A Strong Convergence Theorems for the Split Feasibility Problem with Applications

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Abstract

In this paper, the following split feasibility problem is investigated in uniformly convex and 2-uniformly smooth Banach spaces: Find a point $x^* \in VI(f, C)$ such that $y^* = Fx^* \in Fix(T)$ where $f$ is an inverse strongly accretive operator of $E_1$, $T$ is a nonexpansive mapping of $E_2$ and $F : E_1 \to E_2$ is a bounded linear operator. Based on the idea of viscosity approximation method, a new algorithm is introduced to solve this problem and the strong convergence of the iterative scheme presented is obtained. Finally, the main result is applied to solve equilibrium problem and zero point problem.

Keywords: split feasibility problem; variational inequality problem; accretive operator; strong convergence; Banach spaces

1 Introduction

The variational inequality problem (shortly, VIP) is an important problem in the field of nonlinear analysis and optimization. Over the years, it has been developed not only in pure science but also in applied science. Its theory is closely related to other nonlinear problems, such as fixed point problem, zero point problem, equilibrium problem, etc. For more details, see [1–8] and the references therein.

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In 1964, Stampacchia [9] initially introduced \( VIP \) for modeling in the mechanics problem. Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( f: C \rightarrow H \) be a given nonlinear mapping, the \( VIP \) in Hilbert spaces is defined as finding a point \( x^* \in C \) such that

\[
\langle f(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C,
\]

we denote the solution set of \( VIP(1.1) \) by \( VI(f, C) \).

It’s well known that \( VI(f, C) \) is equivalent to the solution set of the following fixed point equation:

\[
x^* = P_C(I - \lambda f)x^*,
\]

where \( P_C \) denotes the metric projection from \( H \) onto \( C \), \( \lambda \) is a positive number. To solve \( VIP(1.1) \), Goldstein et al. [2] introduced the following projection method. For a fixed point \( x_0 \in C \),

\[
x_{n+1} = P_C(x_n - \lambda f(x_n)), \quad n \in N.
\]

When \( f \) is inverse strongly monotone and Lipschitz continuous, they proved the strong convergence of the algorithm (1.2). In order to relax the requirement of operator \( f \), Korpelevich [3] introduced the following famous extragradient method to solve saddle point problems in Euclidean space \( \mathbb{R}^n \):

\[
\begin{align*}
y_n &= P_C(x_n - \lambda f(x_n)), \\
x_{n+1} &= P_C(x_n - \lambda f(y_n)),
\end{align*}
\]

where \( f \) is monotone and \( L \)-Lipschitz continuous and \( \lambda \in (0, \frac{1}{L}) \) with \( L \) is the Lipschitz constant. Under some wild conditions, they proved \( \{x_n\} \) converges to a point in \( VI(f, C) \).

In 2006, Aoyama et al. [10] investigated the following generalized variational inequality problem (shortly, \( GVIP \)) in Banach space.

Let \( E \) be a smooth Banach space with norm \( \| \cdot \| \), \( E^* \) denote the dual space of \( E \), \( C \) be a nonempty closed convex subset of \( E \), \( \langle x, f^* \rangle \) denote the value of \( f^* \in E^* \) at \( x \in E \), and \( f \) be an accretive operator of \( C \) into \( E \). Find a point \( x^* \in C \) such that

\[
\langle f(x^*), J(x - x^*) \rangle \geq 0, \quad \forall x \in C,
\]

where \( J \) is the duality mapping of \( E \) into \( E^* \). They introduced the following iterative scheme to solve the \( GVIP(1.4) \): \( x_0 \in C \),

\[
x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Q_C(x_n - \lambda_n f(x_n)), \quad n \in N,
\]

where \( Q_C \) is a sunny nonexpansive retraction from \( E \) onto \( C \), \( \{\alpha_n\} \) is a sequence in \( [0, 1] \), and \( \{\lambda_n\} \) is a sequence of positive real numbers. They obtained a weak convergence result.
In 2011, Yao and Maruster [11] constructed the following new simple algorithm for solving (1.4): $x_0 \in C$

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)Q_C((1 - \alpha_n)(x_n - \lambda f x_n)), \quad n \in \mathbb{N}, \quad (1.5)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$, and $\lambda \in (0, \frac{\alpha}{\|f\|^2})$ is a constant. It is worth mentioning that their algorithm has strong convergence under certain appropriate conditions.

In 1994, Censor and Elfving [12] introduced split feasibility problem (shortly, SFP), which was defined as follows:

Let $C$ and $D$ be nonempty closed convex subsets of real Hilbert spaces $H_1$ and $H_2$, respectively, $F : H_1 \to H_2$ be a bounded linear operator. Finding a point $x^* \in C$ such that

$$Fx^* \in D. \quad (1.6)$$

To solve SFP, Byrne [13] proposed the following algorithm

$$x_{n+1} = P_C(I - \gamma F^*(I - P_D)F)x_n, \quad (1.7)$$

for $n \in \mathbb{N}$ and $\lambda \in (0, \frac{2}{\|F\|^2})$, and they obtained a weak convergence result.

In 2010, based on the idea of SFP, Censor et al. [14] firstly proposed split variational inequality problem (shortly, SVIP) in Hilbert spaces. Obviously, SVIP is a natural extension of the VIP and the SFP. Subsequently, this issue attracted a lot of attention (see [15–20] and the references therein).

The so-called SVIP is defined as follows in Hilbert spaces: find $x^* \in C$ such that

$$\langle f(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C,$$

and such that $y^* = Fx^* \in D$ satisfies

$$\langle g(y^*), y - y^* \rangle \geq 0, \quad \forall y \in D, \quad (1.8)$$

where $C$ and $D$ are nonempty closed convex subsets of real Hilbert spaces $H_1$ and $H_2$, respectively, $f : C \to H_1$, $g : D \to H_2$ are two nonlinear operators, and $F : H_1 \to H_2$ is a bounded linear operator. To solve it, Censor et al. [14] provided the following algorithm: $x_0 \in C$

$$x_{n+1} = P_C(I - \lambda f)(x_n - \gamma F^*(I - P_D(I - \mu g))Fx_n), \quad n \in \mathbb{N}.$$

They showed that $P_C(I - \lambda f)$ and $P_D(I - \mu g)$ are nonexpansive, when $f$ and $g$ are inverse strongly monotone operators in Hilbert spaces. In fact, it is easy to prove that they are also nonexpansive in uniformly convex and 2-uniformly smooth Banach spaces.
In 2017, by combining extragradient method and algorithm (1.7), Tian et al. [21] considered the following iteration method for solving the SVIP (1.8) in Hilbert spaces:

\[
\begin{align*}
    y_n &= P_C(x_n - \gamma_n F^*(I - P_Q(I - \mu g))Fx_n), \\
    t_n &= P_C(y_n - \lambda_n f(y_n)), \\
    x_{n+1} &= P_C(y_n - \lambda f(t_n)), \\
\end{align*}
\]

they got a weak convergence theorem.

