Pythagorean Triangle and Geometrical Interpretation of Prime Numbers

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Abstract

A geometrical interpretation of prime numbers could be useful in attempting related, still unresolved problems. The current investigation is conceived as a step towards this direction. First, Pythagorean table in $\mathbb{R}^2$ is extended to products of $n$ factors in $\mathbb{R}^n$. Hence Pythagorean triangle is defined and extended from $\mathbb{R}^2$ to $\mathbb{R}^n$, and related properties are shown. Then Pythagorean triangle and its basis are considered within the framework of polytopes, and natural points i.e., with coordinates equal to natural numbers, are counted with regard to special regions. An application is made to 12-note (extended to $M$-note) equal temperament musical scale and to inclined, regular $m$-tethraedron in $\mathbb{R}^{m-1}$. Finally, a geometrical interpretation of prime numbers, in connection with Pythagorean triangle, is presented and extended to $[1/(n - 1)]$-prime numbers in $\mathbb{R}^n$, where the reason for which unity cannot be prime number, or $[1/(n - 1)]$-prime number, is clearly explained. Prime numbers are generated in sequence starting from 7 via a simple algorithm. Pythagorean triangle is represented in a reduced form, where the infinity of natural numbers is trapped within a limited region of the plane, by use of suitable transformations. Goldbach’s conjecture is reformulated in three different versions: two, involving Pythagorean triangle; a third one, involving a semiaxis carrying natural numbers and its counterpart where the orientation is reversed. In connection with the former alternative, it is shown Bertrand’s postulate (Chebyshev’s theorem) makes a necessary condition for the validity of Goldbach’s conjecture.

Keywords: Pythagorean table; Pythagorean triangle; Pythagorean polytope; prime numbers

1 Introduction

Euclidean geometry can be conceived as an idealization of real physical space and closely relates to physical sciences e.g., cristallography, atomic physics, general relativity. Accordingly, a geometrical interpretation could help in solving open questions, keeping in mind science discoveries are owing to mere intuition, while methodology is used later in theory formulation. In particular, a considerable amount of problems are currently debated with regard to prime numbers, in spite of their conceptual simplicity e.g., Riemann hypothesis, Goldbach’s conjecture, twin primes conjecture.

The present investigation concerns a geometrical interpretation of prime numbers involving, among others, Pythagorean table, Pythagorean triangle, Pythagorean polytope. The main steps are listed below.

I. Pythagorean table is redefined as correspondence between codomain and domain of the product function, and then extended from Euclidean plane, \( \mathbb{R}^2 \), to Euclidean \( n \)-dimension plane, \( \mathbb{R}^n \).

II. Pythagorean triangle is extracted from Pythagorean table, an extension is made from Euclidean plane, \( \mathbb{R}^2 \), to Euclidean \( n \)-dimension plane, \( \mathbb{R}^n \), and related properties are derived.

III. Pythagorean triangle and selected regions, extended from Euclidean plane, \( \mathbb{R}^2 \), to Euclidean \( n \)-dimension plane, \( \mathbb{R}^n \), are considered as special cases of polytopes (Pythagorean polytopes), related properties are derived, and natural points, i.e. with coordinates equal to natural numbers, are counted for each selected region.

IV. A geometric interpretation of prime numbers in \( \mathbb{R}^2 \) is presented in connection with Pythagorean triangle and extended to \( \lfloor 1/(n - 1) \rfloor \)-prime numbers in \( \mathbb{R}^n \), where the reason for which unity is not prime number, or \( \lfloor 1/(n - 1) \rfloor \)-prime number, is clearly explained. In addition, an algorithm is formulated for determining the sequence of prime numbers starting from 7.

V. A reduced representation of Pythagorean triangle is shown, where the infinity of natural numbers is trapped within a limited region of the plane.

VI. Goldbach’s conjecture is reformulated in a twofold manner in connection with (a) Pythagorean triangle (two versions), and (b) the semiaxis carrying natural numbers and its counterpart with reversed orientation (one version). In the former alternative, it is shown Bertrand’s postulate (Chebyshev’s theorem) makes a necessary condition for the validity of Goldbach’s conjecture.
2 Pythagorean table

In extending usual Pythagorean table, related to the product of 2 factors within $\mathbb{R}^2$, to Pythagorean table, related to the product of $n$ factors within $\mathbb{R}^n$, some notation is needed. Accordingly, entities defined within $\mathbb{R}^2$ shall be preceded by the prefix, 2-, and their counterparts defined within $\mathbb{R}^n$ shall be preceded by the prefix, $n$-. For instance, it shall be written 2-sector instead of bisector, and $n$-sector; 2-ant instead of quadrant, and $n$-ant; $2^{-1}$-perimeter instead of half perimeter, and $2^{-(n-1)}$-perimeter.

In first place, let 2-Pythagorean table be reconsidered.

2.1 2-Pythagorean table

With regard to a Cartesian orthogonal reference frame in $\mathbb{R}^3$, $(Ox_1x_2x_3)$, let $x_3 = f(x_1, x_2) = |x_1x_2|$ be the (absolute) product function, which is defined on the domain, $x_3 = 0$. Minimum value occurs on the coordinate axes, $x_2 = x_3 = 0$, $x_1 = x_3 = 0$, and maximum value occurs for $|x_1| = |x_2|$ with regard to fixed ($|x_1| + |x_2|$), keeping in mind 2-square exhibits larger 2-area with respect to 2-rectangle of equal 2$^{-1}$-perimeter, $p_2 = |x_1| + |x_2|$. More specifically, $p_2$ is an increasing function of 2-rectangle “squareness”, $s_2 = \min(x_1, x_2)/\max(x_1, x_2)$, $0 \leq s_2 \leq 1$, where the lower value relates to 1-rectangle i.e. $\min(x_1, x_2) = 0$. In particular, integer points, $(i_1, i_2)$, integers, make a discrete sub domain of the product function, and related discrete sub codomain is $|i_1i_2|$.

From this point on, attention shall be restricted to positive 2$^2$-ant of the principal plane, $(Ox_1x_2)$, where integer points reduce to natural points, $i_1$, $i_2$, naturals, and the point, $(i_1, i_2)$, is related to the natural number, $i_1i_2$, as sketched in Fig. 1.

The generic horizontal straight line, $x_1 = i_1$, and the generic vertical straight line, $x_2 = i_2$, are nothing but $i_1$ and $i_2$ multiplication tables, respectively. The 2-sector of positive 2$^2$-ant, $x_1 = x_2$, carries the sequence of perfect 2-squares, starting from 0.

The 1-triangle of vertexes, $[0, (k + 1)], [(k + 1), 0]$, normal to the 2-sector of positive 2$^2$-ant, carries 2-areas of perfect 2-rectangles exhibiting 2$^{-1}$-perimeter, $p_2 = k + 1$.

The above mentioned 2-areas are equal for symmetric points with respect to the 2-sector, i.e. basis and height lengths of perfect 2-rectangle are interchanged, monotonically increasing from null values on coordinate axes to largest values on points less distant from the 2-sector on each side. Symmetry
Figure 1: Representation of 2-Pythagorean table as correspondence between discrete codomain and domain of the product function, \( f(i_1, i_2) = i_1i_2 \), where the orientation of coordinate axes conforms to usual notation of standard Pythagorean table. 1-triangles of vertexes, \([0, (k + 1)], [(k + 1), 0] \), \( 2 \leq k + 1 \leq 21 \), normal to 2-sector of positive \( 2^2 \)-ant, make the locus of points where \( 2^{-1} \)-perimeter of perfect 2-rectangle, \( p_2 = i_1 + i_2 = k + 1 \), is constant. 1-triangles of the kind considered, carrying points not appearing in the figure, are not shown. The 2-sector of positive \( 2^2 \)-ant carries the sequence of perfect 2-squares.
is owing to 2-area invariance with respect to the exchange between basis and height length.

Straight lines parallel to the 2-sector of positive 2\(^2\)-ant, \(x_2 = x_1 \mp \ell\), carry 2-areas of perfect 2-rectangles exhibiting 2\(^{-1}\)-perimeter, 2\(i_1 \mp \ell\).

The situation is shown in Fig. 2 where product values, \(i_1i_2\), related to codomain of the product function, are placed on points of the domain, \((i_1, i_2)\), that is 2-Pythagorean table.

### 2.2 3-Pythagorean table

With regard to a Cartesian orthogonal reference frame in \(\mathbb{R}^4\), \((Ox_1x_2x_3x_4)\), let \(x_4 = f(x_1, x_2, x_3) = |x_1x_2x_3|\) be the (absolute) product function, which is defined on the domain, \(x_4 = 0\). Minimum value occurs on the coordinate axes, \(x_2 = x_3 = x_4 = 0\), \(x_1 = x_3 = x_4 = 0, x_1 = x_2 = x_4 = 0\), and maximum value occurs for \(|x_1| = |x_2| = |x_3|\) with regard to assigned \((|x_1| + |x_2| + |x_3|)\), keeping in mind 3-square exhibits larger 3-area with respect to 3-rectangle of equal 2\(^{-2}\)-perimeter, \(p_3 = |x_1| + |x_2| + |x_3|\).

More specifically, \(p_3\) is an increasing function of 3-rectangle “squareness”, \(s_3 = \min(x_1, x_2, x_3)/\max(x_1, x_2, x_3), 0 \leq s_3 \leq 1\), where the lower value relates to 1-rectangle and 2-rectangle i.e. \(\min(x_1, x_2, x_3) = 0\). In particular, integer points, \((i_1, i_2, i_3)\), \(i_1, i_2, i_3\), integers, make a discrete sub domain of the product function, and related discrete sub codomain is \(|i_1i_2i_3|\).

From this point on, attention shall be restricted to positive 2\(^3\)-ant of the principal 3-plane, \((Ox_1x_2x_3)\), where integer points reduce to natural points, \(i_1, i_2, i_3\), naturals, and the point, \((i_1, i_2, i_3)\), is related to the natural number, \(i_1i_2i_3\). The related representation may be considered as 3-dimension Pythagorean table, or 3-Pythagorean table.

The 3-sector of positive 2\(^3\)-ant, \(x_1 = x_2 = x_3\), carries the sequence of perfect 3-squares, starting from 0.

The equilateral 2-triangle of vertexes, \([0,0,(k+1)], [0,(k+1),0], [(k+1),0,0]\), normal to the 3-sector of positive 2\(^3\)-ant, carries 3-areas of perfect 3-rectangles exhibiting 2\(^{-2}\)-perimeter, \(p_3 = k + 2\).

The above mentioned 3-areas are equal for symmetric points with respect to the 3-sector, i.e. basis and height length of perfect 3-rectangle are interchanged, monotonically increasing from null values on 1-rectangles and 2-rectangles to largest values on points less distant from the 3-sector on each side. Symmetry is owing to 3-area invariance with respect to the exchange between basis and height length.

Straight lines parallel to the 3-sector of positive 2\(^3\)-ant, \(x_3 = x_2 \mp \ell_2 = x_1 \mp \ell_1\), carry 3-areas of perfect 3-rectangles exhibiting 2\(^{-2}\)-perimeter, 3\(i_1 \mp \ell_1 \mp \ell_2\).
Figure 2: 2-Pythagorean table as correspondence between codomain and domain of the natural product function, \( f(i_1, i_2) = i_1 i_2 \), where \( i_1, i_2 \) are natural numbers and axis orientation relates to usual Pythagorean table. 1-triangles of vertexes \((0, k + 1), (k + 1, 0), 2 \leq k + 1 \leq 21\), normal to the 2-sector of positive \(2^2\)-ant, carry 2-areas where related perfect 2-rectangles exhibit constant \(2^{-1}\)-perimeter, \( p_2 = i_1 + i_2 = k + 1 \). The 2-sector of positive \(2^2\)-ant carries the sequence of perfect 2-squares.
2.3 $n$-Pythagorean table

With regard to a Cartesian orthogonal reference frame in $\mathbb{R}^{n+1}$, $(Ox_1 x_2 ... x_{n+1})$, let $x_{n+1} = f(x_1, x_2, ..., x_n) = |x_1 x_2 ... x_n|$, be the (absolute) product function, which is defined on the domain, $x_{n+1} = 0$. Minimum value occurs on the coordinate axes, $x_2 = x_3 = ... = x_{n+1} = 0$, $x_1 = x_3 = ... = x_{n+1} = 0$, ..., $x_1 = x_2 = ... = x_{n+1} = 0$, and maximum value occurs for $x_1 = x_2 = ... = x_n$ with regard to assigned $(|x_1| + |x_2| + ... + |x_n|)$, keeping in mind $n$-square exhibits larger $n$-area with respect to $n$-rectangle of equal $2^{-(n-1)}$-perimeter, $p_n = |x_1| + |x_2| + ... + |x_n|$. More specifically, $p_n$ is an increasing function of $n$-rectangle “squareness”, $s_n = \min(x_1, x_2, ..., x_n) / \max(x_1, x_2, ..., x_n)$, $0 \leq s_n \leq 1$, where the lower value relates to 1-rectangle, 2-rectangle, ... $(n-1)$-rectangle, i.e. $\min(x_1, x_2, ..., x_n) = 0$. In particular, integer points, $(i_1, i_2, ..., i_n)$, $i_1, i_2, ..., i_n$ integers, make a discrete sub domain of the product function, and related discrete sub codomain is $[i_1 i_2 ... i_n]$. From this point on, attention shall be restricted to positive $2^n$-ant of the principal $n$-plane, $(Ox_1 x_2 ... x_n)$, where integer points reduce to natural points, $i_1, i_2, ..., i_n$ naturals, and the point, $(i_1, i_2, ..., i_n)$, is related to the natural number, $i_1 i_2 ... i_n$. The related representation may be considered as $n$-dimension Pythagorean table, or $n$-Pythagorean table. The principal planes, $(Ox_j x_k)$, $1 \leq j < k \leq n$, carry each a 2-Phythagorean table. The $n$-sector of positive $2^n$-ant, $x_1 = x_2 = ... = x_n$, carries the sequence of perfect $n$-squares, starting from 0. The equilateral $(n-1)$-triangle of vertexes, $[0, 0, ..., (k+1)], [0, ..., (k+1), 0], ..., [(k+1), 0, ..., 0]$, normal to the $n$-sector of positive $2^n$-ant, carries $n$-areas of perfect $n$-rectangles exhibiting $2^{-(n-1)}$-perimeter, $p_n = k + (n-1)$. The above mentioned $n$-areas are equal for symmetric points with respect to the $n$-sector, i.e. basis and height length of perfect $n$-rectangle are interchanged, monotonically increasing from null values on $(n-1)$-rectangles, $(n-2)$-rectangles, ..., 1-rectangles, to largest values on points less distant from the $n$-sector on each side. Symmetry is owing to $n$-area invariance with respect to the exchange between basis and height length. Straight lines parallel to the $n$-sector of positive $2^n$-ant, $x_n = x_{n-1} \mp \ell_{n-1} = ... = x_2 \mp \ell_2 = x_1 \mp \ell_1$, carry $n$-areas of perfect $n$-rectangles exhibiting $2^{-(n-1)}$-perimeter, $n i_1 \mp \ell_1 \mp \ell_2 \mp ... \mp \ell_{n-1}$. Each boundary of 2-Pythagorean table, as shown in Fig. 2, may be conceived as 1-Pythagorean table, i.e. carrying products of a single factor, that is the multiplication table of 1. The empty set may be conceived as 0-Pythagorean table, i.e. carrying products of no factor. In conclusion, Pythagorean table can be extended from 2 to $n$ factors, $n \geq 0$, and defined as $n$-Pythagorean table. Geometrical interpretation is performed within a $n$-plane, via correspondance between codomain and domain.
of the discrete product function, \( f(i_1, i_2, ..., i_n) = i_1 i_2 ... i_n \), restricted to positive \( 2^n \)-ant. 0-Pythagorean table relates to the empty set. 1-Pythagorean table relates to the sequence of natural numbers.

3 Pythagorean triangle

Pythagorean triangle, inferred from Pythagorean table, can be extended to \( n \)-Pythagorean triangle, inferred from \( n \)-Pythagorean table. In general, \( n \)-areas of perfect \( n \)-rectangles of assigned basis and height length may be conceived as ordered \( n \)-tuples of factors related to natural numbers, or \( n \)-products.

In the following subsections, 2-Pythagorean triangle shall first be defined and related properties shall be described. The same shall be done for 3-Pythagorean triangle and 4-Pythagorean triangle, where some connection with ordinary space still remains. Finally, \( n \)-Pythagorean triangle shall be taken into consideration.

3.1 2-Pythagorean triangle

With regard to 2-Pythagorean table shown in Fig. 2, a suitable rigid rotation of the Cartesian orthogonal reference frame, (\( O x_1 x_2 \)), makes 2-sector of positive \( 2^2 \)-ant aligned with the vertical and positive coordinate semiaxes displaced like slope roof. Accordingly, 2-Pythagorean table attains 2-triangle configuration which is bounded on the top and unbounded on the bottom.

Let 2-Pythagorean triangle of order, \( k \), be defined as right isosceles 2-triangle carrying the 2-area, 1, on the right vertex and the 2-area, \( k \), on the remaining vertexes, where hypotenuse makes basis and cathetuses make oblique sides.

Basis of 2-Pythagorean triangle of order, \( \ell \), \( 1 \leq \ell \leq k \), is sub basis of 2-Pythagorean triangle of order, \( k \). Sub basis, \( \ell \), carries 2-areas of perfect 2-rectangles of \( 2^{-1} \)-perimeter, \( p_{2, \ell} = \ell + 1 \), symmetrically displaced with respect to height which, in turn, carries the sequence of perfect 2-squares.

Sub heights, \((j_1, j_2)\), parallel and symmetrically displaced with respect to height, carry the sequence of perfect 2-rectangles where \( 2^{-1} \)-perimeter is incremented by two units moving downwards. Sub sides, symmetrical with respect to the height and crossing on it, carry the multiplication table of the starting 2-area on each side. The 2-Pythagorean triangle of order, \( k = 16 \), is reproduced in Fig. 3.

2-Pythagorean triangle of order, \( k \), exhibits the following properties.

I. The total number of sub bases is \( k \).

II. The total number of sub heights equals the number of 2-products placed on lateral 1-faces.
Figure 3: Representation of the 2-Pythagorean triangle of order, $k = 16$. Sub basis, $\ell$, carries 2-areas of perfect 2-rectangles exhibiting $2^{-1}$-perimeter, $p_{2,\ell} = \ell + 1$, symmetrically placed with respect to height which, in turn, carries the sequence of perfect 2-squares. Sub heights, $(j_1, j_2)$, symmetrically placed with respect to height, carry the sequence of perfect 2-rectangles exhibiting $2^{-1}$-perimeter incremented by two units moving downwards. Sub sides, symmetric with respect to height and crossing on it, carry multiplication tables of the initial number, shown behind each side.
III. Sub basis, \( \ell \), carries \( \ell \) 2-areas, which is the number of 2-tuples, \( \{i_1, i_2\} \), where \( i_1, i_2 \), are nonzero natural numbers satisfying \( i_1 + i_2 = \ell + 1 \).

IV. Sub heights, \( (j_1, j_2) \), branching off from the boundary of sub basis, \( j = j_1 + j_2 - 1 \), carry each \( \{1 + \text{Int}[\ell - j/2]\} \) 2-areas, up to sub basis, \( \ell \).

V. Sub heights cross (i.e. have one 2-area in common with) sub bases two by two, starting from the boundary of sub basis where sub height branches off. Accordingly, odd/even sub heights cross odd/even sub bases.

VI. 2-area can be defined as intersection between sub basis, \( \ell \), \( 1 \leq \ell \leq k \), and sub height, \( (j_1, j_2) \), \( \min(j_1, j_2) = 1 \), \( \max(j_1, j_2) = j \leq \ell \), and related coordinates read \( (\ell, j_1, j_2) \).

VII. 2-area, \( (\ell, j_1, j_2) \), is separated from nearest lateral 1-face by an equal amount of 2-areas along both sub basis, \( \ell \), and sub height, \( (j_1, j_2) \).

