi} q(s)ds - \int\limits_{t+a}^{\pi} q(s - t)q(s)ds; t \in [a, \pi - a] \\ 0; t \in [0, a) \cup (\pi - a, \pi] \end{array} \right\}$$

It is obvious that $\int_a^{\pi-a} K(s)ds = -I_2$, accordingly

$$\int_a^t K(s)ds = -I_2 - \int_t^{\pi-a} K(s)ds \tag{8}$$

Substituting expression from (8) in (6),(7) and after arranging we come to

$$2zF_0(z) - 2z\sin(z\pi) + 2z\cos(z(\pi - a))I_1 = \widetilde{a_c}(z) + L_c^*(z) \tag{9}$$

$$-2F_1(z) + 2z\cos(z\pi) + 2\sin(z(\pi - a))I_1 = \widetilde{a_s}(z) + L_s^*(z) \tag{10}$$

where

$$L_c^*(z) = \int\limits_a^{\pi-a}\int\limits_t^{\pi-a} K(s)\cos(z(\pi-2t))dsdt, L_s^*(z) = \int\limits_a^{\pi-a}\int\limits_t^{\pi-a} K(s)\sin(z(\pi-2t))dsdt.$$

We use a notation

$$\begin{array}{c} A(z) = 2zF_0(z) - 2z\sin(z\pi) + \cos(z(\pi - a))I_1, \\ B(z) = -2F_1(z) + 2z\cos(z\pi) + \sin(z(\pi - a))I_1. \end{array}$$

According to (9) and (10) for $z = m, m \in \mathbb{Z} \setminus \{0\}$ we get:

$$(-1)^m A(m) = \int\limits_{\frac{a}{2}}^{\pi-\frac{a}{2}} \widetilde{q}(t)\cos(2mt)dt + \int\limits_a^{\pi-a}\int\limits_t^{\pi-a} K(s)\cos(2mt)dsdt \tag{11}$$

$$(-1)^{m+1}B(m)=\int\limits_{\frac{a}{2}}^{\pi-\frac{a}{2}}\widetilde{q}(t)\sin(2mt)dt+\int\limits_{a}^{\pi-a}\int\limits_{t}^{\pi-a}K(s)\sin(2mt)dsdt \tag{12}$$

Also from (9) we have

$$\lim_{z\to 0}A(z)=\int\limits_{\frac{a}{2}}^{\pi-\frac{a}{2}}\widetilde{q}(t)dt+\int\limits_{a}^{\pi-a}\int\limits_{t}^{\pi-a}K(s)dsdt \tag{13}$$

Multiplying (13) with $\frac{1}{\pi}$, (11) with $\frac{1}{\pi}e^{2imt}$ and (12) with $\frac{-i}{\pi}e^{2imt}$ for $m\in\mathbb{Z}\setminus\{0\}$, then summing we obtain the integral equation

$$\widetilde{q}(t)+\int\limits_{t}^{\pi-a}K(s)ds\mathbf{1}_{(a,\pi-a)}(t)=f(t) \tag{14}$$

where

$$f(t)=\tfrac{1}{\pi}\lim_{z\to 0}A(z)+\tfrac{1}{\pi}\textstyle\sum_{m\in\mathbb{Z}\setminus\{0\}}((-1)^mA(m)+i(-1)^mB(m))e^{2imt}.$$

Function $f(t)$ is uniquely determined by spectra $(\lambda_{nk})_{n=1}^{\infty}$ of $L_k$, $k=0,1$.
Putting $t=x-\frac{a}{2}$ in (14) we obtain (see [4], eq.(25)):
for $x\in(\frac{3a}{2},\pi-\frac{a}{2})$

$$q(x)+\int\limits_{a}^{x-\frac{a}{2}}q_1(s)\int\limits_{x+\frac{a}{2}}^{\pi}q_2(t)dtds-\int\limits_{a}^{\pi-x+\frac{a}{2}}q_1(s)\int\limits_{s+x-\frac{a}{2}}^{\pi}q_2(t)dtds=f\left(x-\frac{a}{2}\right) \tag{15}$$

and

$$q(x)=f(x-\frac{a}{2}),\;\; x\in(a,\frac{3a}{2})\cup(\pi-\frac{a}{2},\pi)$$

where $q_1(x)=q(x)$, for $x\in(a,\pi-a)$, $q_2(x)=q(x)$ for $x\in(2a,\pi)$.