Very recently, Tian et al. [22] again proposed the following algorithm to solve a class split feasibility problem with variational problem:

\[
\begin{align*}
    y_n &= P_C(x_n - \gamma_n F^*(I - T)Fx_n), \\
    t_n &= P_C(y_n - \lambda_n f(y_n)), \\
    w_n &= P_C(y_n - \lambda f(t_n)), \\
    x_{n+1} &= \alpha_n h((x_n) + (1 - \alpha_n)w_n), \\
\end{align*}
\]

where \( f \) is a monotone and Lipschitz continuous mapping, \( T \) is a nonexpansive mapping, they proved a strong convergence theorem.

In this paper, we study the following SFP in Banach spaces:

Finding \( x^* \in VIP(1.4) \) such that \( y^* = Fx^* \in Fix(T) \) (1.9)

where \( T \) is a nonexpansive mapping and \( Fix(T) \) is the fixed point set of \( T \).

Based on the idea of viscosity approximation method, a new algorithm is proposed to solve it and a strong convergence theorem is obtained. Some related applications in the field of nonlinear analysis and optimization are also given.

## 2 Preliminaries

Let \( E^* \) be the duality space of a real Banach space \( E \). We denote \( x_n \rightharpoonup x \) to indicate that the sequence \( \{x_n\} \) converges weakly to \( x \), and \( x_n \to x \) to indicate that the sequence \( \{x_n\} \) converges strongly to \( x \). Denote by \( Fix(T) \) the fixed point set of a self mapping \( T \) defined on \( E \) (i.e., \( Fix(T) = \{x \in E : Tx = x\} \)).

Let \( U = \{x \in E : \|x\| = 1\} \). A Banach space \( E \) is said to be uniformly convex if for every \( \epsilon \) with \( 0 \leq \epsilon \leq 2 \), there exists \( \delta > 0 \) such that for any \( x, y \in U \), if \( \|x - y\| \geq \epsilon \), then

\[
\left\| \frac{x + y}{2} \right\| \leq 1 - \delta. \tag{2.1}
\]

It’s well known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space space \( E \) is said to be smooth if for any \( x, y \in U \), the limit

\[
\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}, \tag{2.2}
\]
exists. If the limit (2.2) is attained uniformly for \( x, y \in U \), then \( E \) is said to be uniformly smooth. The modulus of smoothness of \( E \) is defined as follows:

\[
\rho(\tau) = \sup \left\{ \frac{1}{2} (\| x + y \| + \| x - y \|) - 1 : x, y \in E, \| x \| = 1, \| y \| = \tau \right\}.
\]

Note that \( E \) is uniformly smooth if and only if \( \lim_{\tau \to 0} \frac{\rho(\tau)}{\tau} = 0 \). Let \( q \) be a fixed real number with \( 1 < q \leq 2 \). Then \( E \) is said to be \( q \)-uniformly smooth if there exists a constant \( c > 0 \) such that

\[
\rho(\tau) \leq c \tau^{q} \quad \text{for all } \tau > 0.
\]

For \( q > 1 \), the generalized duality mapping \( J_q : E \to 2^{E^*} \) is defined by

\[
J_q(x) = \{ j_q(x) \in E^* : \langle j_q(x), x \rangle = \| x \|^{q}, \| j_q(x) \| = \| x \|^{q-1} \}.
\]

In particular, when \( q = 2 \), the mapping \( J_2 \) is the normalized duality mapping, denoted by \( J \). The normalized duality mapping has the following properties [23]:

(i) if \( E \) is smooth, then \( J \) is single-valued, which is denoted by \( j \);
(ii) if \( E \) is strictly convex, then \( J \) is strictly monotone;
(iii) if \( E \) is reflexive, then \( J \) is surjective;
(iv) if \( E \) is uniformly smooth, then \( J \) is uniformly norm-to-norm continuous on each bounded subset of \( E \).

In order to prove our results, we recall the following definitions and lemmas.

**Definition 2.1** Let \( C \) be a nonempty closed convex set of a real Banach space \( E \). Let \( T : E \to E \), \( f : C \to E \) be two nonlinear operators.

(i) \( T \) is said to be \( \alpha \)-contractive if for any \( x, y \in E \), there exists \( \alpha \in (0, 1) \) such that

\[
\| Tx - Ty \| \leq \alpha \| x - y \|.
\]

(ii) \( T \) is said to be nonexpansive if for any \( x, y \in E \),

\[
\| Tx - Ty \| \leq \| x - y \|.
\]

(iii) \( T \) is said to be \( L \)-Lipschitz continuous, with \( L > 0 \), if for any \( x, y \in E \),

\[
\| Tx - Ty \| \leq L \| x - y \|.
\]

(iv) \( f \) is said to be accretive if for any \( x, y \in C \),

\[
\langle fx - fy, j(x - y) \rangle > 0,
\]

an accretive operator \( f \) is said to be \( m \)-accretive if \( R(I + \lambda f) = X, \forall \lambda > 0 \).

(v) \( f \) is said to be \( \beta \)-strongly accretive, with \( \beta > 0 \), if for any \( x, y \in C \),

\[
\langle fx - fy, j(x - y) \rangle \geq \beta \| x - y \|^{2}.
\]
(vi) $f$ is said to be $\beta$-inverse strongly accretive (\(\beta\)-ism), with $\beta > 0$, if for any $x, y \in C$, 
\[
\langle fx - fy, j(x - y) \rangle \geq \beta \|fx - fy\|^2.
\]

Note that a $\beta$-inverse strongly accretive mapping is accretive and $\frac{1}{\beta}$-Lipschitz continuous, with $\beta > 0$, but the inverse is not true. In addition, it is well-known that $\text{Fix}(T)$ is a closed convex subset if $T$ is nonexpansive and $\text{Fix}(T)$ is nonempty.

Lemma 2.2 ([24]) If $E$ is $q$-uniformly smooth, $q \in (1, 2]$, then there is a constant $k > 0$ such that 
\[
\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x) \rangle + 2\|ky\|^q, \quad \forall x, y \in E,
\]
where $k$ is the $q$-uniform smoothness coefficient of $E$.

Lemma 2.3 ([25]) Let $E$ be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $p : [0, 2r] \to [0, +\infty)$ such that $p(0) = 0$ and 
\[
\|tx + (1 - t)y\|^2 \leq t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)p(\|x - y\|)
\]
for all $x, y \in B_r$ and $t \in [0, 1]$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Let $C$ be a nonempty closed convex subset of $E$ and $K$ be a nonempty subset of $C$. Let $Q$ be a mapping from $C$ onto $K$, if $Qx = x$ for every $x \in K$, we say that $Q$ is a retraction. Note that in a uniformly convex Banach space, any nonempty closed convex subset has a retraction. $Q$ is said to be sunny, if for any $x \in C$ and $t \geq 0$, 
\[
Q(tx + (1 - t)Qx) = Qx,
\]
where $tx + (1 - t)Qx \in C$. Furthermore, $Q$ is a sunny nonexpansive retraction from $C$ onto $K$ if $Q$ is a retraction from $C$ onto $K$ which is also sunny and nonexpansive. Actually, a sunny nonexpansive retraction mapping can play similar roles in Banach space as a metric projection mapping does in a Hilbert space.