Let \( p_{2,\ell} \) be \( 2^{-1} \)-perimeter of generic, perfect 2-rectangle, where related 2-area is placed on sub basis, \( \ell \), of 2-Pythagorean triangle of order, \( k \), and let \( i_1, i_2 \), be the length of each side. Then the following relation holds:

\[
p_{2,\ell} = i_1 + i_2 = \ell + 1 ; \quad 1 \leq \ell \leq k
\]

and related 2-area can be expressed as:

\[
i_1i_2 = i_1(p_{2,\ell} - i_1) = i_1(\ell + 1 - i_1) ; \quad 1 \leq i_1 \leq \ell
\]

where 2-product displacement is determined by a single factor, \( i_1 \).

More specifically, the sequence of 2-areas on sub basis, \( \ell \), can be determined starting from the 2-area of perfect 2-rectangle exhibiting unit height, increasing height length and decreasing basis length by a unit in both cases, up to the 2-area of perfect 2-rectangle exhibiting unit basis.

Perfect 2-rectangle 2-area may be specified as:

\[
i_1i_2 = i_1(2L + 2 - i_1) ; \quad 1 \leq i_1 \leq 2L + 1
\]  \hspace{1cm} (1)

with regard to odd sub basis, \( \ell = 2L + 1 \);

\[
i_1i_2 = i_1(2L + 1 - i_1) ; \quad 1 \leq i_1 \leq 2L
\]  \hspace{1cm} (2)

with regard to even sub basis, \( \ell = 2L \).

It can be seen maximum 2-area relates to perfect 2-square or perfect 2-rectangle where basis and height length differ by unit, according if they are displaced on odd or even sub bases, respectively. Related sequences take place as:

\[
(2L + 1) \cdot 1, \; 2L \cdot 2, \; (2L - 1) \cdot 3, \; ..., \; [2L - (L - 1)] \cdot [2L - (L - 1)], \; ..., \\
3 \cdot (2L - 1), \; 2 \cdot 2L, \; 1 \cdot (2L + 1) ;
\]

\[
2L \cdot 1, \; (2L - 1) \cdot 2, \; ..., \; [2L - (L - 1)] \cdot L, \; (2L - L) \cdot (L + 1), \; ..., \\
2 \cdot (2L - 1), \; 1 \cdot 2L ;
\]
where basis and height length of perfect 2-square, displaced on the centre of odd sub basis, \((2L + 1)\), equals \((L + 1)\); and basis and height length of perfect 2-rectangle, displaced on the neighbourhood of the centre of even sub basis, \(2L\), equals \(L, (L + 1); (L + 1), L\).

Keeping in mind products with no even factor yield odd numbers while products with at least one even factor yield even numbers, 2-areas displaced on the boundary of odd sub bases are odd and alternate with even 2-areas passing from the boundary to the centre. On the other hand, 2-areas displaced on even sub bases are even. The intersection between sub bases and sub heights (in the sense that a 2-area exists belonging to both) can occur only between odd sub basis and odd sub height, or even sub basis and even sub height.

To gain more insight, 2-Pythagorean triangle of order, \(k\), has to be conceived as made of ordered 2-tuples of factors, or 2-products, \(\{i_1, i_2\} = i_1 \cdot i_2\), instead of 2-areas, \(i_1i_2\), passing from the representation of Fig. 3 to the representation of Fig. 4, where brackets are omitted to save space.

For 2-products displaced on a selected sub basis, the sequence of factors relates to the sequence of natural numbers, from the right to the left and from the left to the right for factors on the left and on the right, respectively, preserving factor sum, \(i_1 + i_2\). For 2-products displaced on a selected sub height, the sequence of factors relates to the sequence of natural numbers, starting from unity and from the number on the boundary of sub basis from which sub height branches off, respectively, preserving factor difference, \(\max(i_1, i_2) - \min(i_1, i_2)\). For 2-products displaced on a selected sub side, the sequence of factors relates to the sequence of natural numbers and the number on the boundary of sub basis from which sub side branches off, respectively, preserving the last.

With regard to 2-Pythagorean triangle of order, \(k\), let \(\{i_1, i_2\}\) be 2-product of coordinates, \((\ell, j_1, j_2)\), displaced on the intersection between sub basis, \(\ell\), and sub height, \((j_1, j_2)\); \(\min(j_1, j_2) = 1\); \(\max(j_1, j_2) = j \leq \ell; j_1 + j_2 = j + 1\). Factors within 2-product can be set in \(2! = 2\) different ways, which implies \(2\) different inequalities, \(i_1 \geq i_2\) and \(i_2 \geq i_1\). Accordingly, sub basis can be divided into two equivalent regions, as shown in Fig. 4. Factors belonging to 2-products displaced on the same region are ordered in the same way, while the contrary holds for factors belonging to 2-products displaced on different regions.

For 2-products displaced on sub basis, \(\ell\), the following can be established: \(\{i_1, i_2\} = \{(\ell - \ell'), (\ell' + 1)\}, 0 \leq \ell' \leq \ell - 1\), keeping in mind related 2-1-perimeter reads \(p_{2\ell} = (\ell - \ell') + (\ell' + 1) = \ell + 1\).

For 2-products displaced on sub-height, \((j_1, j_2)\), the following can be established: \(\{i_1, i_2\} = \{(j_1 + j'), (j_2 + j')\}, 0 \leq j' \leq j - 1\), \(\min(j_1, j_2) = 1\), \(\max(j_1, j_2) = j\), keeping in mind the difference, \(\max([j_1 + j'), (j_2 + j')] - \min([j_1 + j'), (j_2 + j')] = \max(j_1, j_2) - \min(j_1, j_2)\), remains unchanged.
Figure 4: Representation of the 2-Pythagorean triangle of order, $k = 16$, in terms of 2-products, $\{i_1, i_2\} = i_1 \cdot i_2$, where brackets are omitted to save space. For 2-products displaced on a selected sub basis, the sequence of factors relates to the sequence of natural numbers, from the right to the left and from the left to the right for factors on the left and on the right, respectively, preserving factor sum, $i_1 + i_2$. For 2-products displaced on a selected sub height, the sequence of factors relates to the sequence of natural numbers, starting from unity and from the number on the boundary of sub basis from which sub height branches off, respectively, preserving factor difference, $\max(i_1, i_2) - \min(i_1, i_2)$. For 2-products displaced on a selected sub side, the sequence of factors relates to the sequence of natural numbers and the number on the boundary of sub basis from which sub side branches off, respectively, preserving the last.
In the case under discussion, 2-products are displaced on the intersection between sub basis, \( \ell \), and sub height, \((j_1,j_2)\). Accordingly, the following relations hold:

\[
\begin{align*}
(\ell - \ell')(\ell' + 1) &= (j_1 + j')(j_2 + j') ; \\
\{ \ell - \ell' = j_1 + j' ; \} & \ell = j_2 + j' - 1 + j_1 + j' ; \\
\{ \ell' + 1 = j_2 + j' ; \} & \ell' = j_2 + j' - 1 ; \\
\{ \ell = j + 2j' ; \} & \ell' = j + 1 - j_1 + j' - 1 ; \\
\{ \ell' = j_2 + j' - 1 ; \} & \ell'' = j - j_1 + \ell - j_2 ;
\end{align*}
\]

from which the explicit expression of the first factor can be inferred as:

\[
i_1 = \ell - \ell' = \ell - j + j_1 - \frac{\ell - j}{2} = \ell - j + j_1 - 2\frac{\ell - j}{2} + \frac{\ell - j}{2} =
\]

\[
= \ell - j + j_1 - \ell + j + \frac{\ell - j}{2} ;
\]

and the following relations hold:

\[
i_1 = \ell - \ell' = j_1 + \frac{\ell - j}{2} ; \quad (3a) \\
i_2 = \ell' + 1 = j_2 + \frac{\ell - j}{2} ; \quad (3b) \\
i_1 + i_2 = \ell + 1 ; \quad j_1 + j_2 = j + 1 ; \quad (3c) \\
\min(j_1,j_2) = 1 ; \quad \max(j_1,j_2) = j ; \quad (3d) \\
i_k - j_k = \frac{\ell - j}{2} ; \quad k = 1, 2 ; \quad (3e)
\]

accordingly, values of 2-products can be determined from the knowledge of related coordinates.

Let 2-product, \(\{i_1,i_2\}\), of coordinates, \((\ell,j_1,j_2)\), be displaced on right vertex of right isosceles triangle, where cathetuses lie on sub basis, \(\ell\), and on sub height, \((j_1,j_2)\), respectively, and hypotenuse lies on nearest lateral 1-face implying \(i_1 \geq i_2\) with no loss of generality, hence \(j_1 = j, j_2 = 1\). Accordingly, remaining vertexes carry 2-products, \(\{\ell,1\}, \{j,1\}\), of coordinates, \((\ell,\ell,1), (j,j,1)\), respectively, that is on nearest lateral 1-face from which sub basis, \(\ell\), and sub height, \((j_1,j_2)\), respectively, branch off.

Cathetus lying on sub basis, \(\ell\), carries 2-products:

\[
\{(i_1 + i_0), (i_2 - i_0)\} ; \quad 1 \leq i_0 \leq i_2 - 1 ; \quad (4)
\]

passing from \(\{(i_1 + 1), (i_2 - 1)\}\) to \(\{\ell,1\}\) for increasing \(i_0\), the last displaced on nearest sub basis side. Using the above results yields the explicit expression:

\[
\left(j + \frac{\ell - j}{2} + i_0\right) \left(\frac{\ell - j}{2} + 1 - i_0\right) =
\]
\[ \frac{j + \ell - j}{2} \left( \frac{j - j}{2} + 1 \right) + i_0 \left( \frac{j - j}{2} + 1 \right) - i_0 \left( j + \frac{j - j}{2} \right) - i_0^2 = \]
\[ = \left( j + \frac{j - j}{2} \right) \left( \frac{j - j}{2} + 1 \right) + i_0 \left( \frac{j - j}{2} + 1 - j - \frac{j - j}{2} - i_0 \right) ; \]
\[ 1 \leq i_0 \leq i_2 - 1 = \frac{\ell - j}{2} ; \]
in conclusion:
\[ \left( j + \frac{j - j}{2} + i_0 \right) \left( \frac{j - j}{2} + 1 - i_0 \right) = \left( j + \frac{j - j}{2} \right) \left( \frac{j - j}{2} + 1 \right) + \]
\[ + i_0 (1 - j - i_0) ; \quad 1 \leq i_0 \leq \frac{\ell - j}{2} ; \quad (5) \]
where, in the case under discussion, \( i_0 \)-th place starting from \( \{i_1, i_2\} \) is equivalent to \( (i_2 - i_0) \)-th place starting from \( \{\ell, 1\} \).

Cathetus lying on sub height, \( (j, 1) \), carries 2-products:
\[ \{(j + j_0 - 1), j_0\} ; \quad 1 \leq j_0 \leq i_2 - 1 ; \quad (6) \]
passing from sub height boundary, \( \{j, 1\} \), to \( \{(i_1 - 1), (i_2 - 1)\} \), for increasing \( j_0 \). In the case under discussion, \( j_0 \)-th place starting from \( \{i_1, i_2\} \) is equivalent to \( (i_2 - j_0) \)-th place starting from \( \{j, 1\} \).

The sum of generic pairs of products of the kind considered (i.e. factors are displaced on different cathetuses), for which \( i_0 = j_0 \), yields:
\[ \left( j + \frac{j - j}{2} + i_0 \right) \left( \frac{j - j}{2} + 1 - i_0 \right) + (j + j_0 - 1)j_0 = \]
\[ = \left( j + \frac{j - j}{2} \right) \left( \frac{j - j}{2} + 1 \right) + i_0 (1 - j - i_0) - (1 - j - i_0)i_0 = \]
\[ = \left( j + \frac{j - j}{2} \right) \left( \frac{j - j}{2} + 1 \right) ; \quad (7) \]
which equals related product displaced on the right vertex, regardless from the pair considered, as can be inferred from Fig.4.

The particularization of the above results to perfect 2-square, \( \{i_1, i_2\} = \{(L + 1), (L + 1)\} \), displaced on the centre of a generic odd sub basis, \( \ell = 2L + 1 \), discloses remaining 2-products displaced therein can be determined as difference of perfect 2-squares, \( (L + 1)^2 - (L')^2 \), \( 1 \leq L' \leq L \), as can be inferred from Fig.3.

The particularization of the above results to perfect 2-rectangle, \( \{i_1, i_2\} = \{(L + 1), L\} \), displaced on the neighbourhood of the centre of a generic even sub basis, \( \ell = 2L \), discloses remaining 2-products displaced therein can be determined as difference of perfect 2-rectangles, \( (L + 1)L - (L' + 1)L' \), \( 1 \leq L' \leq L - 1 \), as can be inferred from Fig.3 and Fig.4.
3.2 3-Pythagorean triangle

With regard to 3-Pythagorean table, a suitable rigid rotation of the Cartesian orthogonal reference frame, \((Ox_1x_2x_3)\), makes 3-sector of positive \(2^3\)-ant aligned with the vertical and positive coordinate semiaxes displaced like slope roof. Accordingly, 3-Pythagorean table attains 3-triangle configuration which is bounded on the top and unbounded on the bottom.

Let 3-Pythagorean triangle of order, \(k\), be defined as right isosceles 3-triangle carrying the 3-area, 1, on the right vertex and the 3-area, \(k\), on the remaining vertexes, where hypotenuse makes basis and cathetuses make oblique sides.

Basis of 3-Pythagorean triangle of order, \(\ell\), \(1 \leq \ell \leq k\), is sub basis of 3-Pythagorean triangle of order, \(k\). Sub basis, \(\ell\), carries 3-areas of perfect 3-rectangles of \(2^{-2}\)-perimeter, \(p_{3,\ell} = \ell + 2\), symmetrically displaced with respect to height which, in turn, carries the sequence of perfect 3-squares.

Sub heights, \((j_1,j_2,j_3)\), parallel and symmetrically displaced with respect to height, carry the sequence of perfect 3-rectangles where \(2^{-2}\)-perimeter is incremented by three units moving downwards. Lateral 2-faces carry each a 2-Pythagorean triangle of order, \(k\). The 3-Pythagorean triangle of order, \(k = 7\), is reproduced in Fig. 5.

3-Pythagorean triangle of order, \(k\), exhibits the following properties.

I. The total number of sub bases is \(k\).

II. The total number of sub heights equals the number of 3-products placed on lateral 2-faces.

III. Sub basis, \(\ell\), carries \([\ell + 1/2]\) 3-areas, which is the number of 3-tuples, \(\{i_1,i_2,i_3\}\), where \(i_1, i_2, i_3\), are nonzero natural numbers satisfying \(i_1 + i_2 + i_3 = \ell + 2\).

IV. Sub heights, \((j_1,j_2,j_3)\), branching off from the boundary of sub basis, \(j = j_1 + j_2 + j_3 - 2\), carry each \(\{1 + \text{Int}[\ell/3]\}\) 3-areas, up to sub basis, \(\ell\).

V. Sub heights cross (i.e. have one 3-area in common with) sub bases three by three, starting from the boundary of sub basis where sub height branches off. Accordingly, sub heights cross alternately odd and even sub bases.

VI. 3-area can be defined as intersection between sub basis, \(\ell, 1 \leq \ell \leq k\), and sub height, \((j_1,j_2,j_3)\), \(\min(j_1,j_2,j_3) = 1\), \(\max(j_1,j_2,j_3) \leq j \leq \ell\), and related coordinates read \((\ell,j_1,j_2,j_3)\).

VII. 3-area, \((\ell,j_1,j_2,j_3)\), is separated from nearest lateral 2-face by an equal amount of 3-areas along both sub basis, \(\ell\), and sub height, \((j_1,j_2,j_3)\).

Let \(p_{3,\ell}\) be \(2^{-2}\)-perimeter of generic, perfect 3-rectangle, where related 3-area is placed on sub basis, \(\ell\), of 3-Pythagorean triangle of order, \(k\), and let \(i_1\),
Figure 5: Representation of the 3-Pythagorean triangle of order, $k = 7$, seen head-on. The right vertex, carrying unity, is the trace of 3-sector of positive $2^3$-ant on figure plane. Sub basis, $\ell$, carries 3-areas of perfect 3-rectangles exhibiting $2^{-2}$-perimeter, $p_{3,\ell} = \ell + 2$, symmetrically placed with respect to height which, in turn, carries the sequence of perfect 3-squares. Sub heights, $(j_1, j_2, j_3)$, symmetrically placed with respect to height, carry the sequence of perfect 3-rectangles exhibiting $2^{-2}$-perimeter incremented by three units moving downwards. Lateral 2-faces carry each a 2-Pythagorean triangle of order, $k = 7$. To gain clarity, only 3-areas displaced on the boundary of sub bases are shown.
$i_2, i_3$, be the length of each side. Then the following relation holds:

$$p_{3,\ell} = i_1 + i_2 + i_3 = \ell + 2 ; \quad 1 \leq \ell \leq k ;$$

and related 3-area can be expressed as:

$$i_1i_2i_3 = i_1i_2(p_{3,\ell} - i_1 - i_2) = i_1i_2(\ell + 2 - i_1 - i_2) ; \quad 2 \leq i_1 + i_2 \leq \ell + 1 ;$$

where 3-product displacement is determined by two factors, $i_1, i_2$.

More specifically, the sequence of 3-areas on sub basis, $\ell$, can be determined along the following steps.

1. Start from the 3-area of perfect 3-rectangle exhibiting unit height and unit minor basis, $(\ell \cdot 1 \cdot 1)$, displaced on a vertex.
2. Increase minor basis length and decrease major basis length by a unit in both cases, and maintain height, up to the 3-area, $(1 \cdot \ell \cdot 1)$, displaced on the next vertex.
3. Increase height length and decrease major basis length by a unit in both cases, and maintain minor basis, up to the 3-area, $(1 \cdot 1 \cdot \ell)$, displaced on the next vertex.
4. Increase previously preserved basis length and decrease height by a unit in both cases, and maintain previously decreased basis, up to the 3-area, $(\ell \cdot 1 \cdot 1)$, displaced on the starting vertex.

Passing from the boundary of sub basis, $\ell$, to the first inner contour, a similar procedure can be used starting from the 3-area of perfect 3-rectangle exhibiting 2 unit height length and 2 unit minor basis length, $[(\ell - 2) \cdot 2 \cdot 2]$, displaced on a vertex. In general, passing from $(\ell_0 - 2)$-th to $(\ell_0 - 1)$-th inner contour, a similar procedure can be used starting from the 3-area of perfect 3-rectangle exhibiting $\ell_0$ unit height length and $\ell_0$ unit minor basis length, $[(\ell - 2\ell_0 + 2) \cdot \ell_0 \cdot \ell_0]$. If the condition $\ell_0 \leq \ell - 2\ell_0 + 2$, or $3\ell_0 \leq \ell + 2$, no longer holds, the whole amount of 3-areas have been displaced on sub basis, $\ell$.

Perfect 3-rectangle 3-area may be specified as:

$$i_1i_2i_3 = i_1i_2(2L + 3 - i_1 - i_2) ; \quad 2 \leq i_1 + i_2 \leq 2L + 2 ; \quad (8)$$

with regard to odd sub basis, $\ell = 2L + 1$;

$$i_1i_2i_3 = i_1i_2(2L + 2 - i_1 - i_2) ; \quad 2 \leq i_1 + i_2 \leq 2L + 1 ; \quad (9)$$

with regard to even sub basis, $\ell = 2L$.

Keeping in mind products with no even factor yield odd numbers while products with at least one even factor yield even numbers, 3-areas displaced on vertexes of odd sub bases are odd and alternate with even 3-areas on sub basis.
side. On the other hand, 3-areas displaced on even sub bases are even. The same holds passing from the boundary to the centre: inner contours where odd 3-areas are displaced on vertexes carry odd 3-areas alternate with even 3-areas; inner contours where even 3-areas are displaced on vertexes carry even 3-areas only.

With regard to the 3-Pythagorean triangle of order, \( k = 10 \), sub bases are shown in Fig. 6.