**Theorem 2.1** *(Theorem in [4]) The potential* $q(x)$, $x\in[0,\frac{3a}{2})\cup(\pi-a,2a)\cup(\pi-\frac{a}{2},\pi]$, *is uniquely determined by spectra* $(\lambda_{nk})_{n=1}^{\infty}$ *of boundary spectral problems* $L_k$, $k=0,1$.

**Theorem 2.2** *Let* $q(x)=-q(\pi-x+\frac{a}{2})$ *for* $x\in(\frac{3a}{2},\pi-a)$ *then the potential* $q(x)$ *is uniquely determined by spectra* $(\lambda_{nk})_{n=1}^{\infty}$ *of boundary spectral problems* $L_k, k=0,1$.

**Proof:** Using the fact that $q(x)$ is known for $x \in [0, \frac{3a}{2}) \cup (\pi - a, 2a) \cup (\pi - \frac{a}{2}, \pi)$, integral equation (15) can be transformed (see [4], equations (27) and (28)), for $x \in (\frac{3a}{2}, \pi - a)$

$$q_1(x) + \int\limits_{x+\frac{a}{2}}^{\pi-\frac{a}{2}} q_2(s) K_1(s,x) ds - \int\limits_{\frac{3a}{2}}^{\pi-x+\frac{a}{2}} q_1(s) K_2(s + x - \frac{a}{2}) ds = Q_1(x) \qquad (16)$$

where $K_1(s,x) = \int\limits_{s-x+\frac{a}{2}}^{x-\frac{a}{2}} q_1(t)dt$, $K_2(s + x - \frac{a}{2}) = \int\limits_{s+x-\frac{a}{2}}^{\pi} q_2(t)dt$ and $Q_1(x)$ are known. For $x \in (2a, \pi - \frac{a}{2})$

$$q_2(x) + K_2(x + \frac{a}{2}) \int\limits_{\frac{3a}{2}}^{x-\frac{a}{2}} q_1(s)\, ds = Q_2(x) \qquad (17)$$

where $K_2(x + \frac{a}{2}) = \int\limits_{x+\frac{a}{2}}^{\pi} q_2(t)dt$ and $Q_2(x)$ are known. Since $\int_{\frac{3a}{2}}^{\pi-a} q_1(s)ds = 0$, from (17) we obtain $q_2(x) = K_2(x + \frac{a}{2}) \int\limits_{x-\frac{a}{2}}^{\pi-a} q_1(s)\, ds + Q_2(x)$.

Now putting $q_2(x)$ from last equation in (16) and using information that $q(x) = -q(\pi - x + \frac{a}{2})$, $x \in (\frac{3a}{2}, \pi - a)$, we obtain the Volterra integral equation

$$q_1(x) + \int_x^{\pi-a} q_1(t) \int_{x+\frac{a}{2}}^{t+\frac{a}{2}} K_2(s + \frac{a}{2}) K_1(s,x) ds dt + \int_x^{\pi-a} q_1(t) K_2(\pi - t + x) dt$$

$$= -\int_{x+\frac{a}{2}}^{\pi-\frac{a}{2}} Q_2(s) K_1(s,x) ds + Q_1(x) \qquad (18)$$

Since Volterra integral equation (18) has unique solution, we can calculate $q_1(x)$ for $x \in (\frac{3a}{2}, \pi - a)$. Then from (17) we calculate $q_2(x)$ for $x \in (2a, \pi - \frac{a}{2})$. □

**Remark 3.3** The similar result with the same method can be obtained if $q(x) = q(\pi - x + a/2)$, for $x \in (\frac{3a}{2}, \pi - a)$ and integral $\int_{\frac{3a}{2}}^{\pi-a} q_1(x)dx$ is known.

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