Lemma 2.4 ([26]) Let $C$ be a nonempty closed convex subset of a smooth Banach space $E$. Let $K$ be a nonempty subset of $C$ and $Q : C \to K$ be a retraction. Then $Q$ is sunny nonexpansive if and only if one of the following inequalities holds:

(i) $\langle x - Qx, j(y - Qx) \rangle \leq 0$, $\forall x \in C$, $y \in K$;
(ii) $\|Qx - Qy\|^2 \leq \langle x - y, j(Qx - Qy) \rangle$, $\forall x \in C$.

Lemma 2.5 ([10]) Let $C$ be a nonempty closed convex subset of a smooth Banach space $E$. Let $Q_C$ be a sunny nonexpansive retraction from $E$ onto $C$ and let $f$ be an accretive operator of $C$ into $E$. Then for all $\lambda > 0$,
\[
\text{VI}(f, C) = \text{Fix}(Q_C(I - \lambda f)),
\]
where $VI(f, C) = \{u \in C : \langle fu, J(v - u) \rangle \geq 0, \forall v \in C\}$.

**Lemma 2.6** ([27]) Let $E$ be a real smooth and uniformly convex Banach space and let $r > 0$, $B_r = \{z \in E : \|z\| \leq r\}$. Then there exists a strictly increasing and continuous convex function $g : [0, 2r] \to [0, +\infty)$ such that $g(0) = 0$ and $g(\|x - y\|) \leq \|x\|^2 - 2\langle x, y \rangle + \|y\|^2$, for all $x, y \in B_r$.

**Lemma 2.7** ([28]) Let $C$ be a nonempty bounded closed convex subset of a uniformly convex Banach space $E$ and let $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. If $\{x_n\}$ is a sequence of $C$ converging weakly to $x$ and if $\{(I - T)x_n\}$ converges strongly to $y$, then $(I - T)x = y$.

**Lemma 2.8** ([29]) Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n$, where $\{\gamma_n\} \subset (0, 1)$ and $\{\delta_n\}$ are two sequences satisfying: (i) $\sum_{n=0}^{\infty} \gamma_n = \infty$ and (ii) $\limsup_{n \to \infty} \frac{\delta_n}{\gamma_n} \leq 0$ or $\sum_{n=0}^{\infty} |\delta_n| < \infty$. Then, $\lim_{n \to \infty} a_n = 0$.

**Lemma 2.9** ([30]) Let $\{\Gamma_n\}$ be a sequence of real numbers such that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_{i+1}}$ for all $i \in N$. Then there exists a nondecreasing sequence $\{\tau(n)\}_{n \geq n_0}$ of $N$, which defined as follows:

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in N$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\tau(n) \leq \tau(n + 1) \leq \cdots$ and $\lim_{n \to \infty} \tau(n) = \infty$;

(ii) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and $\Gamma_n \leq \Gamma_{\tau(n)+1}$, $\forall n \geq n_0$.

### 3 Main Results

**Theorem 3.1** Let $E_1$ and $E_2$ be two uniformly convex and 2-uniformly smooth Banach spaces, $C$ be a nonempty closed convex subset of $E_1$, $J_1 : E_1 \to 2^{E_1^*}$ and $J_2 : E_2 \to 2^{E_2^*}$ be the normalized duality mappings, $F : E_1 \to E_2$ be a bounded linear operator, $h : C \to C$ be $\alpha$-contraction with $\frac{1}{2} < \alpha < 1$, $f : C \to E_1$ be a $\beta$-inverse strongly accretive mapping, $T : E_2 \to E_2$ be a nonexpansive mapping, and $Q_C$ be the sunny nonexpansive retraction from $E_1$ onto $C$. Denote $\Omega = \{z \in VI(f, C) : Fz \in Fix(T)\}$. For fixed point $x_0 \in C$, the sequence $\{x_n\}$ is generated by

$$\begin{align*}
    u_n &= Q_C(x_n - \gamma J_1^{-1}F^*J_2(I - T)Fx_n), \\
    v_n &= Q_C(u_n - \lambda f(u_n)), \\
    w_n &= Q_C(v_n - \lambda f(v_n)), \\
    x_{n+1} &= \alpha_n h(x_n) + (1 - \alpha_n)w_n,
\end{align*}$$

where $J_1^{-1}$ is the inverse mapping of $J_1$, $F^*$ is the adjoint operator of $F$, $\gamma \in (0, \frac{1}{\|K\|})$, $\lambda \in (0, \frac{\beta}{\|K\|})$, $\{\alpha_n\} \subset (0, 1)$, $k$ is the smoothness constant of $E_1$ with $k \in (0, \frac{1}{2})$. If $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\Omega \neq \emptyset$, then $\{x_n\}$ converges
So according to Lemma 2.2, we get $Q$ retraction from $z$ and $206$. So, 

$$
\text{Since } \gamma \parallel \lambda \parallel \gamma \in (0, \frac{1}{2}), \text{ we can easily get } 2k^2(1 + \gamma \parallel F \parallel^2) < 1 \text{ and } \gamma^2 \parallel F \parallel^2 - \gamma < 0, \text{ then, we have }
$$

$$
\|u_n - p\| \leq \|x_n - p\|. \quad (3.2)
$$

On the other hand, since $f$ is $\beta$-inverse strongly accretive and $\lambda \in (0, \frac{\beta}{k^2})$, we know that $Q_C(I - \lambda f)$ is nonexpansive. Indeed, for any $x, y \in C$,

$$
\|Q_C(I - \lambda f)x - Q_C(I - \lambda f)y\|^2 \leq \|(x - y) - \lambda(f(x) - f(y))\|^2 \\
\leq \|x - y\|^2 - 2\lambda\langle f(x) - f(y), J_1(x - y) \rangle \\
+ 2k^2\lambda^2\|f(x) - f(y)\|^2 \\
\leq \|x - y\|^2 + 2\lambda(k^2\lambda - \beta)\|f(x) - f(y)\|^2 \\
\leq \|x - y\|^2. \quad (3.3)
$$

So,

$$
\|v_n - p\| = \|Q_C(I - \lambda f)u_n - Q_C(I - \lambda f)p\| \leq \|u_n - p\| \leq \|x_n - p\|. \quad (3.4)
$$
Similarly,

\[ \|w_n - p\| = \|Q_C(I - \lambda f)v_n - Q_C(I - \lambda f)p\| \leq \|v_n - p\| \leq \|x_n - p\|. \]  \hfill (3.5)