To gain more insight, 3-Pythagorean triangle of order, \( k \), has to be conceived as made of ordered 3-tuples of factors, or 3-products, \( \{i_1, i_2, i_3\} = i_1 \cdot i_2 \cdot i_3 \), instead of 3-areas, \( i_1i_2i_3 \), passing from the representation of Fig. 6 to the representation of Fig. 7, where brackets and commas are omitted to save space, keeping in mind each factor relates to a single figure with the exception of 10, denoted as 0.

An inspection of Fig. 7 discloses sub basis, \( \ell \), can be assembled through the following steps, where increments and decrements are intended by a unit.

1. Start from the vertex on the left, \( (\ell, 1, 1) \), along horizontal side decrementing the first factor, incrementing the second, preserving the third, up to the vertex on the right, \( (1, \ell, 1) \).
2. Start from the vertex on the left, \( (\ell, 1, 1) \), along oblique side decrementing the first factor, preserving the second, incrementing the third, up to the vertex on the top, \( (1, 1, \ell) \).
3. Start from the vertex on the top, \( (1, 1, \ell) \), along oblique side preserving the first factor, incrementing the second, decrementing the third, up to the vertex on the right, \( (1, \ell, 1) \).
4. A similar procedure is repeated with regard to \( (\ell_0 - 1) \)-th inner contour starting from the vertex on the left, \( [(\ell - 2\ell_0 + 2), \ell_0, \ell_0] \), for all values within the range of natural numbers, \( 1 < \ell_0 \leq \text{Int}[\ell + 2)/3] \), where \( \ell_0 = 1 \) relates to sub basis boundary and \( \ell_0 = \text{Int}[\ell + 2)/3] \) relates to nearest inner contour with respect to sub basis centre.

For 3-products displaced on a selected sub basis, factor sum, \( i_1 + i_2 + i_3 \), is preserved. For 3-products displaced on a selected sub height, factor difference, \( \max(i_1, i_2, i_3) - \min(i_1, i_2, i_3) \), is preserved. For 3-products displaced on lateral 2-faces, factors exhibit same properties as in 2-Pythagorean triangle of order, \( k \).

With regard to 3-Pythagorean triangle of order, \( k \geq \ell \), an inspection of Fig. 7 and Fig. 4 discloses sub basis, \( \ell \), can be assembled starting from 2-Pythagorean triangle of order, \( \ell \), putting the factor, \( i \), on the third place of each 2-product displaced on sub basis, \( (\ell - i + 1) \), for values of \( i \) satisfying \( 1 \leq i \leq \ell \). The range, \( 1 \leq \ell \leq 8 \), is represented in Fig. 8.
Figure 6: Representation of the 3-Pythagorean triangle of order, $k = 10$, through sub bases starting from the vertex, where unit 3-area is displaced, up to the basis. Sub basis, $\ell$, carries 3-areas of perfect 3-rectangles exhibiting $2^{-2}$-perimeter, $p_{3, \ell} = \ell + 2$, symmetrically placed with respect to the height which, in turn, carries the sequence of perfect 3-squares. Sub heights, $(j_1, j_2, j_3)$, symmetrically placed with respect to the height, carry the sequence of perfect 3-rectangles exhibiting $2^{-2}$-perimeter incremented by three units moving downwards.
Figure 7: Representation of the 3-Pythagorean triangle of order, $k = 10$, through sub bases in terms of 3-products, $\{i_1, i_2, i_3\} = i_1 \cdot i_2 \cdot i_3$, where brackets and commas are omitted to save space, keeping in mind each factor relates to a single figure with the exception of 10, denoted as 0. For 3-products displaced on a selected sub basis, factor sum is preserved. For 3-products displaced on a selected sub height, factor difference (between extreme values) is preserved.
Figure 8: With regard to 3-Pythagorean triangle of order, $k \geq \ell$, sub basis, $\ell$ (right panels), can be assembled starting from 2-Pythagorean triangle of order, $\ell$ (left panels), adding the factor, $i$, (in Italics) on the third place of each 2-product displaced on sub basis, $(\ell - i + 1)$, for values of $i$ satisfying $1 \leq i \leq \ell$. The range shown is $1 \leq \ell \leq 8$. Same notation as in Fig. 7.
With regard to sub basis boundary and inner contours, it can be seen sub heights crossing 3-products, displaced on vertexes and 1-face centre, appear in triplets; sub heights crossing 3-products, displaced otherwise, appear in sextets; height crossing 3-products, displaced on sub basis centre i.e. perfect 3-squares, appears in singlet.

With regard to 3-Pythagorean triangle of order, \( k \), let \( \{i_1, i_2, i_3\} \) be 3-product of coordinates, \((\ell, j_1, j_2, j_3)\), that is displaced on the intersection between sub basis, \( \ell \), and sub height, \((j_1, j_2, j_3)\); \( \min(j_1, j_2, j_3) = 1; \max(j_1, j_2, j_3) \leq \ell \leq j_1 + j_2 + j_3 = j + 2 \). Factors within 3-products can be set in \( 3! = 6 \) different ways, which implies 6 different inequalities involving factors. Accordingly, sub basis can be divided into six equivalent regions, as shown in Fig. 7. Factors belonging to 3-products displaced on the same region are ordered in the same way, while the contrary holds for factors belonging to 3-products displaced on different regions.

For 3-products displaced on sub basis, \( \ell \), the following can be established: 
\[
\{i_1, i_2, i_3\} = \{(\ell - \ell_1 - \ell'), (\ell_1 + 1), (\ell' + 1)\}, \quad 0 \leq \ell_1 + \ell' \leq \ell - 2,
\]
keeping in mind related 2\(^2\)-perimeter reads \( p_{3,\ell} = (\ell - \ell_1 - \ell') + (\ell_1 + 1) + (\ell' + 1) \).

For 3-products displaced on sub height, \((j_1, j_2, j_3)\), the following can be established: 
\[
\{i_1, i_2, i_3\} = \{(j_1 + j'), (j_2 + j'), (j_3 + j')\}, \quad 0 \leq j' \leq j - 2,
\]
\( \min(j_1, j_2, j_3) = 1 \), keeping in mind the difference, \( \max[(j_1 + j'), (j_2 + j'), (j_3 + j')] - \min[(j_1 + j'), (j_2 + j'), (j_3 + j')] = \max(j_1, j_2, j_3) - \min(j_1, j_2, j_3) \), remains unchanged.

In the case under discussion, 3-products are displaced on the intersection between sub basis, \( \ell \), and sub height, \((j_1, j_2, j_3)\). Accordingly, the following relations hold:

\[
(\ell - \ell_1 - \ell')(\ell_1 + 1)(\ell' + 1) = (j_1 + j')(j_2 + j')(j_3 + j');
\]
\[
\ell - \ell_1 - \ell' = j_1 + j'; \quad \ell = j_2 + j' - 1 + j_3 + j' - 1 + j_1 + j';
\]
\[
\ell_1 + 1 = j_2 + j'; \quad \ell_1 = j_2 + j' - 1;
\]
\[
\ell' + 1 = j_3 + j'; \quad \ell' = j_3 + j' - 1;
\]
\[
\ell = j + 3j'; \quad \ell = j + 3j';
\]
\[
\ell_1 = j_2 + j' - 1; \quad \ell_1 = j_2 + j' - 1 + \frac{\ell - j}{3};
\]
\[
\ell' = j_3 + j' - 1; \quad \ell' = j_3 + j' - 1 + \frac{\ell - j}{3};
\]

from which the explicit expression of the first factor can be inferred as:

\[
i_1 = \ell - \ell_1 - \ell' = \ell - j_2 + 1 - \frac{\ell - j}{3} - j_3 + 1 - \frac{\ell - j}{3} =
\]
\[
= \ell - j_2 - j_3 + 2 - 2\frac{\ell - j}{3} = \ell + j_1 - j - 3\frac{\ell - j}{3} + \frac{\ell - j}{3} =
\]
\[
= \ell + j_1 - j - \ell + j + \frac{\ell - j}{3};
\]
and the following relations hold:

\[ i_1 = \ell - \ell_1 - \ell' = j_1 + \frac{\ell - j}{3} ; \quad (10a) \]
\[ i_2 = \ell_1 + 1 = j_2 + \frac{\ell - j}{3} ; \quad (10b) \]
\[ i_3 = \ell' + 1 = j_3 + \frac{\ell - j}{3} ; \quad (10c) \]
\[ i_1 + i_2 + i_3 = \ell + 2 ; \quad j_1 + j_2 + j_3 = j + 2 ; \quad (10d) \]
\[ \min(j_1, j_2, j_3) = 1 ; \quad \max(j_1, j_2, j_3) \leq j ; \quad (10e) \]
\[ i_k - j_k = \frac{\ell - j}{3} ; \quad k = 1, 2, 3 ; \quad (10f) \]

accordingly, values of 3-products can be determined from the knowledge of related coordinates.

Let 3-product, \( \{i_1, i_2, i_3\} \), of coordinates, \( (\ell, j_1, j_2, j_3) \), be displaced on right vertex of right isosceles triangle, where cathetuses lie on sub basis, \( \ell \), and on sub height, \( (j_1, j_2, j_3) \), and hypotenuse lies on nearest lateral 2-face implying \( i_1 \geq i_2 \geq i_3 \) with no loss of generality, hence \( j \geq j_1 \geq j_2 \geq j_3 = 1 \). Accordingly, remaining vertexes carry 3-products, \( \{(\ell - j + 1), j, 1\}, \{j_1, j_2, 1\} \), of coordinates, \( [\ell, (\ell - j + 1), j, 1], (j, j_1, j_2, 1) \), respectively, that is on nearest lateral 2-face from which sub basis, \( \ell \), and sub height, \( (j_1, j_2, 1) \), respectively, branch off.

Cathetus lying on sub basis, \( \ell \), carries 3-products:

\[ \{(i_1 + i_0), i_2, (i_3 - i_0)\} ; \quad 1 \leq i_0 \leq i_3 - 1 ; \quad (11) \]

passing from \( \{(i_1 + 1), i_2, (i_3 - 1)\} \) to \( \{(i_1 + i_3 - 1), i_2, 1\} \) for increasing \( i_0 \), the last displaced on nearest sub basis side. Using the above results yields the explicit expression:

\[
\left( j_1 + \frac{\ell - j}{3} + i_0 \right) \left( \frac{\ell - j}{3} + 1 - i_0 \right) = \left( j_1 + \frac{\ell - j}{3} \right) \left( \frac{\ell - j}{3} + 1 \right) + \\
+ i_0 \left( \frac{\ell - j}{3} + 1 \right) - i_0 \left( j_1 + \frac{\ell - j}{3} \right) - i_0^2 = \left( j_1 + \frac{\ell - j}{3} \right) \times \\
\times \left( \frac{\ell - j}{3} + 1 \right) + i_0 \left( \frac{\ell - j}{3} + 1 - j_1 - \frac{\ell - j}{3} - i_0 \right) ; \\
1 \leq i_0 \leq i_3 - 1 = \frac{\ell - j}{3} ;
\]

in conclusion:

\[
\left( j_1 + \frac{\ell - j}{3} + i_0 \right) \left( \frac{\ell - j}{3} + 1 - i_0 \right) = \left( j_1 + \frac{\ell - j}{3} \right) \left( \frac{\ell - j}{3} + 1 \right) + \\
+ i_0 (1 - j_1 - i_0) ; \quad 1 \leq i_0 \leq \frac{\ell - j}{3} ; \quad (12)
\]
where, in the case under discussion, $i_0$-th place starting from $\{i_1, i_2, i_3\}$ is equivalent to $(i_3 - i_0)$-th place starting from $\{(i_1 + i_3 - 1), i_2, 1\}$.

Cathetus lying on sub height, $(j_1, j_2, 1)$, carries 3-products:

$$\{(j_1 + j_0 - 1), (j_2 + j_0), j_0\} ; \quad 1 \leq j_0 \leq i_3 - 1 \quad (13)$$

passing from sub height boundary, $\{j_1, j_2, 1\}$, to $\{(i_1 - 1), (i_2 - 1), (i_3 - 1)\}$, for increasing $j_0$. In the case under discussion, $j_0$-th place starting from $\{i_1, i_2, i_3\}$ is equivalent to $(i_3 - j_0)$-th place starting from $\{j_1, j_2, 1\}$.

The sum of generic pairs of products of the kind considered (i.e. factors are displaced on different cathetuses), for which $i_0 = j_0$, yields:

$$\left( j_1 + \frac{\ell - j}{3} + i_0 \right) \left( \frac{\ell - j}{3} + 1 - i_0 \right) + (j_1 + j_0 - 1)j_0 =$$

$$= \left( j_1 + \frac{\ell - j}{3} - j_1 \right) \left( \frac{\ell - j}{3} + 1 \right) + i_0(1 - j_1 - i_0) - (1 - j_1 - i_0)i_0 =$$

$$= \left( j_1 + \frac{\ell - j}{3} \right) \left( \frac{\ell - j}{3} + 1 \right) ; \quad (14)$$

which equals related product displaced on the right vertex, regardless from the pair considered.

### 3.3 4-Pythagorean triangle

With regard to 4-Pythagorean table, a suitable rigid rotation of the Cartesian orthogonal reference frame, $(Ox_1 x_2 x_3 x_4)$, makes 4-sector of positive 2$^4$-ant aligned with the vertical and positive coordinate semiaxes displaced like slope roof. Accordingly, 4-Pythagorean table attains 4-triangle configuration which is bounded on the top and unbounded on the bottom.

Let 4-Pythagorean triangle of order, $k$, be defined as right isosceles 4-triangle carrying the 4-area, 1, on the right vertex and the 4-area, $k$, on the remaining vertexes, where hypotenuse makes basis and cathetuses make oblique sides, in connection with lateral 2-faces.

Basis of 4-Pythagorean triangle of order, $\ell$, $1 \leq \ell \leq k$, is sub basis of 4-Pythagorean triangle of order, $k$. Sub basis, $\ell$, carries 4-areas of perfect 4-rectangles of 2$^{-3}$-perimeter, $p_{4, \ell} = \ell + 3$, symmetrically displaced with respect to height which, in turn, carries the sequence of perfect 4-squares.

Sub heights, $(j_1, j_2, j_3, j_4)$, parallel and symmetrically displaced with respect to height, carry the sequence of perfect 4-rectangles where 2$^{-3}$-perimeter is incremented by four units moving downwards. Lateral 3-faces carry each a 3-Pythagorean triangle of order, $k$. The Pythagorean triangle of order, $k = 10$, is reproduced in Fig. 9.

Pythagorean triangle of order, $k$, exhibits the following properties.
Figure 9: Representation of the 4-Pythagorean triangle of order, \( k = 10 \), seen head-on. The right vertex, carrying unity, is the trace of 4-sector of positive 2\(^4\)-ant on figure plane. Sub basis, \( \ell \), carries 4-areas of perfect 4-rectangles exhibiting 2\(^{-3}\)-perimeter, \( p_{4, \ell} = \ell + 3 \), symmetrically placed with respect to the height which, in turn, carries the sequence of perfect 4-squares. Sub heights, \((j_1, j_2, j_3, j_4)\), symmetrically placed with respect to the height, carry the sequence of perfect 4-rectangles exhibiting 2\(^{-3}\)-perimeter incremented by four units moving downwards. Lateral 3-faces carry each a 3-Pythagorean triangle of order, \( k = 10 \). To gain clarity, only 4-areas displaced on the boundary of sub bases are shown. Perspective caption: right vertex on the top is placed on the centre of major square; remaining vertexes are placed on major square vertexes; basis sides are placed on major square sides and diagonals; oblique sides are placed on minor squares diagonals, joining each basis vertex with right vertex on the top.
I. The total number of sub bases is $k$.

II. The total number of sub heights equals the total number of 4-products placed on lateral 3-faces.

III. Sub basis, $\ell$, carries $[(\ell + 2)(\ell + 1)\ell/6]$ 4-areas, which is the number of 4-tuples, \{\(i_1, i_2, i_3, i_4\)\}, where $i_1, i_2, i_3, i_4$, are nonzero natural numbers satisfying $i_1 + i_2 + i_3 + i_4 = \ell + 3$.

IV. Sub heights, $(j_1, j_2, j_3, j_4)$, branching off from the boundary of sub basis, $j = j_1 + j_2 + j_3 + j_4 - 3$, carry each \{1 + \text{Int}[\((\ell - j)/4\)]\} 4-areas, up to sub basis, $\ell$.

V. Sub heights cross (i.e. have one 4-area in common with) sub bases four by four, starting from the boundary of sub basis where sub height branchs off. Accordingly, odd sub heights cross odd sub bases and even sub heights cross even sub bases.

VI. 4-area can be defined as intersection between sub basis, $\ell$, $1 \leq \ell \leq k$, and sub height, $(j_1, j_2, j_3, j_4)$, $\min(j_1, j_2, j_3, j_4) = 1$, $\max(j_1, j_2, j_3, j_4) \leq j \leq \ell$, and related coordinates read $(\ell, j_1, j_2, j_3, j_4)$.

VII. 4-area, $(\ell, j_1, j_2, j_3, j_4)$, is separated from nearest lateral 3-face by an equal amount of 4-areas along both sub basis, $\ell$, and sub height, $(j_1, j_2, j_3, j_4)$.

Let $p_{4,\ell}$ be $2^{-3}$-perimeter of generic, perfect 4-rectangle, where related 4-area is placed on sub basis, $\ell$, of 4-Pythagorean triangle of order, $k$, and let $i_1, i_2, i_3, i_4$, be the length of each side. Then the following relation holds:

\[ p_{4,\ell} = i_1 + i_2 + i_3 + i_4 = \ell + 3 ; \quad 1 \leq \ell \leq k ; \]

and related 4-area can be expressed as:

\[ i_1i_2i_3i_4 = i_1i_2i_3(p_{4,\ell} - i_1 - i_2 - i_3) = i_1i_2i_3(\ell + 3 - i_1 - i_2 - i_3) ; \]

\[ 3 \leq i_1 + i_2 + i_3 \leq \ell + 2 ; \]

where 4-product displacement is determined by three factors, $i_1, i_2, i_3$.

More specifically, the sequence of 4-areas on sub basis, $\ell$, can be determined along the following steps.

1. Start from the 4-area of perfect 4-rectangle exhibiting unit height and unit minor bases, $(\ell \cdot 1 \cdot 1 \cdot 1)$, displaced on a vertex.

2. Increase minor basis length and decrease major basis length by a unit in both cases, and maintain remaining minor basis and height, up to the 4-area, $(1 \cdot \ell \cdot 1 \cdot 1)$, displaced on the next vertex.

3. Increase previously preserved minor basis length and decrease major basis length by a unit in both cases, and maintain remaining minor basis and height, up to the 4-area, $(1 \cdot 1 \cdot \ell \cdot 1)$, displaced on the next vertex.
(4) Increase height length and decrease major basis length by a unit in both cases, and maintain remaining minor bases, up to the 4-area, \((1 \cdot 1 \cdot 1 \cdot \ell)\), displaced on the next vertex.

(5) Increase firstly decremented major basis length and decrease height length by a unit in both cases, and maintain remaining minor bases, up to the 4-area, \((\ell \cdot 1 \cdot 1 \cdot 1)\), displaced on the starting vertex.

The above steps relate to sub basis 1-faces i.e. sides: taking a single minor basis length larger than unity, sub basis 2-faces i.e. equilateral triangles can be taken into consideration, and so on, until the sequence of 4-areas on sub basis boundary is completed.

Then a similar procedure is applied to first sub basis inner contour, starting from the 4-area, \([((\ell - 3) \cdot 2 \cdot 2 \cdot 2)]\), displaced on a vertex. In general, a similar procedure is applied to \((\ell_0 - 1)\)-th inner contour, starting from the 4-area, \([((\ell - 3\ell_0 + 3) \cdot \ell_0 \cdot \ell_0 \cdot \ell_0)]\), displaced on a vertex. The sequence of 4-areas on sub basis is completed when inequality, \(\ell_0 \leq \ell - 3\ell_0 + 3\), i.e. \(4\ell_0 \leq \ell + 3\), is no longer satisfied or, in other words, whole sub basis has been assembled.