Since \( h \) is \( \alpha \)-contraction, \( \frac{1}{2} < \alpha < 1 \), and \( 0 < \alpha_n < 1 \), we have

\[ \|x_{n+1} - p\| = \|\alpha_n h(x_n) + (1 - \alpha_n)w_n - p\| \]
\[ \leq \alpha_n\|h(x_n) - p\| + (1 - \alpha_n)\|w_n - p\| \]
\[ \leq \alpha_n\|h(x_n) - h(p)\| + \alpha_n\|h(p) - p\| + (1 - \alpha_n)\|w_n - p\| \]
\[ \leq \alpha_n \cdot \alpha\|x_n - p\| + \alpha_n\|h(p) - p\| + (1 - \alpha_n)\|x_n - p\| \]
\[ = (1 - (1 - \alpha) \cdot \alpha_n)\|x_n - p\| + (1 - \alpha) \cdot \alpha_n \cdot \frac{\|h(p) - p\|}{1 - \alpha} \]
\[ \leq \max\{\|x_n - p\|, \frac{\|h(p) - p\|}{1 - \alpha}\}. \]  \hfill (3.6)

By induction, we have \( \|x_{n+1} - p\| \leq \max\{\|x_0 - p\|, \frac{\|h(p) - p\|}{1 - \alpha}\} \). Hence, \( \{x_n\} \) is bounded. It follows from (3.2), (3.4), (3.5) that \( \{u_n\} \), \( \{v_n\} \) and \( \{w_n\} \) are bounded, too.

On the other hand, it follows from (3.5) and Lemma 2.2 that

\[ \|x_{n+1} - p\|^2 = \|\alpha_n (h(x_n) - p) + (1 - \alpha_n) (w_n - p)\|^2 \]
\[ \leq (1 - \alpha_n)^2 \|w_n - p\|^2 + 2\alpha_n (1 - \alpha_n) \langle h(x_n) - p, J_1(w_n - p) \rangle \]
\[ + 2k^2\alpha_n^2 \|h(x_n) - p\|^2 \]
\[ \leq (1 - \alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n (1 - \alpha_n) \langle h(x_n) - h(p), J_1(w_n - p) \rangle \]
\[ + 2\alpha_n (1 - \alpha_n) \langle h(p) - p, J_1(w_n - p) \rangle + 2k^2\alpha_n^2 \|h(x_n) - p\|^2 \]
\[ = (1 - (1 + (1 - \alpha_n)(1 - 2\alpha))\alpha_n) \|x_n - p\|^2 \]
\[ + \frac{2}{1 + (1 - \alpha_n)(1 - 2\alpha)} \cdot \alpha_n \]
\[ \cdot ((1 - \alpha_n) \langle h(p) - p, J_1(w_n - p) \rangle + k^2\alpha_n \|h(x_n) - p\|^2) \]
\[ = (1 - \tilde{\alpha}_n) \|x_n - p\|^2 + \tilde{\alpha}_n \cdot \frac{2}{1 + (1 - \alpha_n)(1 - 2\alpha)} \]
\[ \cdot ((1 - \alpha_n) \langle h(p) - p, J_1(w_n - p) \rangle + k^2\alpha_n M), \]  \hfill (3.7)

where \( \tilde{\alpha}_n = (1 + (1 - \alpha_n)(1 - 2\alpha))\alpha_n \) and \( M = \sup_{n \in \mathbb{N}} \|h(x_n) - p\|^2 \). By the conditions of \( \frac{1}{2} < \alpha < 1 \) and \( 0 < \alpha_n < 1 \), we can calculate that \( 0 < 1 + (1 - \alpha_n)(1 - 2\alpha) < 1 \) and then, \( \{\tilde{\alpha}_n\} \) satisfies that \( \lim_{n \to \infty} \tilde{\alpha}_n = 0 \) and \( \sum_{n=0}^{\infty} \tilde{\alpha}_n = \infty \).

Now, we show that \( \{x_n\} \) converges strongly to \( z = Q_\Omega h(z) \in \Omega \). We consider the following two cases.

Case 1: Suppose that there exists a natural number \( N \) such that \( \|x_{n+1} - z\| \leq \|x_n - z\| \) for all \( n \geq N \). So \( \lim_{n \to \infty} \|x_n - z\| \) exists. According to \( 0 < \alpha_n < 1 \),
Lemma 2.3 and (3.5), we can get
\[
\|x_{n+1} - z\|^2 = \|\alpha_n (h(x_n) - z) + (1 - \alpha_n)(w_n - z)\|^2 \\
\leq \alpha_n\|h(x_n) - z\|^2 + (1 - \alpha_n)\|w_n - z\|^2 \\
- \alpha_n(1 - \alpha_n)p(\|h(x_n) - w_n\|) \\
\leq \alpha_n\|h(x_n) - z\|^2 + (1 - \alpha_n)\|w_n - z\|^2 \\
\leq \alpha_n\|h(x_n) - z\|^2 + (1 - \alpha_n)\|v_n - z\|^2.
\]

Taking \(\lim \inf\) in the inequality above, since \(\lim \alpha_n = 0\) and \(\{u_n\}, \{v_n\}\) are bounded, we have \(\lim \|x_{n+1} - z\| \leq \liminf_{n \to \infty} \|w_n - z\| \leq \liminf_{n \to \infty} \|v_n - z\|\).

In addition, since \(\|w_n - z\| \leq \|x_n - z\|\), \(\|v_n - z\| \leq \|x_n - z\|\), so we have
\[
\limsup_{n \to \infty} \|w_n - z\| \leq \limsup_{n \to \infty} \|x_n - z\| \text{ and } \limsup_{n \to \infty} \|v_n - z\| \leq \limsup_{n \to \infty} \|x_n - z\|,
\]
which implies that \(\lim \|w_n - z\|\) and \(\lim \|v_n - z\|\) exist, and
\[
\lim_{n \to \infty} \|w_n - z\| = \lim_{n \to \infty} \|x_n - z\|, \tag{3.9}
\]
\[
\lim_{n \to \infty} \|v_n - z\| = \lim_{n \to \infty} \|x_n - z\|. \tag{3.10}
\]

Since \(2k^2(1 + \|F\|^2) < 1\), then it follows from (3.1) and (3.4) that
\[
(\gamma - \gamma^2\|F\|^2)\|(T - I)Fx_n\|^2 \leq 2k^2(1 + \gamma\|F\|^2)\|x_n - z\|^2 - \|u_n - z\|^2 \\
\leq \|x_n - z\|^2 - \|v_n - z\|^2.
\]

Taking \(n \to \infty\) in above inequality, we know from \(\gamma^2\|F\|^2 - \gamma < 0\) and (3.10) that
\[
\lim_{n \to \infty} \|(T - I)Fx_n\| = 0. \tag{3.11}
\]