Perfect 4-rectangle 4-area may be specified as:

\[
\begin{align*}
i_1i_2i_3i_4 &= i_1i_2i_3(2L+4-i_1-i_2-i_3) \ ; \quad 3 \leq i_1+i_2+i_3 \leq 2L+3 \ ; \quad (15) \\
\text{with regard to odd sub basis, } \ell = 2L+1; \\
i_1i_2i_3i_4 &= i_1i_2i_3(2L+3-i_1-i_2-i_3) \ ; \quad 3 \leq i_1+i_2+i_3 \leq 2L+2 \ ; \quad (16) \\
\text{with regard to even sub basis, } \ell = 2L. 
\end{align*}
\]

Keeping in mind products with no even factor yield odd numbers while products with at least one even factor yield even numbers, 4-areas displaced on vertexes of odd sub bases are odd and alternate with even 4-areas on sub basis boundary. On the other hand, 4-areas displaced on even sub bases are even.

The same holds passing from sub basis boundary to sub basis boundary centre: inner sub basis boundary contours where odd 4-areas are displaced on vertexes carry odd 4-areas alternate with even 4-areas; inner sub basis boundary contours where even 4-areas are displaced on vertexes carry even 4-areas only.

The same holds passing from sub basis boundary to sub basis centre: inner sub basis contours, where odd 4-areas are displaced on vertexes, carry odd 4-areas alternate with even 4-areas; inner sub basis contours, where even 4-areas are displaced on vertexes, carry even 4-areas only.

With regard to the 4-Pythagorean triangle of order, \(k = 8\), sub bases are shown in Fig. 10, where 3-triangles are opened to make lateral 2-faces lie on figure plane, with regard to the boundary and inner contours \((\ell \geq 5)\). Each
side being in common between adjacent 2-faces, 4-areas displaced therein must appear only once. Three pairs of sides superimposing after 2-face closure, 4-areas appearing on a side of each pair (in Italics) are not to be counted, as inferred from the geometric figure.

To gain more insight, 4-Pythagorean triangle of order, $k$, has to be conceived as made of ordered 4-tuples of factors, or 4-products, \( \{i_1, i_2, i_3, i_4\} = i_1 \cdot i_2 \cdot i_3 \cdot i_4 \), instead of 4-areas, $i_1 i_2 i_3 i_4$, passing from the representation of Fig. 10 to the representation of Fig. 11, restricted to $k = 7$ to save space, where both brackets and commas are omitted for the same reason, keeping in mind each factor relates to a single figure.

An inspection of Fig. 11 discloses sub basis, $\ell$, can be assembled through the following steps, where increments and decrements are intended by a unit.

1. With regard to basis 2-face, start from the vertex on the left, $(\ell, 1, 1, 1)$, along horizontal side decrementing the first factor, incrementing the second, preserving remainings, up to the vertex on the right, $(1, \ell, 1, 1)$.

2. With regard to basis 2-face, start from the vertex on the left, $(\ell, 1, 1, 1)$, along oblique side decrementing the first factor, incrementing the third, preserving remainings, up to the vertex on the top, $(1, 1, \ell, 1)$.

3. With regard to basis 2-face, start from the vertex on the top, $(1, 1, \ell, 1)$, along oblique side decrementing the third factor, incrementing the second, preserving remainings, up to the vertex on the right, $(1, \ell, 1, 1)$.

4. With regard to front lateral 2-face, start from the vertex on the bottom, $(1, 1, 1, \ell)$, along each oblique side decrementing the fourth factor, incrementing the first or the second, preserving remainings, up to the opposite vertex, $(\ell, 1, 1, 1)$, or $(1, \ell, 1, 1)$, respectively.

5. With regard to left back lateral 2-face, start from the vertex on the left, $(1, 1, 1, \ell)$, along horizontal side decrementing the fourth factor, incrementing the third, preserving remainings, up to the opposite vertex, $(1, 1, \ell, 1)$; then along oblique side decrementing the third factor, incrementing the first, preserving remainings, up to the opposite vertex, $(\ell, 1, 1, 1)$.

6. With regard to right back lateral 2-face, start from the vertex on the right, $(1, 1, 1, \ell)$, along horizontal side decrementing the fourth factor, incrementing the third, preserving remainings, up to the opposite vertex, $(1, 1, \ell, 1)$; then along oblique side decrementing the third factor, incrementing the second, preserving remainings, up to the opposite vertex, $(1, \ell, 1, 1)$.

7. Having 4-areas been displaced on basis boundary sides, keep in mind three among them coincide with their counterparts after lateral 2-face closure for making sub basis, that is a 3-triangle.

8. With regard to $(\ell_0 - 1)$-th basis 2-face inner contour, repeat the above procedure (1)-(6) starting from the vertex on the left, $[(\ell - 2\ell_0 + 2), \ell_0, \ell_0, 1]$. 
Figure 10: Representation of the 4-Pythagorean triangle of order, \( k = 8 \), through sub bases (opened to make lateral 2-faces lie on figure plane) starting from the vertex, where unit 4-area is displaced, up to the basis. Sub basis, \( \ell \), carries 4-areas of perfect 4-rectangles exhibiting \( 2^{-3} \)-perimeter, \( p_{4,\ell} = \ell + 3 \), symmetrically placed with respect to the height which, in turn, carries the sequence of perfect 4-squares. Sub heights, \((j_1, j_2, j_3, j_4)\), symmetrically placed with respect to the height, carry the sequence of perfect 4-rectangles exhibiting \( 2^{-3} \)-perimeter incremented by four units moving downwards. Inner contour \((\ell \geq 5)\) is shown on the right of related boundary. Each side being in common between adjacent 2-faces, 4-areas displaced therein must appear only once. Three pairs of sides superimposing after 2-face closure, 4-areas appearing on a side of each pair (in Italics) are not to be counted, as inferred from the geometric figure.
Figure 11: Representation of the 4-Pythagorean triangle of order, \( k = 7 \), through sub bases (opened to make lateral 2-faces lie on figure plane) in terms of 4-products, \( \{i_1, i_2, i_3, i_4\} = i_1 \cdot i_2 \cdot i_3 \cdot i_4 \), where brackets and commas are omitted to save space, keeping in mind each factor relates to a single figure. For 4-products displaced on a selected sub basis, factor sum is preserved. For 4-products displaced on a selected sub height, factor difference (between extreme values) is preserved.
for \( \ell_0 \) values within the range of natural numbers, 1 < \( \ell_0 \) ≤ \( \text{Int}[(\ell + 2)/3] \), where \( \ell_0 = 1 \) relates to 2-face boundary and \( \ell_0 = \text{Int}[(\ell + 2)/3] \) to 2-face inner contour nearest to 2-face centre.

(9) With regard to each lateral 2-face inner contour, act as in (8) keeping in mind unit factor appears in 4-areas displaced on lateral 2-faces, but on different places (within the product) for different 2-faces, until sub basis boundary is completely assembled.

(10) With regard to \((\ell_0 − 1)\)-th sub basis inner contour, repeat the above procedure (1)-(9) starting from the vertex on the left, \([((\ell − 3\ell_0 + 3), \ell_0, \ell_0, \ell_0), \ell_0 = 1 \) relates to sub basis boundary and \( \ell_0 = \text{Int}[(\ell + 3)/4] \) to inner contour nearest to sub basis centre, until sub basis is completely assembled.

For 4-products displaced on a selected sub basis, factor sum, \( i_1 + i_2 + i_3 + i_4 \), is preserved. For 4-products displaced on a selected sub height, factor difference, \( \max(i_1, i_2, i_3, i_4) − \min(i_1, i_2, i_3, i_4) \), is preserved. For 4-products displaced on lateral 3-faces, factors exhibit same properties as in 3-Pythagorean triangle of order, \( k \).

With regard to 4-Pythagorean triangle of order, \( k \geq \ell \), an inspection of Fig.11 and Fig.7 discloses sub basis, \( \ell \), can be assembled starting from 3-Pythagorean triangle of order, \( \ell \), putting the factor, \( i \), on the fourth place of each 3-product displaced on sub basis, \( (\ell − i + 1) \), for values of \( i \) satisfying \( 1 \leq i \leq \ell \). The range, \( 1 \leq \ell \leq 5 \), is represented in Fig.12.

With regard to sub basis boundary and inner contours, it can be seen sub heights crossing 4-products, displaced on vertexes and 2-faces centres, appear in quadruplets; sub heights crossing 4-products, displaced on 1-face centre, appear in sextuplets; sub heights crossing 4-products, displaced otherwise, appear in dozentuplets; height crossing 4-products, displaced on sub basis centre i.e. perfect 4-squares, appears in singlet.

With regard to 4-Pythagorean triangle of order, \( k \), let \( \{i_1, i_2, i_3, i_4\} \) be 4-product of coordinates, \((\ell, j_1, j_2, j_3, j_4)\), that is displaced on the intersection between sub basis, \( \ell \), and sub height, \((j_1, j_2, j_3, j_4)\); \( \min(j_1, j_2, j_3, j_4) = 1 \); \( \max(j_1, j_2, j_3, j_4) \leq j \leq \ell \); \( j_1 + j_2 + j_3 + j_4 = j + 3 \). Factors within 4-products can be set in \( 4! = 24 \) different ways, which implies \( 24 \) different inequalities involving factors. Accordingly, sub basis can be divided into twenty four equivalent regions, as shown in Fig.13. Factors belonging to 4-products displaced on the same region are ordered in the same way, while the contrary holds for factors belonging to 4-products displaced on different regions.

For 4-products displaced on sub basis, \( \ell \), the following can be established: \( \{i_1, i_2, i_3, i_4\} = \{(\ell − \ell_1 − \ell_2 − \ell'), (\ell_1 + 1), (\ell_2 + 1), (\ell' + 1)\}, 0 \leq \ell_1 + \ell_2 + \ell' \leq \ell − 3 \), keeping in mind related \( 2^{-3}\)-perimeter reads \( p_{4,\ell} = (\ell − \ell_1 − \ell_2 − \ell') + (\ell_1 + 1) + (\ell_2 + 1) + (\ell' + 1) \).
Figure 12: With regard to 4-Pythagorean triangle of order, $k \geq \ell$, sub basis, $\ell$ (right panels), can be assembled starting from 3-Pythagorean triangle of order, $\ell$ (left panels), adding the factor, $i$ (in Italics), on the fourth place of each 3-product displaced on sub basis, $(\ell - i + 1)$, for values of $i$ satisfying $1 \leq i \leq \ell$. The range shown is $1 \leq \ell \leq 5$. Same notation as in Fig. 11.
Figure 13: With regard to 4-Pythagorean triangle of order, $k$, sub basis can be partitioned into 24 equivalent tetrahedrons, where factors are similarly or dissimilarly ordered according if parent 4-products belong to same or different regions, respectively. Sub basis 2-surface is shown open, making lateral 2-faces lie on figure plane. Numbered regions relate to tetrahedron bases, opposite to common vertex on sub basis centre, which trace on figure plane coincides with basis 2-face centre.
For 4-products displaced on sub height, \((j_1, j_2, j_3, j_4)\), the following can be established: \(\{i_1, i_2, i_3, i_4\} = \{(j_1 + j')(j_2 + j'), (j_3 + j'), (j_4 + j')\}\), 0 \(\leq j' \leq j - 3\), \(\min(j_1, j_2, j_3, j_4) = 1\), keeping in mind the difference, \(\max[(j_1 + j'), (j_2 + j'), (j_3 + j'), (j_4 + j')] - \min[(j_1 + j'), (j_2 + j'), (j_3 + j'), (j_4 + j')] = \max(j_1, j_2, j_3, j_4) - \min(j_1, j_2, j_3, j_4)\), remains unchanged.

In the case under discussion, 4-products are displaced on the intersection between sub basis, \(\ell\), and sub height, \((j_1, j_2, j_3, j_4)\). Accordingly, the following relations hold:

\[
(\ell - \ell_1 - \ell_2 - \ell')(\ell_1 + 1)(\ell_2 + 1)(\ell' + 1) = \\
= (j_1 + j')(j_2 + j')(j_3 + j')(j_4 + j') \\
\begin{cases}
\ell - \ell_1 - \ell_2 - \ell' = j_1 + j' \\
\ell_1 + 1 = j_2 + j' \\
\ell_2 + 1 = j_3 + j' \\
\ell' + 1 = j_4 + j'
\end{cases} \\
\ell = j_2 + j' - 1 + j_3 + j' - 1 + j_4 + j' - 1 + j_1 + j' = \\
= j_1 + j_2 + j_3 + j_4 - 3 + 4j' \\
\begin{cases}
\ell = j + 4j' \\
\ell_1 = j_2 - 1 + \frac{\ell - j}{4} \\
\ell_2 = j_3 - 1 + \frac{\ell - j}{4} \\
\ell' = j_4 - 1 + \frac{\ell - j}{4}
\end{cases}
\]

from which the explicit expression of the first factor can be inferred as:

\[
i_1 = \ell - \ell_1 - \ell_2 - \ell' = \\
= \ell - j_2 + 1 - \frac{\ell - j}{4} - j_3 + 1 - \frac{\ell - j}{4} - j_4 + 1 - \frac{\ell - j}{4} = \\
= \ell - j_2 - j_3 - j_4 + 3 - 3\frac{\ell - j}{4} = \ell - j + j_1 - 4\frac{\ell - j}{4} + \frac{\ell - j}{4} = \\
= \ell - j + j_1 - \ell + j + \frac{\ell - j}{4}
\]

and the following relations hold:

\[
\begin{aligned}
i_1 &= \ell - \ell_1 - \ell_2 - \ell' = j_1 + \frac{\ell - j}{4} ; \\
i_2 &= \ell_1 + 1 = j_2 + \frac{\ell - j}{4} ; \\
i_3 &= \ell_2 + 1 = j_3 + \frac{\ell - j}{4} ; \\
i_4 &= \ell' + 1 = j_4 + \frac{\ell - j}{4} ; \\
i_1 + i_2 + i_3 + i_4 &= \ell + 3 ; \\
j_1 + j_2 + j_3 + j_4 &= j + 3 
\end{aligned}
\]

(17a) (17b) (17c) (17d) (17e)
\[ \min(j_1, j_2, j_3, j_4) = 1 \; ; \; \max(j_1, j_2, j_3, j_4) \leq j \; ; \] (17f)
\[ i_k - j_k = \frac{\ell - j}{4} \; ; \; k = 1, 2, 3, 4 \; ; \] (17g)

accordingly, values of 4-products can be determined from the knowledge of related coordinates.

Let 4-product, \( \{i_1, i_2, i_3, i_4\} \), of coordinates, \((\ell, j_1, j_2, j_3, j_4)\), be displaced on right vertex of right isosceles triangle, where cathetuses lie on sub basis, \( \ell \), and on sub height, \((j_1, j_2, j_3, j_4)\), and hypotenuse lies on nearest lateral 2-face implying \( i_1 \geq i_2 \geq i_3 \geq i_4 \) with no loss of generality, hence \( j \geq j_1 \geq j_2 \geq j_3 \geq j_4 = 1 \). Accordingly, remaining vertexes carry 4-products, \( \{(\ell - j + 1), j_1, j_2, j_3, j_4\} \), of coordinates, \([\ell, (\ell - j + 1), j_1, 1], (j, j_1, j_2, j_3, 1)\), respectively, that is on nearest lateral 2-face from which sub basis, \( \ell \), and sub height, \((j_1, j_2, j_3, 1)\), respectively, branch off.

Cathetus lying on sub basis, \( \ell \), carries 4-products:
\[
\{ (i_1 + i_0), i_2, i_3, (i_4 - i_0) \} \; ; \; 1 \leq i_0 \leq i_4 - 1 \; ;
\] (18)

passing from \( \{(i_1 + 1), i_2, i_3, (i_4 - 1)\} \) to \( \{(i_1 + i_4 - 1), i_2, i_3, 1\} \) for increasing \( i_0 \), the last displaced on nearest sub basis side. Using the above results yields the explicit expression:
\[
\left( j_1 + \frac{\ell - j}{4} + i_0 \right) \left( \frac{\ell - j}{4} + 1 - i_0 \right) = \left( j_1 + \frac{\ell - j}{4} \right) \left( \frac{\ell - j}{4} + 1 \right) + \\
+ i_0 \left( \frac{\ell - j}{4} + 1 \right) - i_0 \left( j_1 + \frac{\ell - j}{4} \right) - i_0^2 = \left( j_1 + \frac{\ell - j}{4} \right) \left( \frac{\ell - j}{4} + 1 \right) + \\
+ i_0 \left( \frac{\ell - j}{4} + 1 - j_1 - \frac{\ell - j}{4} - i_0 \right) ; \\
1 \leq i_0 \leq i_4 - 1 = \frac{\ell - j}{4} ;
\]
in conclusion:
\[
\left( j_1 + \frac{\ell - j}{4} + i_0 \right) \left( \frac{\ell - j}{4} + 1 - i_0 \right) = \left( j_1 + \frac{\ell - j}{4} \right) \left( \frac{\ell - j}{4} + 1 \right) + \\
+ i_0 (1 - j_1 - i_0) ; \; \; 1 \leq i_0 \leq \frac{\ell - j}{4} ;
\] (19)

where, in the case under discussion, \( i_0 \)-th place starting from \( \{i_1, i_2, i_3, i_4\} \) is equivalent to \( (i_4 - i_0) \)-th place starting from \( \{(i_1 + i_4 - 1), i_2, i_3, 1\} \).

Cathetus lying on sub height, \((j_1, j_2, j_3, 1)\), carries 4-products:
\[
\{(j_1 + j_0 - 1), (j_2 + j_0), (j_3 + j_0), j_0 \} \; ; \; 1 \leq j_0 \leq i_4 - 1 ;
\] (20)
passing from sub height boundary, \( \{j_1, j_2, j_3, 1\} \), to \( \{(i_1 - 1), (i_2 - 1), (i_3 - 1), (i_4 - 1)\} \), for increasing \( j_0 \). In the case under discussion, \( j_0 \)-th place starting from \( \{i_1, i_2, i_3, i_4\} \) is equivalent to \( (i_4 - j_0) \)-th place starting from \( \{j_1, j_2, j_3, 1\} \).

The sum of generic pairs of products of the kind considered (i.e. factors are displaced on different cathetuses), for which \( i_0 = j_0 \), yields:

\[
\left( j_1 + \frac{\ell - j}{4} + i_0 \right) \left( \frac{\ell - j}{4} + 1 - i_0 \right) + (j_1 + j_0 - 1)j_0 =
\]

\[
= \left( j_1 + \frac{\ell - j}{4} \right) \left( \frac{\ell - j}{4} + 1 \right) + i_0(1 - j_1 - i_0) - (1 - j_1 - i_0)i_0 =
\]

\[
= \left( j_1 + \frac{\ell - j}{4} \right) \left( \frac{\ell - j}{4} + 1 \right);
\]

which equals related product displaced on the right vertex, regardless from the pair considered.

### 3.4 \( n \)-Pythagorean triangle

With regard to \( n \)-Pythagorean table, a suitable rigid rotation of the Cartesian orthogonal reference frame, \((Ox_1x_2...x_n)\), makes \( n \)-sector of positive \( 2^n \)-ant aligned with the vertical and positive coordinate semiaxes displaced like slope roof. Accordingly, \( n \)-Pythagorean table attains \( n \)-triangle configuration which is bounded on the top and unbounded on the bottom.

Let \( n \)-Pythagorean triangle of order, \( k \), be defined as right isosceles \( n \)-triangle carrying the \( n \)-area, 1, on the right vertex and the \( n \)-area, \( k \), on the remaining vertexes, where hypotenuse makes basis and cathetuses make oblique sides, in connection with lateral \( 2 \)-faces.

Basis of \( n \)-Pythagorean triangle of order, \( \ell \), \( 1 \leq \ell \leq k \), is sub basis of \( n \)-Pythagorean triangle of order, \( k \). Sub basis, \( \ell \), carries \( n \)-areas of perfect \( n \)-rectangles of \( 2^{-(n-1)} \)-perimeter, \( p_{n,\ell} = \ell + (n - 1) \), symmetrically displaced with respect to height which, in turn, carries the sequence of perfect \( n \)-squares.

Sub heights, \( (j_1, j_2, ..., j_n) \), parallel and symmetrically displaced with respect to height, carry the sequence of perfect \( n \)-rectangles where \( 2^{-(n-1)} \)-perimeter is incremented by \( n \) units moving downwards. Lateral \( (n - 1) \)-faces carry each a \((n - 1)\)-Pythagorean triangle of order, \( k \).

\( n \)-Pythagorean triangle of order, \( k \), exhibits the following properties.

**I.** The total number of sub bases is \( k \).

**II.** The total number of sub heights equals the total number of \( n \)-products placed on lateral \((n - 1)\)-faces.

**III.** Sub basis, \( \ell \), carries \([ (\ell + n - 1)(\ell + n - 2)...\ell/(n - 1)! ] \) \( n \)-areas, which is the number of \( n \)-tuples, \( \{i_1, i_2, ..., i_n\} \), where \( i_1, i_2, ..., i_n \), are nonzero natural numbers satisfying \( i_1 + i_2 + ... + i_n = \ell + (n - 1) \).
IV. Sub heights, \((j_1, j_2, \ldots, j_n)\), branching off from the boundary of sub basis, 
\(j = j_1 + j_2 + \ldots + j_n - (n - 1)\), carry each \(\{1 + \text{Int}[k - j]/n\}\) \(n\)-areas up to 
sub basis, \(\ell\).

V. Sub heights cross (i.e. have one \(n\)-area in common with) sub bases \(n\) by 
\(n\), starting from the boundary of sub basis where sub height branches off.

VI. \(n\)-area can be defined as intersection between sub basis, \(\ell\), \(1 \leq \ell \leq k\), and 
sub height, \((j_1, j_2, \ldots, j_n)\), \(\min(j_1, j_2, \ldots, j_n) = 1\), \(\max(j_1, j_2, \ldots, j_n) \leq j \leq \ell\), and 
related coordinates read \((\ell, j_1, j_2, \ldots, j_n)\).

VII. \(n\)-area, \((\ell, j_1, j_2, \ldots, j_n)\), is separated from nearest lateral \((n-1)\)-face by an 
equal amount of \(n\)-areas along both sub basis, \(\ell\), and sub height, \((j_1, j_2, \ldots, j_n)\).

Let \(p_{n,\ell}\) be \(2^{-(n-1)}\)-perimeter of generic, perfect \(n\)-rectangle, where related 
\(n\)-area is placed on sub basis, \(\ell\), of \(n\)-Pythagorean triangle of order, \(k\), and let 
i_1, i_2, \ldots, i_n, be the length of each side. Then the following relation holds:

\[
p_{n,\ell} = i_1 + i_2 + \ldots + i_n = \ell + (n - 1) ; \quad 1 \leq \ell \leq k ;
\]

and related \(n\)-area can be expressed as:

\[
i_1i_2\ldots i_n = i_1i_2\ldots i_{n-1}(p_{n,\ell} - i_1 - i_2 - \ldots - i_{n-1}) =
\]

\[
i_1i_2\ldots i_{n-1}(\ell + n - 1 - i_1 - i_2 - \ldots - i_{n-1}) ;
\]

\[
n - 1 \leq i_1 + i_2 + \ldots + i_{n-1} \leq \ell + n - 2 ;
\]

where \(n\)-product displacement is determined by \((n-1)\) factors, \(i_1, i_2, \ldots, i_{n-1}\).

More specifically, the sequence of \(n\)-areas on the boundary of sub basis, \(\ell\), 
can be determined along the following steps.

1) Start from the \(n\)-area of perfect \(n\)-rectangle exhibiting unit height and unit 
minor bases, \((\ell \cdot 1 \cdot 1 \cdot \ldots \cdot 1)\), displaced on a vertex.

2) Increase minor basis length and decrease major basis length by a unit in 
both cases, and maintain remaining minor bases and height, up to the \(n\)-area, 
\((1 \cdot \ell \cdot 1 \cdot \ldots \cdot 1)\), displaced on the next vertex.

3) Increase previously preserved minor basis length and decrease major basis 
length by a unit in both cases, and maintain remaining minor bases and height, 
up to the \(n\)-area, \((1 \cdot 1 \cdot \ell \ldots \cdot 1)\), displaced on the next vertex.

4) Proceed iteratively as shown in (3) up to the \(n\)-area, \((1 \cdot 1 \ldots \cdot \ell \cdot 1)\), 
displaced on a next vertex.

5) Increase height length and decrease major basis length by a unit in both 
cases, and maintain remaining minor bases, up to the \(n\)-area, \((1 \cdot 1 \ldots \cdot 1 \cdot \ell)\), 
displaced on the next vertex.
(6) Increase firstly decremented major basis length and decrease height length by a unit in both cases, and maintain remaining minor bases, up to the \( n \)-area, \((\ell \cdot 1 \cdot 1 \cdot \ldots \cdot 1)\), displaced on the starting vertex.

The above steps relate to sub basis 1-faces i.e. sides: taking a single minor basis length larger than unity, sub basis 2-faces i.e. equilateral triangles can be taken into consideration, and so on, until the sequence of \( n \)-areas on sub basis boundary is completed.

Then a similar procedure is applied to first sub basis inner contour, starting from the \( n \)-area, \([[(\ell-(n-1)) \cdot 2 \cdot 2 \cdot \ldots \cdot 2]]\), displaced on a vertex. In general, a similar procedure is applied to \((\ell_0-1)\)-th inner contour, starting from the \( n \)-area, \([[(\ell-(n-1))\ell_0+(n-1)] \cdot \ell_0 \cdot \ell_0 \cdot \ell_0]\), displaced on a vertex. The sequence of \( n \)-areas on sub basis is completed when inequality, \( \ell_0 \leq \ell-(n-1)\ell_0+(n-1) \), i.e. \( n\ell_0 \leq \ell+(n-1) \), is no longer satisfied or, in other words, whole sub basis has been assembled.

Perfect \( n \)-rectangle \( n \)-area may be specified as:

\[ i_1 i_2 \ldots i_n = i_1 i_2 \ldots i_{n-1} (2L + n - i_1 - i_2 - \ldots - i_{n-1}) \]
\[ n - 1 \leq i_1 + i_2 + \ldots + i_{n-1} \leq 2L + n - 1 \]  \( (22) \)

with regard to odd sub basis, \( \ell = 2L + 1 \);

\[ i_1 i_2 \ldots i_n = i_1 i_2 \ldots i_{n-1} (2L + n - 1 - i_1 - i_2 - \ldots - i_{n-1}) \]
\[ n - 1 \leq i_1 + i_2 + \ldots + i_{n-1} \leq 2L + n - 2 \]  \( (23) \)

with regard to even sub basis, \( \ell = 2L \).

To gain more insight, \( n \)-Pythagorean triangle of order, \( k \), has to be conceived as made of ordered \( n \)-tuples of factors, or \( n \)-products, \( \{i_1, i_2, \ldots, i_n\} = i_1 \cdot i_2 \cdot \ldots \cdot i_n \), instead of \( n \)-areas, \( i_1 i_2 \ldots i_n \).

For \( n \)-products displaced on a selected sub basis, factor sum, \( i_1 + i_2 + \ldots + i_n \), is preserved. For \( n \)-products displaced on a selected sub height, factor difference, \( \max(i_1, i_2, \ldots, i_n) - \min(i_1, i_2, \ldots, i_n) \), is preserved. For \( n \)-products displaced on lateral \((n-1)\)-faces, factors exhibit same properties as in \((n-1)\)-Pythagorean triangle of order, \( k \).

With regard to \( n \)-Pythagorean triangle of order, \( k \geq \ell \), the extension of results found for \( n = 3, 4 \) implies sub basis, \( \ell \), can be assembled starting from \((n-1)\)-Pythagorean triangle of order, \( \ell \), putting the factor, \( i \), on the \( n \)-th place of each \((n-1)\)-product displaced on sub basis, \((\ell - i + 1)\), for values of \( i \) satisfying \( 1 \leq i \leq \ell \).

With regard to sub basis boundary and inner contours, it can be inferred sub heights crossing \( n \)-products, displaced on vertexes and centre of \((n-2)\)-faces, appear in \( \binom{n}{1} \)-tuples; sub heights crossing \( n \)-products, displaced on centre of 1-faces and \((n-3)\)-faces, appear in \( \binom{n}{2} \)-tuples; sub heights crossing
n-products, displaced on centre of 2-faces and \((n - 4)\)-faces, appear in \(\binom{n}{3}\)-tuplets; ...; sub heights crossing n-products, displaced on centre of \((n/2 - 1)\)-faces, appear in \(\binom{n}{n/2}\)-tuplets for even \(n\), and sub heights crossing n-products, displaced on centre of \([\left(\frac{n+1}{2}\right)-1]\)-faces, appear in \(\binom{n}{\left(\frac{n+1}{2}\right)/2}\)-tuplets for odd \(n\); sub heights crossing n-products, displaced otherwise, appear in \(2\binom{n}{2}\)-tuplets; height crossing n-products, displaced on sub basis centre i.e. perfect \(n\)-squares, appears in singlet.

With regard to \(n\)-Pythagorean triangle of order, \(k \geq \ell\), let \(\{i_1, i_2, ..., i_n\}\) be \(n\)-product of coordinates, \((\ell, j_1, j_2, ..., j_n)\), that is displaced on the intersection between sub basis, \(\ell\), and sub height, \((j_1, j_2, ..., j_n)\); \(\min(j_1, j_2, ..., j_n) = 1\); \(\max(j_1, j_2, ..., j_n) \leq j \leq \ell\); \(j_1 + j_2 + ... + j_n = j + (n - 1)\). Factors within n-products can be set in \(n!\) different ways, which implies \(n!\) different inequalities involving factors. Accordingly, sub basis can be divided into \(n!\) equivalent regions i.e. \(n\)-triangles where vertexes are centre of sub basis 0-face, 1-face, ..., \((n - 1)\)-face. Factors belonging to n-products displaced on the same region are ordered in the same way, while the contrary holds for factors belonging to n-products displaced on different regions.

For n-products displaced on sub basis, \(\ell\), the following can be established: \(\{i_1, i_2, ..., i_n\} = \{(\ell - \ell_1 - ... - \ell_{n-2} - \ell'), (\ell_1 + 1), ...,(\ell_{n-2} + 1), (\ell' + 1)\}\), \(0 \leq \ell_1 + ... + \ell_{n-2} + \ell' \leq \ell - (n - 1)\), keeping in mind related \(2^{-(n-1)}\)-perimeter reads \(p_{n,\ell} = (\ell - \ell_1 - ... - \ell_{n-2} - \ell') + (\ell_1 + 1) + ... + (\ell_{n-2} + 1) + (\ell' + 1)\).

For n-products displaced on sub height, \((j_1, j_2, ..., j_n)\), the following can be established: \(\{i_1, i_2, ..., i_n\} = \{(j_1 + j'), (j_2 + j'), ..., (j_n + j')\}\), \(0 \leq j' \leq j - (n - 1)\), keeping in mind the difference, \(\max[(j_1 + j'), (j_2 + j'), ..., (j_n + j')] - \min[(j_1 + j'), (j_2 + j'), ..., (j_n + j')]\) \(= \max(j_1, j_2, ..., j_n) - \min(j_1, j_2, ..., j_n)\), remains unchanged.

In the case under discussion, n-products are displaced on the intersection between sub basis, \(\ell\), and sub height, \((j_1, j_2, ..., j_n)\). Accordingly, the following relations hold:

\[
(\ell - \ell_1 - ... - \ell_{n-2} - \ell')(\ell_1 + 1)...(\ell_{n-2} + 1)(\ell' + 1) = \\
= (j_1 + j')(j_2 + j')...j_{n-1} + j'(j_n + j') \\
\begin{cases}
\ell - \ell_1 - ... - \ell_{n-2} - \ell' = j_1 + j' \\
\ell_1 + 1 = j_2 + j' \\
\vdots \\
\ell_{n-2} + 1 = j_{n-1} + j' \\
\ell' + 1 = j_n + j' \\
\ell = j_2 + j' - 1 + ... + j_{n-1} + j' - 1 + j_n + j' - 1 + j_1 + j' = \\
\quad = j_1 + j_2 + ... + j_n - (n - 1) + nj'
\end{cases}
\]
\[
\begin{align*}
\ell &= j + nj' \\
\ell_1 &= j_2 - 1 + \frac{\ell_2}{n} \\
\vdots \\
\ell_{n-2} &= j_{n-1} - 1 + \frac{\ell_{n-2}}{n} \\
\ell' &= j_n - 1 + \frac{\ell_n}{n}
\end{align*}
\]

from which the explicit expression of the first factor can be inferred as:

\[
i_1 = \ell - \ell_1 - \ldots - \ell_{n-2} - \ell' = \\
= \ell - j_2 + 1 - \frac{\ell - j}{n} - \ldots - j_{n-1} + 1 - \frac{\ell - j}{n} - j_n + 1 - \frac{\ell - j}{n} = \\
= \ell - j_2 - \ldots - j_{n-1} - j_n + (n-1) - (n-1)\frac{\ell - j}{n} = \\
= \ell - j + j_1 - n\frac{\ell - j}{n} + \frac{\ell - j}{n} = \ell - j + j_1 - \ell + j + \frac{\ell - j}{n};
\]

and the following relations hold:

\[
i_1 = \ell - \ell_1 - \ldots - \ell_{n-2} - \ell' = j_1 + \frac{\ell - j}{n}; \quad (24a)
\]

\[
i_2 = \ell_1 + 1 = j_2 + \frac{\ell - j}{n}; \quad (24b)
\]

\[
i_{n-1} = \ell_{n-2} + 1 = j_{n-1} + \frac{\ell - j}{n}; \quad (24c)
\]

\[
i_n = \ell' + 1 = j_n + \frac{\ell - j}{n}; \quad (24d)
\]

\[
i_1 + i_2 + \ldots + i_n = \ell + (n-1); \quad j_1 + j_2 + \ldots + j_n = j + (n-1); \quad (24e)
\]

\[
\min(i_1, i_2, \ldots, i_n) = 1; \quad \max(i_1, i_2, \ldots, i_n) \leq j; \quad (24f)
\]

\[
i_k - j_k = \frac{\ell - j}{n}; \quad k = 1, 2, \ldots, n; \quad (24g)
\]

accordingly, values of \(n\)-products can be determined from the knowledge of related coordinates.

Let \(n\)-product, \(\{i_1, i_2, \ldots, i_n\}\), of coordinates, \((\ell, j_1, j_2, \ldots, j_n)\), be displaced on right vertex of right isosceles triangle, where cathetuses lie on sub basis, \(\ell\), and on sub height, \((j_1, j_2, \ldots, j_n)\), and hypotenuse lies on nearest lateral 2-face implying \(i_1 \geq i_2 \geq \ldots \geq i_n\) with no loss of generality, hence \(j \geq j_1 \geq j_2 \geq \ldots \geq j_n = 1\). Accordingly, remaining vertexes carry \(n\)-products, \(\{(\ell - j + 1), j, 1, \ldots, 1\}, \{j_1, j_2, \ldots, j_{n-1}, 1\}\), of coordinates, \([\ell, (\ell - j + 1), j, 1, \ldots, 1], (j, j_1, j_2, \ldots, j_{n-1}, 1)\), respectively, that is on nearest lateral 2-face from which sub basis, \(\ell\), and sub height, \((j_1, j_2, \ldots, j_n)\), respectively, branch off.

Cathetus lying on sub basis, \(\ell\), carries \(n\)-products:

\[
\{(i_1 + i_0), i_2, \ldots, i_{n-1}, (i_n - i_0)\}; \quad 1 \leq i_0 \leq i_n - 1; \quad (25)
\]
passing from \( \{(i_1 + 1), i_2, \ldots, i_{n-1}, (i_n - 1)\} \) to \( \{(i_1 + i_n - 1), i_2, \ldots, i_{n-1}, 1\} \) for increasing \( i_0 \), the last displaced on nearest sub basis side. Using the above results yields the explicit expression:

\[
\left( j_1 + \frac{\ell - j}{n} + i_0 \right) \left( \frac{\ell - j}{n} + 1 - i_0 \right) = \left( j_1 + \frac{\ell - j}{n} \right) \left( \frac{\ell - j}{n} + 1 \right) + \\
+ i_0 \left( \frac{\ell - j}{n} + 1 \right) - i_0 \left( j_1 + \frac{\ell - j}{n} \right) = \left( j_1 + \frac{\ell - j}{n} \right) \left( \frac{\ell - j}{n} + 1 \right) + \\
+ i_0 \left( \frac{\ell - j}{n} + 1 - j_1 - \frac{\ell - j}{n} - i_0 \right) ; \quad 1 \leq i_0 \leq i_n - 1 = \frac{\ell - j}{n} ;
\]

in conclusion:

\[
\left( j_1 + \frac{\ell - j}{n} + i_0 \right) \left( \frac{\ell - j}{n} + 1 - i_0 \right) = \left( j_1 + \frac{\ell - j}{n} \right) \left( \frac{\ell - j}{n} + 1 \right) + \\
+ i_0(1 - j - i_0) ; \quad 1 \leq i_0 \leq \frac{\ell - j}{n} ; \tag{26}
\]

where, in the case under discussion, \( i_0 \)-th place starting from \( \{i_1, i_2, \ldots, i_n\} \) is equivalent to \( (i_n - i_0) \)-th place starting from \( \{(i_1 + i_n - 1), i_2, \ldots, i_{n-1}, 1\} \).

Cathetus lying on sub height, \( (j_1, j_2, \ldots, j_{n-1}, 1) \), carries \( n \)-products:

\[
\{(j_1 + j_0 - 1), (j_2 + j_0), \ldots, (j_{n-1} + j_0), j_0\} ; \quad 1 \leq j_0 \leq i_n - 1 ; \tag{27}
\]

passing from sub height boundary, \( \{j_1, j_2, \ldots, j_{n-1}, 1\} \), to \( \{(i_1 - 1), (i_2 - 1), \ldots, (i_{n-1} - 1)\} \), for increasing \( j_0 \). In the case under discussion, \( j_0 \)-th place starting from \( \{i_1, i_2, \ldots, i_n\} \) is equivalent to \( (i_n - j_0) \)-th place starting from \( \{j_1, j_2, \ldots, j_{n-1}, 1\} \).

The sum of generic pairs of products of the kind considered (i.e. factors are displaced on different cathetuses), for which \( i_0 = j_0 \), yields:

\[
\left( j_1 + \frac{\ell - j}{n} + i_0 \right) \left( \frac{\ell - j}{n} + 1 - i_0 \right) = (j_1 + j_0 - 1)j_0 = \\
= \left( j_1 + \frac{\ell - j}{n} \right) \left( \frac{\ell - j}{n} + 1 \right) + i_0(1 - j_1 - i_0) - (1 - j_1 - i_0)i_0 = \\
= \left( j_1 + \frac{\ell - j}{n} \right) \left( \frac{\ell - j}{n} + 1 \right) ; \tag{28}
\]

which equals related product displaced on the right vertex, regardless from the pair considered.

With regard to 1-Pythagorean table, a suitable rigid rotation of the Cartesian orthogonal reference frame, \((Ox_1)\), makes 1-sector of positive 2\(^{l}\)-ant aligned with the vertical and positive coordinate semiaxis displaced like slope roof. Accordingly, 1-Pythagorean table attains 1-triangle configuration which is bounded on the top and unbounded on the bottom.
Let 1-Pythagorean triangle of order, $k$, be defined as vertical segment carrying the 1-area, 1, on the top vertex and the 1-area, $k$, on the bottom vertex.

To gain more insight, 1-Pythagorean triangle of order, $k$, has to be conceived as made of ordered 1-tuples of factors, or 1-products, $\{i_1\} = i_1$, instead of 1-areas, $i_1$. The difference, of course, is purely conceptual.