Since \(x_n \in C\) and \(Q_C\) is nonexpansive, we have
\[
\|u_n - x_n\| = \|Q_C(x_n - \gamma J_I^{-1}F^*J_2(I - T)Fx_n) - Q_C(x_n)\| \\
\leq \|x_n - \gamma J_I^{-1}F^*J_2(I - T)Fx_n - x_n\| \\
\leq \gamma\|F\|\|(I - T)Fx_n\|.
\]

Then, it follows from (3.11) that
\[
\lim_{n \to \infty} \|u_n - x_n\| = 0. \tag{3.12}
\]

On the other hand, similar to (3.3), we can obtain
\[
\|w_n - z\|^2 = \|Q_C(I - \lambda f)(v_n) - Q_C(I - \lambda f)(z)\|^2 \\
\leq \|v_n - z\|^2 + 2\lambda(k^2\lambda - \beta)\|f(v_n) - f(z)\|^2. \tag{3.13}
\]
Hence, according to (3.8) and (3.13), we have
\[
\|x_{n+1} - z\|^2 \leq \alpha_n \|h(x_n) - z\|^2 + (1 - \alpha_n) \|w_n - z\|^2 \\
\leq \alpha_n \|h(x_n) - z\|^2 + (1 - \alpha_n) \|v_n - z\|^2 \\
+ 2\lambda(1 - \alpha_n)(k^2\lambda - \beta)\|f(v_n) - f(z)\|^2.
\]

Then, it follows from 0 < \alpha_n < 1, \lambda \in (0, \frac{\beta}{k^2}) and (3.4) that
\[
2\lambda(1 - \alpha_n)(\beta - k^2\lambda)\|f(v_n) - f(z)\|^2 \leq \alpha_n \|h(x_n) - z\|^2 + (1 - \alpha_n) \|v_n - z\|^2 \\
- \|x_{n+1} - z\|^2 \\
\leq \alpha_n \|h(x_n) - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2.
\]

Taking limit in both two sides, it is easy to get
\[
\lim_{n \to \infty} \|f(v_n) - f(z)\| = 0. \quad (3.14)
\]

Similarly, since \(\|v_n - z\|^2 \leq \|u_n - z\|^2 + 2\lambda(k^2\lambda - \beta)\|f(u_n) - f(z)\|^2\) and \(\|x_{n+1} - z\|^2 \leq \alpha_n \|h(x_n) - z\|^2 + (1 - \alpha_n) \|v_n - z\|^2\), we have
\[
\lim_{n \to \infty} \|f(u_n) - f(z)\| = 0. \quad (3.15)
\]

Let \(r_1 = \sup_{n \geq 0} \{\|v_n - z\|, \|w_n - z\|\}\), according to Lemma 2.4(ii) and Lemma 2.6, we can get
\[
\|w_n - z\|^2 = \|Q_C(I - \lambda f)(v_n) - Q_C(I - \lambda f)(z)\|^2 \\
\leq \langle v_n - \lambda f(v_n) - z + \lambda f(z), J_1(w_n - z) \rangle \\
= \langle v_n - z, J_1(w_n - z) \rangle - \lambda \langle f(v_n) - f(z), J_1(w_n - z) \rangle \\
\leq \frac{1}{2} \|v_n - z\|^2 + \|w_n - z\|^2 - g_1(\|v_n - z\| - (v_n - w_n)) \\
+ \lambda \langle f(z) - f(v_n), J_1(w_n - z) \rangle \\
\leq \frac{1}{2} \|v_n - z\|^2 + \frac{1}{2} \|w_n - z\|^2 - \frac{1}{2} g_1(\|v_n - w_n\|) \\
+ \lambda \|f(z) - f(v_n)\| \|w_n - z\|,
\]

which implies that
\[
\|w_n - z\|^2 \leq \|v_n - z\|^2 - g_1(\|v_n - w_n\|) + 2\lambda \|f(z) - f(v_n)\| \|w_n - z\| \\
\leq \|x_n - z\|^2 - g_1(\|v_n - w_n\|) + 2\lambda \|f(z) - f(v_n)\| \|w_n - z\|,
\]

then,
\[
g_1(\|v_n - w_n\|) \leq \|x_n - z\|^2 - \|w_n - z\|^2 + 2\lambda \|f(z) - f(v_n)\| \|w_n - z\|.
\]
Let $n \to \infty$, we can get from (3.9) and (3.14) that $\lim_{n \to \infty} g_1(\|v_n - w_n\|) = 0$. Based on the property of $g_1$, we have

$$\lim_{n \to \infty} \|v_n - w_n\| = 0. \quad (3.16)$$

In the same way, let $r_2 = \sup_{n \geq 0} \{\|u_n - z\|, \|v_n - z\|\}$, we can easily get the

$$\lim_{n \to \infty} \|u_n - v_n\| = 0. \quad (3.17)$$

From (3.12), (3.16) and (3.17), we obtain

$$\lim_{n \to \infty} \|x_n - w_n\| = 0, \quad (3.18)$$

and

$$\lim_{n \to \infty} \|x_n - v_n\| = 0. \quad (3.19)$$

Now we show that $\limsup_{n \to \infty} \langle h(z) - z, J_1(w_n - z) \rangle + k^2 \alpha_n M \leq 0$. To show this, we can choose a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that

$$\limsup_{n \to \infty} \langle h(z) - z, J_1(w_n - z) \rangle = \lim_{j \to \infty} \langle h(z) - z, J_1(w_{n_j} - z) \rangle.$$

As $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to $\bar{z}$. Since (3.18) holds, we may assume without loss of generality that $\{w_{n_j}\} \to \bar{z}$. We show that $\bar{z} \in \Omega$ firstly. Since $Q_C(I - \lambda f)$ is nonexpansive, we have

$$\|Q_C(I - \lambda f)(x_{n_i}) - x_{n_i}\| \leq \|Q_C(I - \lambda f)(x_{n_i}) - v_{n_i}\| + \|v_{n_i} - x_{n_i}\|
= \|Q_C(I - \lambda f)(x_{n_i}) - Q_C(I - \lambda f)(u_{n_i})\| + \|v_{n_i} - x_{n_i}\|
\leq \|x_{n_i} - u_{n_i}\| + \|v_{n_i} - x_{n_i}\|.$$

Taking $n \to \infty$, it follows from (3.12) and (3.19) that

$$\lim_{n \to \infty} \|(I - Q_C(I - \lambda f))(x_{n_i})\| = 0.$$

Therefore, we know from Lemma 2.7 that $(I - Q_C(I - \lambda f))\bar{z} = 0$, so, $\bar{z} \in Fix(Q_C(I - \lambda f))$. Furthermore, we can get from Lemma 2.5 that $\bar{z} \in VI(f, C)$.

On the other hand, since $F$ is a bounded linear operator, we know that $\{Fx_{n_i}\}$ converging weakly to $F\bar{z}$, and from (3.11) we know $\lim_{n \to \infty} \|(I - T)F(x_{n_i})\| = 0$. Then, according to Lemma 2.7, we have $\|(I - T)F\bar{z}\| = 0$, that is, $F\bar{z} \in Fix(T)$. In other word, $\bar{z} \in \Omega$.

Now from Lemma 2.4(i), we observe that

$$\limsup_{n \to \infty} \langle h(z) - z, J_1(w_n - z) \rangle = \lim_{j \to \infty} \langle h(z) - z, J_1(w_{n_j} - z) \rangle = \langle h(z) - z, J_1(\bar{z} - z) \rangle \leq 0.$$
Since \( \lim_{n \to \infty} \alpha_n = 0 \), then
\[
\limsup_{n \to \infty} \left( (1 - \alpha_n) \langle h(z) - z, J_1(w_n - z) \rangle + k^2 \alpha_n M \right)
\leq \limsup_{n \to \infty} (1 - \alpha_n) \langle h(z) - z, J_1(w_n - z) \rangle + \limsup_{n \to \infty} k^2 \alpha_n M
= \limsup_{n \to \infty} \langle h(z) - z, J_1(w_n - z) \rangle
\leq 0.
\]

It follows from Lemma 2.8 and (3.7) that \( \lim_{n \to \infty} \|x_n - z\|^2 = 0 \), which means that \( \{x_n\} \) converges strongly to \( z \).

Case 2: Let \( \Gamma_n = \|x_n - z\|^2 \) for all \( n \in \mathbb{N} \). Suppose that there exists a subsequence \( \Gamma_{n_k} \) of \( \Gamma_n \) such that \( \Gamma_{n_k} \leq \Gamma_{n_{k+1}} \) for all \( k \in \mathbb{N} \). We define \( \tau : \mathbb{N} \to \mathbb{N} \) by \( \tau_n = \max \{ k \leq n : \Gamma_k < \Gamma_{k+1} \} \). So, by Lemma 2.9, we have \( \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1} \) and \( \Gamma_n \leq \Gamma_{\tau(n)+1} \).

Arguing as in Case 1, we have \( \lim_{n \to \infty} \|x_n - w_n\| = 0 \), and
\[
\limsup_{n \to \infty} \left( \frac{2}{1 + (1 - \alpha_n)(1 - 2\alpha)} \right) ((1 - \alpha_n) \langle h(z) - z, J_1(w_{\tau(n)} - z) \rangle + k^2 \alpha_n M) \leq 0.
\]

Moreover, from (3.7), similarly we have
\[
\|x_{\tau(n)+1} - z\|^2 \leq (1 - \tilde{\alpha}_n) \|x_{\tau(n)} - z\|^2 + \tilde{\alpha}_n \cdot \frac{2}{1 + (1 - \alpha_n)(1 - 2\alpha)} \cdot ((1 - \alpha_n) \langle h(z) - z, J_1(w_{\tau(n)} - z) \rangle + k^2 \alpha_n M),
\]
where \( \tilde{\alpha}_n = (1 + (1 - \alpha_n)(1 - 2\alpha))\alpha_n \) and \( M = \sup_{n \in \mathbb{N}} \|h(x_n) - p\|^2 \). In addition, we have \( \lim_{n \to \infty} \tilde{\alpha}_n = 0 \) and \( \Sigma_{n=0}^{\infty} \tilde{\alpha}_n = \infty \). Then, we know from Lemma 2.8 that \( \lim_{n \to \infty} \Gamma_{\tau(n)} = 0 \).

Since \( \Gamma_{\tau(n)} < \Gamma_{\tau(n)+1} \), we have
\[
\|x_{\tau(n)+1} - z\|^2 \leq (1 - \tilde{\alpha}_n) \|x_{\tau(n)+1} - z\|^2 + \tilde{\alpha}_n \cdot \frac{2}{1 + (1 - \alpha_n)(1 - 2\alpha)} \cdot ((1 - \alpha_n) \langle h(z) - z, J_1(w_{\tau(n)} - z) \rangle + k^2 \alpha_n M).
\]

Then
\[
\|x_{\tau(n)+1} - z\|^2 \leq \frac{2}{1 + (1 - \alpha_n)(1 - 2\alpha)} ((1 - \alpha_n) \langle h(z) - z, J_1(w_{\tau(n)} - z) \rangle + k^2 \alpha_n M).
\]

And it follows from \( \Gamma_n \leq \Gamma_{\tau(n)+1} \) that
\[
\|x_n - z\|^2 \leq \frac{2}{1 + (1 - \alpha_n)(1 - 2\alpha)} ((1 - \alpha_n) \langle h(z) - z, J_1(w_{\tau(n)} - z) \rangle + k^2 \alpha_n M).
\]
As $n \to \infty$, we obtain
\[ \limsup_{n \to \infty} \|x_n - z\|^2 \leq 0. \]
Hence, $\lim_{n \to \infty} \|x_n - z\|^2 = 0$, i.e., $x_n \to z = Q\Omega h(z)$. This completes the proof.

Now, we consider the following split variational inequality problem (shortly, \textit{SVIP}):

Let $E_1$ and $E_2$ be two real Banach spaces, $C$ and $D$ be nonempty closed convex subsets of $E_1$ and $E_2$, respectively, $f : C \to E_1$ be a $\beta$-inverse strongly accretive mapping and $q : E_2 \to E_2$ be a $\delta$-inverse strongly accretive mapping, and $F : E_1 \to E_2$ be a bounded linear operator. $J_1 : E_1 \to 2^{E_1}$, $J_2 : E_2 \to 2^{E_2}$ represent the duality mapping of $E_1$ and $E_2$, respectively. Find a point $x^* \in C$ such that
\[ \langle f(x^*), J_1(x - x^*) \rangle \geq 0, \quad \forall x \in C \]
and such that $y^* = Fx^* \in D$ solves
\[ \langle q(y^*), J_2(y - y^*) \rangle \geq 0, \quad \forall y \in D. \] (3.20)

According to Lemma 2.5, we know that $VI(q, D)$ is equivalent to the fixed point set of $Q_D(I - \mu q)$. In addition, it follows from (3.3) that $Q_D(I - \mu q)$ is nonexpansive when $\mu \in (0, \frac{\delta}{k_2^2})$, where $k_2$ is the smoothness constant of $E_2$.
So, we can get the following corollary by replacing the nonexpansive mapping $T$ in the Theorem 3.1 with $Q_D(I - \mu q)$.