With regard to 0-Pythagorean table, any rigid rotation of the Cartesian orthogonal reference frame, $(O)$, makes 0-sector of positive 2$^0$-ant aligned with the vertical. Accordingly, 0-Pythagorean table attains 0-triangle configuration which is bounded on the top and on the bottom (coinciding with the top).

Let 0-Pythagorean triangle of order, $k$, be defined as point carrying 0-areas related to the sequence of natural numbers from 1 to $k$.

To gain more insight, 0-Pythagorean triangle of order, $k$, has to be conceived as made of ordered 0-tuples of factors, or 0-products, $\{\emptyset\} = 0$, instead of 0-areas, 0. The difference, of course, is purely conceptual.

## 4 Pythagorean polytope

With regard to a Cartesian, orthogonal reference frame in $\mathbb{R}^m$, $(Ox_1x_2...x_m)$, let integer lattice, $\mathbb{Z}^m \subset \mathbb{R}^m$, be defined as subset of $\mathbb{R}^m$ made of all points exhibiting integer coordinates, or integer points. Let natural lattice, $\mathbb{N}^m \subset \mathbb{Z}^m \subset \mathbb{R}^m$, be defined as subset of $\mathbb{Z}^m$ made of all points exhibiting nonnegative integer coordinates, or natural points.

Let rational polyhedron be defined as subset of $\mathbb{R}^m$, the boundary of which can be defined via a finite number of linear inequalities with integer coefficients. Let polytope be defined as rational polyhedron, the boundary of which is a closed $(m-1)$-surface. In particular, let integer polytope and natural polytope be defined as polytopes where vertex coordinates are integer and nonnegative integer numbers, respectively. For further details, an interested reader is addressed to specific publications on the subject e.g., [1]. From this point on, attention shall be restricted to special classes of natural polytopes.

Let $(m+1)$ points in $\mathbb{R}^m$ be defined as $m$-misaligned if $m$-tuples involving different points lie on different $(m-1)$-planes. Let $(m+1)$-hedron in $\mathbb{R}^m$, or more concisely $(m+1)$-hedron, be defined as geometric figure resulting from connection of $(m+1)$ $m$-misaligned points via segments. In the special case of ordinary Euclidean space, $m = 3$, 4-hedrons are tetrahedrons. For further details, an interested reader is addressed to specific publications on the subject e.g., [2][3].

According to above definitions, $n$-Pythagorean triangle of order, $k$, is a special case of natural $(n+1)$-hedron in $\mathbb{R}^n$, and sub basis, $\ell$, $1 \leq \ell \leq k$, is a special case of natural $n$-hedron in $\mathbb{R}^{n-1}$. The set of $n$-products displaced therein belongs to a special kind of natural polytopes, hereafter referred to as $n$-Pythagorean polytope of order, $k$, and basis $n$-Pythagorean polytope,
ℓ, respectively. Warning: in the last definitions, n relates to dimensions of Euclidean space where polytope is placed instead of polytope dimensions.

With regard to n-Pythagorean triangle of order, k, results of Section 3 imply the following properties.

I. The number of n-products displaced on lateral (n−1)-faces equals the sum of the numbers of n-products displaced on the boundary of each sub basis, 1 ≤ ℓ ≤ k, where at least one unit factor appears.

II. The number of n-products displaced within n-Pythagorean triangle equals the sum of the numbers of n-products displaced on each sub basis, 1 ≤ ℓ ≤ k.

III. The number of (n−1)-products displaced on the boundary of (n−1)-Pythagorean triangle equals the number of n-products displaced on the boundary of n-Pythagorean triangle of equal order.

IV. The number of (n−1)-products displaced within (n−1)-Pythagorean triangle equals the number of n-products displaced on basis of n-Pythagorean triangle of equal order.

Next considerations are aimed to count natural points within special regions of n-Pythagorean polytope of order, k. More specifically, the number of natural points, \( N_{nS}(ℓ), N_{nB}(ℓ) \), as a whole and on the boundary, respectively, of basis n-Pythagorean polytope, ℓ, is determined. Summing on sub bases yields the number of natural points, \( N_{nP}(k), N_{nL}(k), N_{nF}(k) \), as a whole, on the lateral boundary, and on the boundary, respectively, of n-Pythagorean polytope of order, k.

The above properties imply the following relations:

\[
\begin{align*}
N_{nB}(ℓ) &= N_{(n-1)P}(ℓ) ; \quad 1 \leq ℓ \leq k ; \quad (29) \\
N_{nS}(ℓ) &= N_{(n-1)P}(ℓ) ; \quad 1 \leq ℓ \leq k ; \quad (30) \\
N_{nP}(k) &= 1 + \sum_{ℓ=2}^{k} N_{nS}(ℓ) = 1 + \sum_{ℓ=2}^{k} N_{(n-1)P}(ℓ) ; \quad (31) \\
N_{nL}(k) &= 1 + \sum_{ℓ=2}^{k} N_{nB}(ℓ) = 1 + \sum_{ℓ=2}^{k} N_{(n-1)F}(ℓ) ; \quad (32) \\
N_{nF}(k) &= N_{nL}(k) + N_{nS}(k) - N_{nB}(k) ; \quad (33)
\end{align*}
\]

where, concerning Eqs. (31)-(32), top vertex belongs to lateral boundary of n-Pythagorean triangle (n > 0) and, concerning Eq. (33), basis boundary is in common between basis (n−1)-surface and lateral (n−1)-surface of n-Pythagorean triangle (n > 1).
4.1 Case \( n = 0 \)

0-Pythagorean triangle of order, \( k \), in \( \mathbb{R}^0 \) is a point where it is superimposed the sequence of natural numbers from 1 to \( k \). 0-product defines coordinate of natural point. Vertex, basis, and sub bases may be conceived as coincident.

Sub basis, \( \ell \), \( 1 \leq \ell \leq k \), includes natural point carrying 0-product, \( \{ \ell \} \), necessarily devoid of unit factors.

The number of natural points, \( N_{0S}(\ell) \), \( N_{0B}(\ell) \), as a whole and on the boundary, respectively, of basis 0-Pythagorean polytope, \( \ell \), is conventionally assigned as:

\[
N_{0S}(\ell) = \delta_{1\ell} \quad ; \quad N_{0B}(\ell) = \delta_{1\ell} ;
\]

according to above considerations.

The number of natural points, \( N_{0P}(k) \), \( N_{0L}(k) \), \( N_{0F}(k) \), as a whole, on the lateral boundary, and on the boundary, respectively, of 0-Pythagorean polytope of order, \( k \), is conventionally assigned as:

\[
N_{0P}(k) = 1 \quad ; \quad N_{0L}(k) = \delta_{1k} \quad ; \quad N_{0F}(k) = \delta_{1k} ;
\]

according to above considerations. The expression of \( N_{0F}(k) \), \( N_{0P}(k) \), has been inferred from Eqs. (29), (30), respectively, and (36) below. The expression of \( N_{0S}(\ell) \), \( N_{0B}(\ell) \), \( N_{0L}(k) \), has been inferred from Eqs. (33), (34), (35) keeping in mind \( N_{0S}(\ell) = N_{0B}(\ell) \), concerning a single natural point, in the case under discussion.

4.2 Case \( n = 1 \)

1-Pythagorean triangle of order, \( k \), in \( \mathbb{R}^1 \) is a vertical segment where it is displaced the sequence of natural numbers from 1, on the top vertex, to \( k \), on the bottom vertex. 1-products define coordinates of natural points. Basis and sub bases carry a single natural point which, concerning vertexes, coincides with sub basis boundary.

Sub basis, \( \ell \), \( 1 \leq \ell \leq k \), includes natural point carrying 1-product, \( \{ \ell \} \).

The number of natural points, \( N_{1S}(\ell) \), \( N_{1B}(\ell) \), as a whole and on the boundary, respectively, of basis 1-Pythagorean polytope, \( \ell \), reads:

\[
N_{1S}(\ell) = 1 \quad ; \quad N_{1B}(\ell) = \delta_{1\ell} ;
\]

and the number of basis internal points, \( N_{1I}(\ell) \), is obtained by difference as:

\[
N_{1I}(\ell) = N_{1S}(\ell) - N_{1B}(\ell) = 1 - \delta_{1\ell} ;
\]

keeping in mind only the top vertex carries unit factor. On the other hand, lateral boundary of 1-triangle is made of the top vertex, where 1-product, \( \{ 1 \} \),
is displaced. Whole boundary, in addition, is made of the top and the bottom vertex where 1-product, \( \{k\} \), \( k > 1 \), is displaced.

The number of natural points, \( N_{1P}(k) \), \( N_{1L}(k) \), \( N_{1F}(k) \), as a whole, on the lateral boundary, and on the boundary, respectively, of 1-Pythagorean polytope of order, \( k \), reads:

\[
N_{1P}(k) = k \quad ; \quad N_{1L}(k) = 1 \quad ; \quad N_{1F}(k) = 2 - \delta_{1k} \quad ; \quad (38)
\]

according to Eqs. (31), (32), (33), particularized to the case under discussion.

### 4.3 Case \( n = 2 \)

2-Pythagorean triangle of order, \( k \), in \( \mathbb{R}^2 \) is a right isosceles triangle where hypotenuse is basis and cathetuses are oblique sides. 2-products define coordinates of natural points. Basis and each oblique side carry an equal amount of 2-products where factor sum preserves along basis, and one factor equals unity while the other one follows the sequence of natural numbers from 1 to \( k \) along oblique sides.

Sub basis, \( \ell \), includes natural points carrying 2-products, \( \{i_1, i_2\}; i_1 + i_2 = \ell + 1 \); and related boundary includes natural points carrying 2-products with at least one unit factor. The number of natural points, \( N_{2S}(\ell) \), \( N_{2B}(\ell) \), as a whole and on boundary, respectively, of basis 2-Pythagorean polytope, \( \ell \), reads:

\[
N_{2S}(\ell) = \ell \quad ; \quad N_{2B}(\ell) = 2 - \delta_{1\ell} \quad ; \quad (39)
\]

and the number of basis internal natural points, \( N_{2I}(\ell) \), is obtained by difference as:

\[
N_{2I}(\ell) = N_{2S}(\ell) - N_{2B}(\ell) = (\ell - 2) + \delta_{1\ell} \quad ; \quad (40)
\]

according to above considerations and results.

The number of natural points, \( N_{2P}(k) \), \( N_{2L}(k) \), \( N_{2F}(k) \), as a whole, on the lateral boundary, and on the boundary, respectively, of 2-Pythagorean polytope of order, \( k \), reads:

\[
N_{2P}(k) = \sum_{\ell=1}^{k} \ell \quad ; \quad N_{2L}(k) = 1 + 2(k - 1) \quad ; \\
N_{2F}(k) = [1 + 2(k - 1)] + k - (2 - \delta_{1k})
\]

in conclusion:

\[
N_{2P}(k) = \frac{(k + 1)k}{2} \quad ; \quad N_{2L}(k) = 2k - 1 \quad ; \quad N_{2F}(k) = 3k - 3 + \delta_{1k} \quad ; \quad (41)
\]

according to Eqs. (31), (32), (33), particularized to the case under discussion. Concerning the sum of first \( k \) natural numbers see e.g., [5], Cap. 19, §19.9, cfr. Appendix A. Comparison between above results and their counterparts for \( n = 1 \) is in agreement with Eqs. (29), (30).
4.4 Case $n = 3$

3-Pythagorean triangle of order, $k$, in $\mathbb{R}^3$ is a right isosceles tetrahedron where hypotenuse is basis side and cathetuses are oblique sides, with regard to lateral 2-faces. 3-products define coordinates of natural points. Basis and each lateral 2-face carry an equal amount of 3-products, where factor sum preserves along basis, and two factors equal unity while the third follows the sequence of natural numbers from 1 to $k$ along oblique sides, see Fig. 7.

Sub basis, $\ell$, includes natural points carrying 3-products, \(\{i_1, i_2, i_3\}; i_1+i_2+i_3 = \ell + 2\); and related boundary includes natural points carrying 3-products with at least one unit factor. The number of natural points, $N_{3S}(\ell)$, $N_{3B}(\ell)$, as a whole and on the boundary, respectively, of basis 3-Pythagorean polytope, $\ell$, reads:

\[
N_{3S}(\ell) = \frac{(\ell + 1)\ell}{2} ; \quad N_{3B}(\ell) = \delta_{1\ell} + 3(\ell - 1) ;
\]

and the number of basis internal natural points, $N_{3I}(\ell)$, is obtained by difference as:

\[
N_{3I}(\ell) = N_{3S}(\ell) - N_{3B}(\ell) = \frac{(\ell + 1)\ell}{2} - \delta_{1\ell} - 3(\ell - 1) = \\
= \frac{\ell^2 + \ell - 6\ell + 6}{2} - \delta_{1\ell} = \frac{\ell^2 - 5\ell + 6}{2} - \delta_{1\ell} ;
\]

in conclusion:

\[
N_{3I}(\ell) = \frac{(\ell - 2)(\ell - 3)}{2} - \delta_{1\ell} ;
\]

according to above considerations and results.

The number of natural points, $N_{3P}(k)$, $N_{3L}(k)$, $N_{3F}(k)$, as a whole, on the lateral boundary, and on the boundary, respectively, of 3-Pythagorean polytope of order, $k$, reads:

\[
N_{3P}(k) = \sum_{\ell=1}^{k} \frac{(\ell + 1)\ell}{2} = \frac{1}{2} \left[ \sum_{\ell=1}^{k} \ell^{2} + \sum_{\ell=1}^{k} \ell \right] = \\
= \frac{1}{2} \left[ k(k+1)(2k+1) + \frac{k(k+1)}{2} \right] = \frac{1}{2} k(k+1) \left[ \frac{2k+1}{3} + 1 \right] = \\
= \frac{k(k+1)}{4} \frac{2k+3}{3} = \frac{k(k+1)}{4} \frac{2k+4}{3} ;
\]

\[
N_{3L}(k) = 1 + \sum_{\ell=2}^{k} 3(\ell - 1) = 1 + 3 \sum_{i=1}^{k-1} i = 1 + \frac{3k(k-1)}{2} ;
\]

\[
N_{3F}(k) = \left[ 1 + \frac{3(k-1)}{2} \right] + \frac{(k+1)k}{2} - [\delta_{1k} + 3(k-1)] =
\]
\[ N_{3P}(k) = \frac{(k+2)(k+1)k}{6} \];  
\[ N_{3L}(k) = \frac{3k^2 - 3k}{2} + 1 \];  
\[ N_{3F}(k) = \frac{4k^2 - 8k + 6}{2} + 4 - \delta_{1k} \];

in conclusion:

4.5 Case \( n = 4 \)

4-Pythagorean triangle of order, \( k \), in \( \Re^4 \) is a right isosceles 4-dimension tetrahedron where hypotenuse is basis side and cathetuses are oblique sides, with regard to lateral 2-faces. 4-products define coordinates of natural points. Basis and each lateral 3-face carry an equal amount of 4-products where factor sum preserves along basis, and three factors equal unity while the fourth follows the sequence of natural numbers from 1 to \( k \) along oblique sides, see Fig. 11.

Sub basis, \( \ell \), includes natural points carrying 4-products, \( \{i_1, i_2, i_3, i_4\}; i_1 + i_2 + i_3 + i_4 = \ell + 3 \); and related boundary includes natural points carrying 3-products with at least one unit factor. The number of natural points, \( N_{4S}(\ell) \), \( N_{4B}(\ell) \), as a whole and on the boundary, respectively, of basis 4-Pythagorean polytope, \( \ell \), reads:

\[ N_{4S}(\ell) = \sum_{i=1}^{\ell} \frac{(i+1)i}{2} = \frac{1}{2} \left[ \sum_{i=1}^{\ell} i^2 + \sum_{i=1}^{\ell} i \right] \];

\[ N_{4B}(\ell) = \left[ 1 + 3 \sum_{i=1}^{\ell-1} i \right] + \left[ \frac{(\ell - 2)(\ell - 3)}{2} - \delta_{1\ell} \right] = \]
\[ = 1 - \delta_{1\ell} + \frac{3\ell(\ell - 1) + (\ell - 2)(\ell - 3)}{2} = \]
\[ = \frac{2 + 3\ell^2 - 3\ell + \ell^2 - 2\ell - 3\ell + 6}{2} - \delta_{1\ell} = \]
\[ = \frac{4\ell^2 - 8\ell + 8}{2} - \delta_{1\ell} \];
in conclusion:

\[ N_{4S}(\ell) = \frac{(\ell + 2)(\ell + 1)\ell}{6} ; \quad (47) \]

\[ N_{4B}(\ell) = \frac{4\ell^2 - 8\ell}{2} + 4 - \delta_{1\ell} ; \quad (48) \]

and the number of basis internal natural points, \( N_{4I}(\ell) \), is obtained by difference as:

\[
N_{4I}(\ell) = N_{4S}(\ell) - N_{4B}(\ell) = \\
\frac{(\ell + 2)(\ell + 1)\ell}{6} - \frac{4\ell^2 - 8\ell}{2} - 4 + \delta_{1\ell} = \\
\frac{\ell^3 + 2\ell^2 + \ell^2 + 2\ell - 12\ell^2 + 24\ell}{6} - 4 + \delta_{1\ell} = \\
\frac{\ell^3 - 9\ell^2 + 26\ell - 24}{6} + \delta_{1\ell} ;
\]

in conclusion:

\[ N_{4I}(\ell) = \frac{(\ell - 2)(\ell - 3)(\ell - 4)}{6} + \delta_{1\ell} ; \quad (49) \]

according to above considerations and results.

The number of natural points, \( N_{4P}(k) \), \( N_{4L}(k) \), \( N_{4F}(k) \), as a whole, on the lateral boundary, and on the boundary, respectively, of 4-Pythagorean polytope of order, \( k \), reads:

\[
N_{4P}(k) = \sum_{\ell=1}^{k} \frac{(\ell + 2)(\ell + 1)\ell}{6} = \frac{1}{6} \sum_{\ell=1}^{k} (\ell^3 + 2\ell^2 + \ell^2 + 2\ell) = \\
= \frac{1}{6} \left[ \sum_{\ell=1}^{k} \ell^3 + 3 \sum_{\ell=1}^{k} \ell^2 + 2 \sum_{\ell=1}^{k} \ell \right] = \\
= \frac{1}{6} \left[ \frac{(k + 1)^2 k^2}{4} + 3 \frac{(k + 1)k(2k + 1)}{6} + 2 \frac{(k + 1)k}{2} \right] = \\
= \frac{(k + 1)k^2 k^2 + k + 4k + 2 + 4}{6} = \\
= \frac{(k + 1)k^2 k^2 + 5k + 6}{6} = \frac{(k + 1)k (k + 3)(k + 2)}{4} ;
\]

\[
N_{4L}(k) = 1 + \sum_{\ell=2}^{k} \left[ \frac{4\ell^2 - 8\ell}{2} + 4 - \delta_{1\ell} \right] = 1 + 2 \sum_{\ell=2}^{k} \ell^2 - 4 \sum_{\ell=2}^{k} \ell + 4(k - 1) = \\
= 1 - 2 + 2 \sum_{\ell=1}^{k} \ell^2 + 4 - 4 \sum_{\ell=1}^{k} \ell + 4k - 4 = 
\]
Pythagorean triangle and geometrical interpretation of prime numbers

\[ = -1 + 2 \frac{(k + 1)k(2k + 1)}{6} - 4 \frac{(k + 1)k}{2} + 4k = \]
\[ = -1 + \frac{2k^3 + 2k^2 + k^2 + k + 1}{3} - 2k^2 - 2k + 4k = \]
\[ = -3 + 2k^3 + 3k^2 + k - 6k^2 + 6k = \frac{2k^3 - 3k^2 + 7k - 3}{3} ; \]
\[ N_{4F}(k) = \frac{3}{3} \left( \frac{(k+2)(k+1)k}{6} - \frac{3}{2} - 4 + \delta_1k \right) \]
\[ = \frac{2k^3 - 3k^2 + 7k - 3}{3} + \frac{k^3 + 2k^2 + k + 2}{6} - \frac{4k^2 - 8k + 8}{2} + \]
\[ + \delta_1k = \frac{5k^3 - 15k^2 + 40k - 30}{6} + \delta_1k ; \]

in conclusion:

\[ N_{4P}(k) = \frac{(k + 3)(k + 2)(k + 1)k}{24} ; \] (50)
\[ N_{4L}(k) = \frac{2k^3 - 3k^2 + 7k}{3} - 1 ; \] (51)
\[ N_{4F}(k) = \frac{5k^3 - 15k^2 + 40k}{6} - 5 + \delta_1k ; \] (52)

according to Eqs. (31), (32), (33), particularized to the case under discussion. Concerning the sum of first cube \( k \) natural numbers see e.g., [5], Cap. 19, §19.11, cfr. Appendix A. Comparison between above results and their counterparts for \( n = 3 \) is in agreement with Eqs. (29), (30).