**Corollary 3.2** Let $E_1$ and $E_2$ be two uniformly convex and 2-uniformly smooth Banach spaces, $C$ and $D$ be two nonempty closed convex subsets of $E_1$ and $E_2$, respectively, $J_1 : E_1 \to 2^{E_1}$ and $J_2 : E_2 \to 2^{E_2}$ be the normalized duality mappings, $F : E_1 \to E_2$ be a bounded linear operator, $h : C \to C$ be a $\alpha$-contractive mapping with $\frac{1}{2} < \alpha < 1$, $f : C \to E_1$ be a $\beta$-inverse strongly accretive mapping and $q : E_2 \to E_2$ be a $\delta$-inverse strongly accretive mapping, and $Q_C, Q_D$ be the sunny nonexpansive retractions from $E_1$ onto $C$ and from $E_2$ onto $D$. Denote $\Omega = \{ z \in VI(f, C) : Fz \in VI(q, D) \}$. For fixed point $x_0 \in C$, the sequence $\{x_n\}$ is generated by
\[
\begin{cases}
    u_n = Q_C(x_n - \gamma J_1^{-1}F^*J_2(I - Q_D(I - \mu q))Fx_n), \\
    v_n = Q_C(u_n - \lambda f(u_n)), \\
    w_n = Q_C(v_n - \lambda f(v_n)), \\
    x_{n+1} = \alpha_n h(x_n) + (1 - \alpha_n)w_n,
\end{cases}
\]
where $J_1^{-1}$ is the inverse mapping of $J_1$, $F^*$ is the adjoint operator of $F$, $\gamma \in (0, \frac{1}{\|F\|^2})$, $\lambda \in (0, \frac{\beta}{k_1^2})$, $\mu \in (0, \frac{\delta}{k_2^2})$, $k_1$ and $k_2$ are the smoothness constants of $E_1$ and $E_2$ with $0 < k_1 < \frac{1}{2}$ and $0 < k_2 < \frac{1}{2}$, respectively. $\{\alpha_n\}$ is a sequence in $(0, 1)$. If $\lim_{n \to \infty} \alpha_n = 0$, $\Sigma_{n=0}^{\infty} \alpha_n = \infty$ and $\Omega \neq \emptyset$, then $\{x_n\}$ converges strongly to a point $z \in \Omega$, where $z = Q\Omega h(z)$, $Q\Omega$ be the sunny nonexpansive retractions from $C$ onto $\Omega$. 
4 Applications

In this section, we apply the main results to solve equilibrium problem and zero point problem in Banach spaces.

4.1 Equilibrium problem

Let $E$ be a uniformly convex and 2-uniformly smooth Banach space and $C$ be a nonempty closed convex subset of it. Suppose that $G : C \times C \to \mathbb{R}$ be a bifunction, the so-called equilibrium problem (shortly, EP) is to find a point $x^* \in C$ such that

$$G(x^*, y) \geq 0, \quad \forall y \in C.$$

Denote the solution set of EP by $EP(G)$, we need the following assumptions on $G$ to solve EP:

(A1) $G(x, x) = 0$ for all $x \in C$;
(A2) $G$ is monotone, i.e., $G(x, y) + G(y, x) \leq 0$ for all $x, y \in C$;
(A3) For each $x, y, z \in C$, $\lim_{t \downarrow 0} G(tz + (1 - t)x, y) \leq G(x, y)$;
(A4) For each $x \in C$, $y \mapsto G(x, y)$ is convex and lower semicontinuous.

**Lemma 4.1** ([31]) Let $C$ be a closed convex subset of a smooth, strictly convex and reflexive Banach space $E$ and $G : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). For any $r > 0$ and $x \in E$, there exists $z \in C$ such that

$$G(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C.$$

**Lemma 4.2** ([32]) Let $C$ be a closed convex subset of a smooth, strictly convex and reflexive Banach space $E$ and $G : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). For any $r > 0$ and $x \in E$, define the following resolvent $T_r : E \to C$ of $G$:

$$T_r x = \{ z \in C : G(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C \}.$$

It is not difficult to show that:

(i) $T_r$ is single-valued;
(ii) $T_r$ is firmly nonexpansive;
(iii) $Fix(T_r) = EP(G)$;
(IV) $EP(G)$ is closed and convex.

By Lemma 4.2, we know $T_r$ is nonexpansive. So, we may replace the nonexpansive mapping $T$ in the Theorem 3.1 with $T_r$ and from Lemma 4.1, Lemma 4.2, we have the following result.

**Theorem 4.3** Let $E_1, E_2$ be two uniformly convex and 2-uniformly smooth Banach spaces, $C$ and $D$ be nonempty closed convex subsets of $E_1$ and $E_2$, respectively, $J_1 : E_1 \to 2^{E_1^*}$ and $J_2 : E_2 \to 2^{E_2^*}$ be the normalized duality
mappings, $F : E_1 \to E_2$ be a bounded linear operator, $h : C \to C$ be a $\alpha$-contractive mapping with $\frac{1}{2} < \alpha < 1$, $f : C \to E_1$ be $\beta$-inverse strongly accretive, $G : D \times D \to R$ be a bifunction satisfying (A1)-(A4), $T_r : E \to C$ be the resolvent of $G$, and $Q_C$ be the sunny nonexpansive retraction from $E_1$ onto $C$. Denote $\Omega = \{z \in VI(f, C) : Fz \in EP(G)\}$. For fixed point $x_0 \in C$, the sequence $\{x_n\}$ is generated by

$$
\begin{align*}
  u_n &= Q_C(x_n - \gamma J_{1}^{-1}F^*J_2(I - T_r)Fx_n), \\
  v_n &= Q_C(u_n - \lambda f(u_n)), \\
  w_n &= Q_C(v_n - \lambda f(v_n)), \\
  x_{n+1} &= \alpha_n h(x_n) + (1 - \alpha_n)w_n,
\end{align*}
$$

where $J_{1}^{-1}$ is the inverse mapping of $J_1$, $F^*$ is the adjoint operator of $F$, $\gamma \in (0, \frac{1}{\|F^*\|})$, $\lambda \in (0, \frac{\beta}{k^2})$, $k$ is the smoothness constant of $E_1$ with $0 < k < \frac{1}{2}$. $\{\alpha_n\}$ is a sequence in $(0, 1)$. If $\lim_{n \to \infty} \alpha_n = 0$, $\Sigma_{n=0}^\infty \alpha_n = \infty$ and $\Omega \neq \emptyset$, then $\{x_n\}$ converges strongly to a point $z \in \Omega$, where $z = Q_\Omega h(z)$, $Q_\Omega$ is the sunny nonexpansive retraction from $C$ onto $\Omega$.

**4.2 Zero point problem**

Let $E$ be a real Banach space and $B : E \to 2^{E^*}$ be a set-value mapping, the so-called zero point problem is to find a point $x^* \in E$ such that $0 \in Bx^*$. The zero point set of $B$ denoted by $B^{-1}(0)$.

For an accretive operator $B$ with domain $D(B)$, the resolvent $J_r$ of $B$ is defined by $J_r = (I + rB)^{-1}$ for each $r > 0$. It is known that $B^{-1}(0) = Fix(J_r)$ and $J_r$ is a single-valued nonexpansive mapping. So the following result can be obtained from Theorem 3.1.

**Theorem 4.4** Let $E_1$, $E_2$ be two uniformly convex and 2-uniformly smooth Banach spaces, $C$ and $D$ be nonempty closed convex subsets of $E_1$ and $E_2$, respectively, $J_1 : E_1 \to 2^{E_1}$ and $J_2 : E_2 \to 2^{E_2}$ be the normalized duality mappings, $F : E_1 \to E_2$ be a bounded linear operator, $h : C \to C$ be a $\alpha$-contractive mapping with $\frac{1}{2} < \alpha < 1$, $f : C \to E_1$ be $\beta$-inverse strongly accretive, $B : E_2 \to 2^{E_2}$ be $m$-accretive with $D(B) \neq \emptyset$, and $Q_C$ be the sunny nonexpansive retraction from $E_1$ onto $C$. Denote $\bar{\Omega} = \{z \in VI(f, C) : Fz \in B^{-1}(0)\}$. For fixed point $x_0 \in C$, the sequence $\{x_n\}$ is generated by

$$
\begin{align*}
  u_n &= Q_C(x_n - \gamma J_{1}^{-1}F^*J_2(I - J_r)Fx_n), \\
  v_n &= Q_C(u_n - \lambda f(u_n)), \\
  w_n &= Q_C(v_n - \lambda f(v_n)), \\
  x_{n+1} &= \alpha_n h(x_n) + (1 - \alpha_n)w_n,
\end{align*}
$$

where $J_{1}^{-1}$ is the inverse mapping of $J_1$, $F^*$ is the adjoint operator of $F$, $J_r$ is the resolvent of $B$ with $r > 0$, $\gamma \in (0, \frac{1}{\|F^*\|})$, $\lambda \in (0, \frac{\beta}{k^2})$, $k$ is the smoothness constant of $E_1$ with $0 < k < \frac{1}{2}$. $\{\alpha_n\}$ is a sequence in $(0, 1)$. If $\lim_{n \to \infty} \alpha_n = 0$, 

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$\Sigma_{n=0}^{\infty}O_n = \infty$ and $\Omega \neq \emptyset$, then $\{x_n\}$ converges strongly to a point $z \in \Omega$, where $z = Q_\Omega h(z)$, $Q_\Omega$ is the sunny nonexpansive retraction from $C$ onto $\Omega$.

Next, we consider the zero point problem of sum of two accretive operators. Let $E$ be a real Banach space, $B : E \to 2^{E^*}$ be a set-value mapping and $q : E \to E$ be an operator. Find a point $x^* \in E$ such that $0 \in Bx^* + qx^*$.

**Lemma 4.5** ([33]) Let $E$ be a uniformly convex and 2-uniformly smooth Banach space, $k$ be the smoothness constant of $E$, $B : E \to 2^{E^*}$ be an $m$-accretive mapping, and $q : E \to E$ be $\delta$-inverse strongly accretive, for any $r > 0$,

$$T_r := J_r(I - rq) = (I + rB)^{-1}(I - rq),$$

then $Fix(T_r) = (B + q)^{-1}(0)$.

**Lemma 4.6** ([24]) Let $E$ be a uniformly convex and $q$-uniformly smooth Banach space with $q \in (1, 2]$. Assume that $A$ is a single-valued $\alpha$-inverse strongly accretive mapping with $D(B) \neq \emptyset$, $q : E \to E$ be $\delta$-inverse strongly accretive, for any $r > 0$, such that for all $x, y \in B_r$,

$$\|T_r x - T_r y\|^q \leq \|x - y\|^q - r(\alpha q - r^{q-1}k_q)\|Ax - Ay\|^q - \Phi(\|J_r(I - ra)x - (I - J_r)(I - ra)y\|),$$

where $k_q$ is the $q$-uniform smoothness coefficient of $E$.

Let $E$ be a uniformly convex and 2-uniformly smooth Banach space, $k$ be the smoothness coefficient of $E$, $q : E \to E$ be $\delta$-inverse strongly accretive. According to Lemma 4.6, we can easily prove that if $0 < r < \frac{2k}{\delta}$, then $T_r$ is nonexpansive. Indeed, since $\Phi$ is defined on $R^+$, we have

$$\|T_r x - T_r y\|^2 \leq \|x - y\|^2 - r(2\delta + kr)\|qx - qy\|^2 - \Phi(\|J_r(I - ra)x - (I - J_r)(I - ra)y\|) \leq \|x - y\|^2.$$ 

Therefore, we can replace the nonexpansive mapping $T$ in the Theorem 3.1 with $T_r$ in the Lemma 4.5, then the following result holds.

**Theorem 4.7** Let $E_1, E_2$ be two uniformly convex and 2-uniformly smooth Banach spaces, $C$ and $D$ be nonempty closed convex subsets of $E_1$ and $E_2$, respectively, $J_1 : E_1 \to 2^{E_1^*}$ and $J_2 : E_2 \to 2^{E_2^*}$ be the normalized duality mappings, $F : E_1 \to E_2$ be a bounded linear operator, $h : C \to C$ be a $\alpha$-contractive mapping with $\frac{1}{2} < \alpha < 1$, $f : C \to E_1$ be a $\beta$-inverse strongly accretive mapping, $B : E_2 \to 2^{E_2^*}$ be an $m$-accretive mapping with $D(B) \neq \emptyset$, $q : E_2 \to E_2$ be $\delta$-inverse strongly accretive, and $Q_C$ be the sunny nonexpansive retraction from $E_1$ onto $C$. Denote $\Omega = \{z \in VI(f, C) : Fz \in (B + q)^{-1}0\}$. For fixed
point \( x_0 \in C \), the sequence \( \{x_n\} \) is generated by

\[
\begin{align*}
  u_n &= Q_C(x_n - \gamma J_1^{-1}F^*J_2(I - J_r(I - rq))Fx_n), \\
  v_n &= Q_C(u_n - \lambda f(u_n)), \\
  w_n &= Q_C(v_n - \lambda f(v_n)), \\
  x_{n+1} &= \alpha_n h(x_n) + (1 - \alpha_n)w_n,
\end{align*}
\]

where \( J_1^{-1} \) is the inverse mapping of \( J_1 \), \( F^* \) is the adjoint operator of \( F \), \( J_r \) is the resolvent of \( B \) with \( r \in (0, \frac{2k}{k^2}) \), \( \gamma \in (0, \frac{1-2k^2}{\|F\|^2}) \), \( \lambda \in (0, \frac{2}{k^2}) \), \( k \) is the smoothness constant of \( E \) with \( 0 < k < \frac{1}{2} \). \( \{\alpha_n\} \) is a sequence in \((0,1)\). If \( \lim_{n \to \infty} \alpha_n = 0 \), \( \sum_{n=0}^{\infty} \alpha_n = \infty \) and \( \Omega \neq \emptyset \), then \( \{x_n\} \) converges strongly to a point \( z \in \Omega \), where \( z = Q_\Omega h(z) \), \( Q_\Omega \) is the sunny nonexpansive retraction from \( C \) onto \( \Omega \).

**References**


A strong convergence theorems for the split feasibility problem


**Received: February 21, 2020; Published: March 23, 2020**