### 4.6 General case

\( n \)-Pythagorean triangle of order, \( k \), in \( \mathbb{R}^n \) is a right isosceles \( n \)-dimension tetrahedron where hypotenuse is basis side and catheuses are oblique sides, with regard to lateral 2-faces. \( n \)-products define coordinates of natural points. Basis and each lateral \( (n - 1) \)-face carry an equal amount of \( n \)-products, where factor sum preserves along basis, and \( (n - 1) \) factors equal unity while \( n \)-th follows the sequence of natural numbers from 1 to \( k \) along oblique sides.

Sub basis, \( \ell \), includes natural points carrying \( n \)-products, \( \{i_1, i_2, ..., i_n\} \); \( i_1 + i_2 + ... + i_n = \ell + (n - 1) \); and related boundary includes natural points carrying \( n \)-products with at least one unit factor. The number of natural points, \( N_{nS}(\ell) \), \( N_{nI}(\ell) \), \( N_{nB}(\ell) \), as a whole, inside, and on the boundary, respectively, of basis \( n \)-Pythagorean polytope, \( \ell \), is inferred extending results found for \( n = 1, 2, 3, 4, \ldots \).
via Eqs. (29), (30). Related expressions read:

\[ N_{nS}(\ell) = N_{(n-1)P}(\ell) = \binom{\ell + n - 2}{n - 1}; \]  
\[ N_{nI}(\ell) = \binom{\ell - 2}{n - 1} + (-1)^n \delta_{1\ell}; \]  
\[ N_{nB}(\ell) = N_{(n-1)F}(\ell) = N_{nS}(\ell) - N_{nI}(\ell) = \binom{\ell + n - 2}{n - 1} - \binom{\ell - 2}{n - 1} - (-1)^n \delta_{1\ell}; \]

where it is intended that, in general, \( \binom{i}{j} = 0 \) if \( i < j \).

The number of natural points, \( N_{nP}(k) \), \( N_{nL}(k) \), \( N_{nF}(k) \), as a whole, on the lateral boundary, and on the boundary, respectively, of \( n \)-Pythagorean politope of order, \( k \), is inferred extending results found for \( n = 1, 2, 3, 4 \), via Eqs. (31), (32), (33). Related expressions read:

\[ N_{nP}(k) = \sum_{\ell=1}^{k} \binom{\ell + n - 2}{n - 1}; \]
\[ N_{nL}(k) = 1 + \sum_{\ell=2}^{k} \left[ \binom{\ell + n - 2}{n - 1} - \binom{\ell - 2}{n - 1} - (-1)^n \delta_{1\ell} \right] = \]
\[ = 1 + \sum_{i=1}^{k-1} \binom{i + n - 1}{n - 1} - \sum_{i=1}^{n-1} \binom{i - 1}{n - 1} - \sum_{i=n}^{k-1} \binom{i - 1}{n - 1} = \]
\[ = \sum_{i=0}^{k-1} \binom{i + n - 1}{n - 1} - \sum_{i=n}^{k-1} \binom{i - 1}{n - 1}; \]
\[ N_{nF}(k) = \sum_{i=0}^{k-1} \binom{i + n - 1}{n - 1} - \sum_{i=n}^{k-1} \binom{i - 1}{n - 1} = \]
\[ = \sum_{i=0}^{k-1} \binom{i + n - 1}{n - 1} + \binom{k + n - 2}{n - 1} + \binom{k - 2}{n - 1} + (-1)^n \delta_{1k}; \]
in conclusion:

\[ N_{nP}(k) = \binom{k + n - 1}{n}; \]  

\(^{2}\text{More specifically, binomial coefficient may be expressed as } (i \geq j):\]

\[ \binom{i}{j} = \frac{i!}{j!(i-j)!} = \frac{i(i-1)...(i-j+1)(i-j)!}{j!(i-j)!} = \frac{i(i-1)...(i-j+1)}{j!}; \]

where the inequality, \( 0 \leq i \leq j - 1 \), implies one factor on numerator, and then whole fraction, is null.
\[ N_{nL}(k) = \binom{k + n - 1}{n} - \binom{k - 1}{n} ; \quad (57) \]
\[ N_{nF}(k) = \binom{k + n - 1}{n} - \binom{k - 1}{n} + \binom{k - 2}{n - 1} + (-1)^n\delta_{1k} = \]
\[ = \binom{k + n - 1}{n} - \binom{k - 1}{n} + \binom{k - 1}{n} - \binom{k - 2}{n} + (-1)^n\delta_{1k} ; \]
and the final result is:
\[ N_{nF}(k) = \binom{k + n - 1}{n} - \binom{k - 2}{n} + (-1)^n\delta_{1k} . \quad (58) \]

Concerning relations between binomial coefficients see e.g., [5], Cap. 3, §§3.6, 3.9, cfr. Appendix A.

4.7 Application: 12-note equal temperament \( n \)-scale

Within the framework of musical intervals under 12-note equal temperament, musical scale of cardinality, \( n \), or \( n \)-scale, can be conceived as ordered \( n \)-tuple of natural numbers, \( \{i_1, i_2, ..., i_n\} \), satisfying the following conditions:
\[ i_1 + i_2 + ... + i_n = 12 ; \quad i_j > 0 ; \quad 1 \leq j \leq n \leq 12 ; \quad (59) \]
for further details, an interested reader is addressed to specific publications on the subject e.g., [2][3].

Accordingly, \( n \)-scales relate to \( n \)-products where factors coincide with \( n \)-scale musical intervals and, in consequence, basis of \( n \)-Pythagorean triangle of order, \( k = 12 - (n - 1) = 13 - n \), that is \( (n - 1) \)-Pythagorean triangle of equal order, carries the whole amount of \( n \)-scales.

The particularization of Eq. (53) to the case under discussion yields:
\[ N_{nS}(k) = N_{(n-1)P}(k) = \binom{13 - n + n - 2}{n - 1} - \binom{12 - 1}{n - 1} ; \quad (60) \]
\[ k = 13 - n ; \quad 1 \leq n \leq 12 ; \quad 12 \geq k \geq 1 ; \]
where results for different \( n \) are listed in Tab. 1. For further details, an interested reader is addressed to specific publications on the subject e.g., [2][3].

Within the framework of \( M \)-note temperate musical scale, Eq. (59) reads:
\[ i_1 + i_2 + ... + i_n = M ; \quad i_j > 0 ; \quad 1 \leq j \leq n \leq M ; \quad (61) \]
and Eq. (60) is extended as:
\[ N_{nS}(k) = N_{(n-1)P}(k) = \binom{M - n + 1 + n - 2}{n - 1} - \binom{M - 1}{n - 1} ; \quad (62) \]
\[ k = M - (n - 1) = M - n + 1 ; \quad 1 \leq n \leq M ; \quad M \geq k \geq 1 ; \]
where, for fixed \( M \), counterparts of results listed in Tab. 1 can be determined.
Table 1: Total number of $n$-scales determined as total number of $n$-products displaced on basis of $n$-Pythagorean triangle of order, $k = 13 - n$, or in other words within $(n - 1)$-Pythagorean triangle of equal order.

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
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<tbody>
<tr>
<td>$k$</td>
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<td>9</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$N_{ns}(k)$</td>
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<td>55</td>
<td>165</td>
<td>330</td>
<td>462</td>
<td>462</td>
<td>330</td>
<td>165</td>
<td>55</td>
<td>11</td>
<td>1</td>
</tr>
</tbody>
</table>

### 4.8 Application: regular inclined $m$-hedron in $\mathbb{R}^{m-1}$

By definition, $m$-hedrons in $\mathbb{R}^{m-1}$ exhibit $m$ vertexes, or 0-faces. In general, $j$-tuples of vertexes define $m$-hedron $(j - 1)$-faces. The total number of $m$-hedron $(j - 1)$-faces, $F_j$, reads:

$$F_j = \binom{m}{j} = \frac{m(m-1)...(m-j+1)}{j!} ; \quad 0 \leq j \leq m ; \quad (63)$$

where $F_m = 1$ relates to $(m - 1)$-face, coinciding with $m$-hedron, and $F_0 = 1$ relates to $(-1)$-face, coinciding with metacentre.

With regard to orthogonal Cartesian reference frame in $\mathbb{R}^m$, $(Ox_1x_2...x_m)$, let regular inclined $m$-hedron in $\mathbb{R}^{m-1}$ be defined as $m$-hedron of vertex coordinates, $V_\ell \equiv (\delta_{1\ell} k, \delta_{2\ell} k, ..., \delta_{m\ell} k), 1 \leq \ell \leq m$, that is vertexes are lying on positive coordinate semiaxes at a distance, $k$, from origin.

Let metacentre of regular inclined $m$-hedron in $\mathbb{R}^{m-1}$ be defined in $\mathbb{R}^m$ as centre of $m$-sphere carrying on its surface vertexes of regular inclined $m$-hedron, that is coinciding with origin. Metacentre can be conceived as symmetry centre and related to empty set (of points) in $\mathbb{R}^{m-1}$.

With regard to regular inclined $m$-hedrons, $(j - 1)$-face number, $F_j$, $0 \leq j \leq m$, as a function of vertex number, $m = F_1$, is listed in Tab. 2. For further details, cfr. [2][3], from which Tab. 2 has been extracted.

Comparison between Eqs. (63) and (53), (62), discloses the number of $n$-products displaced on $n$-Pythagorean triangle of order, $k = M - (n - 1) = M - n + 1$, equals the number of $(n - 1)$-faces, $F_n$, of regular inclined $(M - 1)$-hedron in $\mathbb{R}^{M-2}$, as shown from inspection of Tab. 1 and Tab. 2 in the special case, $M = 12$.

### 5 Prime numbers

With regard to 2-Pythagorean triangle in $\mathbb{R}^2$, let prime number be defined as 2-area, $(n_p, 1)$, which has no equivalent counterpart in absence of unit factor, that is basis length of perfect 2-rectangle of unit height, which has no equivalent
Table 2: \((j - 1)\)-face number, \(F_j\), \(0 \leq j \leq m\), as a function of vertex number, \(m = F_1\), with regard to regular inclined \(m\)-hedron in \(\mathbb{H}^m\), \(0 \leq m \leq 13\). Addition of \((-1)\)-faces, \(F_0\), related to \(m\)-hedron metacentre, and \((m - 1)\)-faces, \(F_m\), related to \(m\)-hedron itself, implies horizontal and oblique unit line, respectively, giving rise to Pascal’s triangle. For \(j < 10\), \(F_j\) is denoted as \(F_{0j}\) to save aesthetics. Table taken from [2][3].

<table>
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<th>(F_00)</th>
<th>1</th>
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<tbody>
<tr>
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<td>10</td>
<td>11</td>
</tr>
<tr>
<td>(F_{02})</td>
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<td>10</td>
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<td>21</td>
<td>28</td>
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</tr>
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<td>(F_{03})</td>
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<td>20</td>
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<td>56</td>
<td>84</td>
<td>120</td>
<td>165</td>
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<tr>
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<td>84</td>
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<td>252</td>
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counterpart with basis and height length larger than unity. Accordingly, 2-products exhibiting prime factor and unit factor are displaced on oblique sides of 2-Pythagorean triangle.

In particular, perfect 2-square of side, \( \ell \), is equivalent to perfect 2-rectangle of basis, \( \ell^2 \), and unit height, that is \( \ell \cdot \ell = \ell^2 \cdot 1 \), which for perfect 2-square of unit side reduces to \( 1 \cdot 1 = 1^2 \cdot 1 \). Then perfect 2-square of unit side has to be conceived as distinct from perfect 2-rectangle of basis length, \( 1^2 \), and unit height. In other words unity, as perfect 2-square, cannot be prime number.

With regard to \( n \)-Pythagorean triangle in \( \mathbb{R}^n \), let prime number be defined as \( n \)-area, \( (n_{p1} \cdot n_{p2} \cdot \ldots \cdot n_{p(n-1)} \cdot 1) \), which has no equivalent counterpart in absence of unit factors, that is basis length of perfect \( n \)-rectangle of unit remaining bases and height, which has no equivalent counterpart with basis and height length larger than unity. Accordingly, \( n \)-products exhibiting one prime factor and remaining unit factors are displaced on oblique sides of \( n \)-Pythagorean triangle.

In particular, perfect \( n \)-square of side, \( \ell \), is equivalent to perfect \( n \)-rectangle of bases, \( \ell^{n_1}, \ell^{n_2}, \ldots, \ell^{n_{n-1}} \), \( n_1 + n_2 + \ldots + n_{n-1} = n \), and unit height, that is \( \ell \cdot \ell \cdot \ldots \cdot \ell = \ell^{n_1} \cdot \ell^{n_2} \cdot \ldots \cdot \ell^{n_{n-1}} \cdot 1 \), which for perfect \( n \)-square of unit side reduces to \( 1 \cdot 1 \cdot \ldots \cdot 1 = 1^{n_1} \cdot 1^{n_2} \cdot \ldots \cdot 1^{n_{n-1}} \cdot 1 \). Then perfect \( n \)-square of unit side has to be conceived as distinct from perfect \( n \)-rectangle of basis length, \( 1^{n_1}, 1^{n_2}, \ldots, 1^{n_{n-1}} \), and unit height. In other words unity, as perfect \( n \)-square, cannot be \( [1/(n-1)] \)-th number.

It can be seen \( [1/(n-1)] \)-th numbers relate to products of \( (n-1) \) prime numbers and unity. For this reason, from this point on attention shall be restricted to prime numbers in \( \mathbb{R}^2 \) and related geometric and arithmetic entities shall be indicated in standard notation erasing the prefix, 2-.

With regard to Pythagorean triangle of order, \( k \), according to above considerations, products exhibiting one prime factor, \( m \leq k \), and one unit factor, are displaced on oblique sides while products exhibiting factors larger than unity are displaced inside, cfr. Fig. 4.

Even numbers larger than 2, as multiples of 2, are composite (i.e. nonprime)
numbers. Numbers larger than 5, carrying 5 on the last figure, as multiples of 5, are composite numbers. Accordingly, search of prime numbers larger than 5 has to be restricted to odd numbers where 5 does not appear on the last figure, displaced on oblique sides of Pythagorean triangle. In conclusion, a necessary and sufficient condition for a natural number, \( m \), \( 1 < m \leq k \), to be prime, is that related perfect rectangle area lies only on oblique sides of Pythagorean triangle of order, \( k \), that is on sub basis boundary, cfr. Fig. 3.

The following considerations are necessary for establishing if a selected natural number, \( m > 5 \), is prime. First, areas are displaced on each sub basis in increasing order from boundary up to mean point, symmetrically on both sides. In fact, areas displaced on natural points equally distant from mean point relate to perfect rectangles where basis and height are interchanged.

Let \( i^2 \) be maximum perfect square displaced on mean point of odd sub basis for which \( i^2 < m \), and let \( (2k+1) \) be sub basis carrying areas, \( i^2, [(2k+1) \cdot 1] \), on mean point and boundary, respectively. Accordingly, related perfect rectangles have equal half perimeter, \( (2k+1) + 1 = i + i \), hence \( 2k+1 = 2i - 1 \). Then sub basis where area, \( i^2 \), is displaced on mean point, can be expressed in terms of \( i \).

In conclusion, Pythagorean triangle of order, \( (2k+1) \), carries no area equal to \( m \). On the other hand, Pythagorean triangle of order, \( m \), is the one of higher order carrying above mentioned area on its basis, contrary to bases below related to Pythagorean triangles of order, \( k > m \). Accordingly, areas expressed by composite numbers, \( m \), are displaced inside Pythagorean triangle of order, \( m \), on a sub basis, \( \ell, 2k + 2 \leq \ell \leq m - 1 \).

The last limit can be improved taking due account of areas displaced on nearest point to sub basis boundary instead of sub basis boundary. More specifically with regard to sub basis, \( j \), area, \( [(j - 1) \cdot 2] \), has to be considered instead of \( (j \cdot 1) \). Areas equal to \( m \), can be displaced inside sub basis, \( j \), provided inequality, \( 2(j - 1) < m \), holds, hence \( j < m/2 + 1 \), where \( m \) is necessarily odd for establishing if it is prime.

Accordingly, Pythagorean triangle of order, \( j = \text{Int}(m/2) + 1 \), is the one of higher order where areas equal to \( m \) could be displaced within basis, and the above procedure is restricted to sub bases, \( \ell, 2i \leq \ell \leq j \). For instance, \( m = 19 \) implies \( 2i = 8, j = 10 \), but area equal to 19 is not displaced within sub bases, \( \ell, 8 \leq \ell \leq 10 \), then 19 is prime number, cfr. Fig. 3. An algorithm in the above mentioned sense is formulated in GWBASIC for determining the sequence of prime numbers starting from 7, as shown in Appendix C. Outputs have been checked with available results e.g., [4], up to 10853.

To get further insight, it could be useful a finite representation of Pythagorean triangle of order, \( k \to +\infty \), as well as a finite representation of natural numbers within unit interval is obtained via the transformation, \( n \to 1/n \), by extension of natural numbers to rational numbers, \( \mathcal{N} \to \mathcal{Q} \).
With regard to Pythagorean triangle of odd order, \((2k + 1)\), let \(i_1i_2\) be a generic area displaced on basis which, on mean point, carries a perfect square, \((k + 1)^2\), according to above results. Keeping in mind half perimeter of perfect rectangles displaced on basis is preserved, the particularization to perfect rectangle displaced on basis boundary and within basis, respectively, yields:

\[
i_1 + i_2 = (2k + 1) + 1 = 2(k + 1) ; \quad i_2 = 2(k + 1) - i_1 ; \quad 2k + 1 \geq i_1 \geq k + 1 ; \quad S(i_1) = i_1i_2 = i_1[2(k + 1) - i_1] ;
\]

and reduced area, \(s(i_1)\), is defined as:

\[
s(i_1) = \frac{S(i_1)}{S(k + 1)} = \frac{i_1i_2}{(k + 1)^2} = \frac{i_1[2(k + 1) - i_1]}{(k + 1)^2} = \frac{i_1}{k + 1} \left[2 - \frac{i_1}{k + 1}\right] \tag{64}
\]

\[
s(k + 1) = 1 ; \quad s(2k + 1) = s(1) = \frac{2(k + 1) - 1}{(k + 1)^2} = \frac{2k + 1}{(k + 1)^2} \tag{65}
\]

which implies a reduced representation of Pythagorean triangle of odd order.

With regard to Pythagorean triangle of even order, \(2k\), let \(i_1i_2\) be a generic area displaced on basis which, in the neighbourhood of mean point, carries on both sides a perfect rectangle, \((k + 1)k, k(k + 1)\), respectively, according to above results. Keeping in mind half perimeter of perfect rectangles displaced on basis is preserved, the particularization to perfect rectangle displaced on basis boundary and within basis, respectively, yields:

\[
i_1 + i_2 = 2k + 1 ; \quad i_2 = 2k + 1 - i_1 ; \quad 2k \geq i_1 \geq k + 1 ; \quad S(i_1) = i_1i_2 = i_1(2k + 1 - i_1) ;
\]

and reduced area, \(s(i_1)\), is defined as:

\[
s(i_1) = \frac{S(i_1)}{S(k + 1)} = \frac{i_1i_2}{k(k + 1)} = \frac{i_1(2k + 1 - i_1)}{k(k + 1)} = \frac{i_1}{k + 1} \left[2 - \frac{i_1 - 1}{k}\right] \tag{66}
\]

\[
s(k + 1) = s(k) = 1 ; \quad s(2k) = s(1) = \frac{2}{k + 1} \tag{67}
\]

which implies a reduced representation of Pythagorean triangle of even order.

A reduced representation of Pythagorean triangle of order, \(k \to +\infty\), can be performed using the above results. Accordingly, unit reduced area is displaced on basis mean point; finite reduced area is displaced on natural points sufficiently close to basis mean point provided \([1 - i_1i_2/(k + 1)^2] \ll 1\); infinitesimal reduced area is displaced on natural points coinciding with, or sufficiently close to, vertexes opposite to oblique sides provided \([i_1i_2/(k + 1)^2] \ll 1\).

With regard to generic sub basis, related transformation reads:

\[
i_1[2(\ell + 1) - i_1] \to \frac{i_1}{\ell + 1} \left[2 - \frac{i_1}{\ell + 1}\right] ; \quad 2\ell + 1 \to \frac{2\ell + 1}{(\ell + 1)^2} \tag{68}
\]

\[
i_1(2\ell + 1 - i_1) \to \frac{i_1}{\ell + 1} \left[2 - \frac{i_1 - 1}{\ell}\right] ; \quad 2\ell \to \frac{2\ell + 2}{(\ell + 1)^2} \tag{69}
\]
concerning odd and even sub bases, respectively.

In particular, reduced areas displaced on boundary of sub basis, \( m, 1 \leq m \leq 12 \), that is on oblique sides of reduced Pythagorean triangle, are:

\[
s(m) = 1, 1, 3, 2, 5, 1, 7, 2, 9, 1, 11, 2,
\]

\[
m = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12;
\]

according to Eqs. (68), (69). On the other hand, reduced areas displaced on natural points within sub basis can be inferred via Eqs. (64), (66).

With regard to reduced Pythagorean triangle of order, \( k \to +\infty \), reduced areas displaced on sub basis, \( m, 1 \leq m \leq 12 \), are shown in Fig. 14 as bullets, where next sub bases, \( 13 \leq m \leq 20 \), are also marked.

6 Conclusion

Aiming to privilege geometric interpretation of prime numbers, Pythagorean table is conceived as projection of codomain of product function on related domain, with extension from ordinary plane to \( n \)-plane. Pythagorean triangle is defined after suitable rotation of orthogonal Cartesian reference frame related to Pythagorean table, with extension from ordinary plane to \( n \)-plane.

More specifically, \( n \)-Pythagorean triangle of order, \( k \), is right isosceles \((n + 1)\)-hedron where right vertex lies on the origin, lateral 2-faces are right isosceles triangles where cathetuses relate to oblique sides and hypotenuse to basis sides. Related basis is inclined regular \( n \)-hedron where vertexes lie on positive coordinate semiaxes and distance from origin equals \( k \).

Sub basis, \( \ell \), carries \( n \)-areas of perfect \( n \)-rectangles of basis and height length, \( i_1, i_2, ..., i_n \), where \( 2^{-(n-1)} \)-perimeter is preserved, \( i_1 + i_2 + ... + i_n = \ell + (n - 1), 1 \leq i_j \leq \ell, j = 1, 2, ..., n \); that is \( n \)-products, \( \{i_1, i_2, ..., i_n\} \). In addition, sub bases occurring \( n \) by \( n \) exhibit similar features.

\( n \)-Pythagorean triangle and sub basis, \( \ell \), can be conceived as a special class of polytopes where coordinates of lattice points are factors of \( n \)-products, that is nonzero natural numbers. Calculations are performed for natural points displaced within \( n \)-Pythagorean triangle, on lateral boundary, on whole boundary, on basis, \( \ell \), and on related boundary.

Within the framework of a geometric interpretation, prime numbers are related to areas of perfect rectangles exhibiting unit basis or height, with no equivalent counterpart exhibiting basis and height length larger than unity, which could be of some utility in connection with still unresolved questions about prime numbers.

For instance, Goldbach’s conjecture:
Figure 14: Reduced representation of Pythagorean triangle of order, $k \to +\infty$, via transformations described in the text. Unit reduced areas are displaced on sub basis mean point, one single in odd sub bases and two superimposed in even sub bases. With regard to basis, $k \to +\infty$, infinitesimal reduced areas are displaced on natural points coinciding with, or sufficiently close to, vertexes opposite to oblique sides provided $i_1i_2/(k+1)^2 \ll 1$, and finite reduced areas are displaced on natural points coinciding with, or sufficiently close to, basis mean point provided $1 - i_1i_2/(k+1)^2 \ll 1$. Reduced areas displaced on sub bases, $m$, $1 \leq m \leq 12$, are denoted as bullets, where vertex opposite to basis carries unit reduced areas, related to $m = 1, 2$. Next sub bases, $13 \leq m \leq 20$, are also shown.
“Multiples of 2 are expressible as sum of two prime numbers”

from the standpoint of Pythagorean triangle translates into:

“With regard of Pythagorean triangle of order, k, there is at least one product, displaced on odd sub basis, \((2\ell + 1), 1 \leq \ell \leq \text{Int}\left[\frac{(k-1)}{2}\right]\), where both factors are prime numbers, \([\ell + 1 \mp j], 0 \leq j \leq \ell\].”

An inspection of Fig. 4 discloses half perimeter of perfect rectangle displaced on odd sub basis, \((2\ell + 1),\) equals factor sum, \((2\ell + 2),\) and carries a perfect square on mean point, that is a product of equal factors.

To gain further insight, Fig. 4 (top panel) is reproduced in Fig. 15 (top panel), where products involving pairs of prime factors satisfying Goldbach’s conjecture, defined as Goldbach products, are marked in bold.

An inspection of Fig. 15 (top panel) discloses Goldbach products are symmetrically displaced on odd sub bases of Pythagorean triangle of order, \(3 \leq 2\ell + 1 \leq 15,\) where \(\{2, 2\}\) is the sole Goldbach product involving the factor, 2.

The above considerations imply a more intuitive reformulation of Goldbach’s conjecture, as:

“With regard to semiaxis carrying natural numbers and its counterpart with opposite orientation, let \((2\ell + 1)\) be an arbitrary odd number larger than unity, \(\ell \geq 1;\) let \(m, 1 \leq m \leq 2\ell + 1,\) be a natural number displaced on first semiaxis; and let \(\left[(2\ell + 1) - (m - 1)\right]\) be the corresponding natural number on the second semiaxis. Then, among \((2\ell + 1)\) pairs of conjugate elements, \(\{m, [(2\ell + 1) - (m - 1)]\},\) there is one at least where prime numbers appear on both places.”

The situation is represented in Fig. 15 (bottom panel), where \(2\ell + 1 = 15, \ell = 7, m + [(2\ell + 1) - (m - 1)] = 2\ell + 2 = 16,\) and pairs of Goldbach factors are \((3, 13), (5, 11), (11, 5), (13, 3).\)

The current investigation is aimed to get better insight on geometrical interpretation of prime numbers, leaving further developments to passionate expert mathematicians with high propension towards multidimensional geometry, to avoid erroneous (in author’s opinion) conclusions e.g., results arise from combinatorics in general and binomial coefficients in particular, or from set theory, instead of intrinsic properties, though previously defined, of Euclidean \(n\)-spaces.

**Acknowledgements** Results presented in the current paper were independently obtained by the author, in absence of quotations. In the case some “discoveries” have been “rediscovered”, the author thanks in advance whoever will provide overlooked references.
Figure 15: Top panel: same representation as in Fig. 4, where Goldbach products are marked in bold. Bottom panel: an equivalent formulation of Goldbach’s conjecture, where Goldbach factors are marked in bold, with regard to $2\ell + 2 = 16$. See text for further details.
References


Appendix

A Most used formulae

To help an interested reader, most used formulae throughout the text are listed below.

Sum of natural numbers from 1 to $k$:

$$
\sum_{\ell=1}^{k} \ell = \frac{k(k+1)}{2} ;
$$

(70)

see e.g., [5], Cap. 19, §19.9.

Sum of square natural numbers from 1 to $k$:

$$
\sum_{\ell=1}^{k} \ell^2 = \frac{k(k+1)(2k+1)}{2} ;
$$

(71)

see e.g., [5], Cap. 19, §19.10.

Sum of cube natural numbers from 1 to $k$:

$$
\sum_{\ell=1}^{k} \ell^3 = \left[ \frac{k(k+1)}{2} \right]^2 ;
$$

(72)

see e.g., [5], Cap. 19, §19.11.
Sum of binomial coefficients with bottom argument, \( m \), and top argument from \( m \) to \( m + k - 1 \):

\[
\sum_{\ell=1}^{k} \binom{\ell + m - 1}{m} = \binom{m}{m} + \binom{m + 1}{m} + ... + \binom{m + k - 1}{m} = \binom{m + k}{m + 1} ; \tag{73}
\]

see e.g., [5], Cap. 3, §39.9.

Recursive formula for binomial coefficients:

\[
\binom{k}{m} = \binom{k - 1}{m - 1} + \binom{k - 1}{m} ; \tag{74}
\]

see e.g., [5], Cap. 3, §39.6.

## B Algebraic and geometric properties

To help an interested reader, some algebraic and geometric properties, mentioned throughout the text, are analyzed in detail below.

**I. With regard to natural numbers, the difference between consecutive perfect squares is the odd number just below double square root of minuend.**

The difference between consecutive perfect squares reads:

\[
n^2 - (n - 1)^2 = n^2 - n^2 + 2n - 1 = 2n - 1 ;
\]

from which an iterative expression of perfect squares is inferred as:

\[
n^2 = (n - 1)^2 + 2n - 1 ; \tag{75}
\]

where \( n \) is a natural number.

**II. With regard to natural numbers, perfect squares equal the sum of all odd numbers less than twice related square roots.**

The inverse sequence of consecutive perfect squares, explicitly expressed, reads:

\[
n^2 = (n - 1)^2 + 2n - 1 ;
\]

\[
(n - 1)^2 = [(n - 1) - 1]^2 + 2(n - 1) - 1 ;
\]

\[
(n - 2)^2 = [(n - 2) - 1]^2 + 2(n - 2) - 1 ;
\]

\[\ldots\]

\[
[n - (n - 3)]^2 = \{[n - (n - 3)] - 1\}^2 + 2[n - (n - 3)] - 1 ;
\]

\[
[n - (n - 2)]^2 = \{[n - (n - 2)] - 1\}^2 + 2[n - (n - 2)] - 1 ;
\]
according to above results. The substitution of latter into former relations yields:

\[ n^2 = (2n - 1) + [2(n - 1) - 1] + [2(n - 2) - 1] + \ldots + \{2[n - (n - 3)] + (n - (n - 2)] - 1\} + \{[n - (n - 2)] - 1\}^2 = \]
\[ = (2n - 1) + (2n - 3) + (2n - 5) + \ldots + 5 + 3 + 1 \ ; \]

in conclusion:

\[ n^2 = \sum_{k=1}^{n}(2k - 1) \ ; \quad (76) \]

which is the final result\(^3\). The situation is sketched in Fig. 16 for \(1 \leq n \leq 25\).

**III.** With regard to perfect rectangles of fixed half perimeter, \(p_2 = 2n\), decreasing sequence of areas equals decreasing sequence of differences between related perfect square and lower perfect square area.

Let perfect rectangle be defined as rectangle where side lengths are expressed by natural numbers. Let perfect square be defined as perfect rectangle of equal sides. With regard to perfect rectangles of fixed half perimeter, \(p_2 = 2n\), the decreasing sequence of areas reads:

\[ nn, (n + 1)(n - 1), (n + 2)(n - 2), \ldots, [n + (n - 1)][n - (n - 1)] \ ; \]

which is equivalent to:

\[ n^2 - 0, n^2 - 1, n^2 - 4, \ldots, n^2 - (n - 1)^2 \ ; \]

where \((\ell + 1)\)-th term can be expressed as:

\[ n^2 - \ell^2 = \sum_{k=\ell+1}^{n}(2k - 1) \ ; \quad (77) \]

that is area of perfect rectangle of half perimeter, \(p_2 = 2n\), and side length, \((n + \ell), (n - \ell)\).

The inclusion of limiting case, \(\ell = n\), implies null area owing to null side, and perfect rectangle reduces to perfect segment i.e. length is expressed by a natural number.

The inclusion of limiting case, \(n = 0\), implies null length owing to null side, and perfect segment reduces to perfect point i.e. length is expressed by zero, intended as natural number. Accordingly, all points are perfect in the above mentioned sense.

\(^3\)A more elegant, though less evident, demonstration can be performed by induction. To this aim, Eq. (76) is verified for \(n = 1\), then it is supposed to hold for \((n - 1)\) and it is verified for \(n\).
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Figure 16: Sequence of odd numbers as difference of consecutive perfect squares. With regard to natural numbers, perfect squares equal the sum of all odd numbers less than twice related square roots, starting from bottom left cell, corresponding to unity. Next boundaries, needed to make the succession of perfect squares, relate to the sequence of odd numbers. Unity is also difference of consecutive perfect squares: $1^2 - 0^2 = 1$. The last difference of consecutive perfect squares, inferred from the figure, is $25^2 - 24^2 = 625 - 576 = 49$. 
Table 3: Expression of odd numbers, \( d_k = 2k + 1, 1 \leq k \leq 29 \), as difference between a convenient odd square, \( d_k^2 \), \( d_n = 2n + 1 = (d_k^2 + 1)/2 \), and nearest lower even square, \( p_n^2 \), \( p_n = 2n = (d_k^2 + 1)/2 - 1 \).

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IV. Odd squares can be expressed as difference between a convenient odd square and nearest lower even square.

The above statement implies the validity of the Diophantine equation\(^4\):

\[
(2k+1)^2 = (2n+1)^2 - (2n)^2 = 4n^2 + 4n + 1 - 4n^2 = 4n + 2 - 1 = 2(2n+1) - 1 ;
\]

in conclusion:

\[
2n + 1 = \frac{(2k + 1)^2 + 1}{2} = 2k^2 + 2k + 1 ; \tag{78}
\]

which is an expression of \( 2n \) in terms of \( k \). Results for odd numbers from 1 up to 59 are listed in Tab. 3.

Let both sides of the above Diophantine equation be multiplied by an arbitrarily selected perfect square, \( N^2 \). The result is:

\[
N^2(2k + 1)^2 = N^2(2n + 1)^2 - N^2(2n)^2 = N^2[2(2n + 1) - 1] ;
\]

or:

\[
N(2n + 1) = \frac{N^2(2k + 1)^2 + N^2}{2N} = N(2k^2 + 2k + 1) ; \tag{79}
\]

which is an expression of \( 2nN \) in terms of \( k, N \). It is worth remembering odd products involve odd factors while even products involve at least one even factor.

---

\(^4\)Let Diophantine equation be defined as equation in one or more unknowns, with integer coefficients, integer solutions of which are needed.
Related results can be inferred from their counterparts listed in Tab. 3, valid for \( N = 1 \), using the following relations:

\[
N(2k + 1) = N d_k ; \quad N(2n + 1) = N d_n ; \quad N(2n) = N p_n ;
\]

according to above considerations. Convenient choices of \( N \) and \( k \) yield any solution of Diophantine equation, \( a^2 + b^2 = c^2 \), where \( a = N d_k \), \( b = N p_n \), \( c = N d_n \).

At this stage a natural question arises, if the difference between perfect \( \ell \)-squares, \( (N + \Delta N)^\ell , N^\ell \), yields a perfect \( \ell \)-square or, in other words, the Diophantine equation:

\[
K^\ell = (N + \Delta N)^\ell - N^\ell = \sum_{j=0}^{\ell} \binom{\ell}{j} N^j (\Delta N)^{\ell-j} - N^\ell = \sum_{j=0}^{\ell-1} \binom{\ell}{j} N^j (\Delta N)^{\ell-j} ;
\]

has integer solutions. According to Fermat-Wiles theorem (last Fermat's theorem), it is not the case for \( \ell > 2 \), hence the right-hand side of above equation never equals a natural number provided \( \ell > 2 \).

V. Sum of sums of first \( \ell \) natural numbers, \( 1 \leq \ell \leq 2n + 1 \), equals sum of odd squares, \( (2\ell + 1)^2 \), \( 0 \leq \ell \leq n \).

Sum of sums of first \( \ell \) natural numbers, \( 1 \leq \ell \leq 2n + 1 \), reads:

\[
\sum_{k=1}^{1} k + \sum_{k=1}^{2} k + \sum_{k=1}^{3} k + ... + \sum_{k=1}^{2i} k + \sum_{k=1}^{2i+1} k + ... + \sum_{k=1}^{2n} k + \sum_{k=1}^{2n+1} k =
\]

\[
= \frac{1 \cdot 2}{2} + \frac{2 \cdot 3}{2} + \frac{3 \cdot 4}{2} + ... + \frac{2i(2i + 1)}{2} + \frac{(2i + 1)(2i + 2)}{2} + ... + \\
+ \frac{2n(2n + 1)}{2} + \frac{(2n + 1)(2n + 2)}{2} =
\]

\[
= 1 + 1 \cdot 3 + 3 \cdot 2 + ... + i(2i + 1) + (2i + 1)(i + 1) + ... + n(2n + 1) + \\
+ (2n + 1)(n + 1) =
\]

\[
= 1 + (1 + 2)3 + ... + (i + i + 1)(2i + 1) + ... + (n + n + 1)(2n + 1) =
\]

\[
= 1^2 + 3^2 + ... + (2i + 1)^2 + ... + (2n + 1)^2 ;
\]

in conclusion:

\[
\sum_{\ell=1}^{2n+1} \sum_{k=1}^{\ell} k = \sum_{i=0}^{n} (2i + 1)^2 ; \quad (80)
\]

as stated above.
C Prime number determination

GWBASIC program used for determining the sequence of prime numbers, starting from 7 and following prescriptions explained in the text, is listed below. Execution has been stopped at 12007 and outputs have been checked up to 10583 by comparison with available results e.g., [4].

10 REM PROGRAMMA PRIMI
20 REM GENERA LA SEQUENZA DEI NUMERI PRIMI SUCCESSIVI A 5
30 REM VERIFICANDO CHE UN NUMERO NON COMPARE
40 REM ALL’INTERNO DEL TRIANGOLO PITAGORICO
50 REM
60 REM
70 REM
80 REM
90 REM
100 DEFDBL A-H, O-Z
110 DEFINT I-N
120 REM DIM DY(6),Y(6),YP(6),F(42)
130 REM ISD=0 NON STAMPA SU DISCO
140 REM ISD=1 STAMPA SU DISCO
150 ISD=1
160 IF ISD=0 GOTO 190
170 OPEN"o",#1,"c:\caimmi\dati\primi.dat"
180 R0=.0000000001#
190 PI=4#*ATN(1#)
200 LMAX=6001
210 FOR L=1 TO LMAX
220 REM STOP
230 IF ABS(L/5-INT(L/5))<R0 GOTO 400
240 I=5+2*L
250 IRMIN=2*INT(SQR(CDBL(I)))-1
260 IRMAX=INT(I/2)+1
270 IFIN=0
280 FOR IR=IRMIN TO IRMAX
290 IF IFIN=1 GOTO 350
300 IF ABS(IR/2-INT(IR/2))<R0 THEN KMAX=IR/2 ELSE KMAX=INT(IR/2)+1
310 FOR K=2 TO KMAX
320 IF ABS(K*(IR-K+1)-I)>R0 GOTO 340
330 IFIN=1
340 NEXT K
350 NEXT IR
360 IF IFIN=1 GOTO 400
370 PRINT USING "####### "; L, I 
380 IF ISD=0 GOTO 400 
390 PRINT#1, USING "####### "; L, I 
400 NEXT L 
410 IF ISD=0 GOTO 430 
420 CLOSE#1 
430 PRINT "FINE" 
440 STOP 
450 END 

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