The Employment of Neutral Approach for Linear
Singular System Stability Study with Additive
Time Varying Delays

Kawtar El Haouti, Noreddine Chaibi and Abdelkrim Amoumou

Laboratory of Industriel Engineering, Departement of Physics
Hassan II University, Faculty of Science Ain Chock
Casablanca, Morroco

This article is distributed under the Creative Commons by-nc-nd Attribution License.
Copyright © 2020 Hikari Ltd.

Abstract

The introduced paper focus on the studies of linear singular systems. Especially, to the stability of this type of systems with two time varying delays. This can be reached through a neutral approach in order to acquire a new stability criteria with less conservative results. Starting with the transformation of singular systems by neutral ones, that approve more flexible computing, then employing an augmented Lyapunov Krasovski and Writing based integral inequality. The derivative of the function will turn into a less complex computing equation. The established criteria will be formulated in term of LMIs, and the generated conditions will be easily solved. This paper will be ended by a numerical example to show the less conservatism stability of singular systems with additive time varying delays by exploring our suggested method.

Keywords: Stability, singular systems, time varying delay, neutral approach

1. Introduction

In automatic analysis, most of works are based on models supposing that there is no singularity in the system. In fact two cases are possible in reality, the first one, consider that the matrix multiplied by the state space vector is inversible, where classical methods of modeling and control may be applied. In the second case, the state space vector is multiple of a singular matrix, and here the majority of works
base their developments on assumptions in order to eliminate the systems singularity. In this paper, the generalized or descriptor state space system will take place for better representation of physical systems [7]. In one hand, the analysis of singular systems does not stop on proving the stability, but also it consists on studying the regularity in the absence of impulses (for continuous singular system) and causality (for discrete singular systems), this justifies the admissibility of the studied system. In the other, it is well known that the category of the comprising delays system allows to have more confident results. These delays are classified into two types, time dependent and the time independent. Underlining that the first type is less conservative than the second, as it can be shown plainly in the transmission example of signals from a point to another passes through segments of networks, where the successive delays are induced due to the fact of varying transmission conditions [11][2]. The reason why it is more interesting to consider different time varying delays $h_1(t), h_2(t)$, in the same state space vector, where $h_1(t)$ is the time varying delay induced from the sensor to the controller, and $h_2(t)$ is the time varying delay induced from the controller to the actuator [16], as it can be seen in figure 1.

![Networked control system](image)

Figure 1. Networked control system.

In the recent literature most works on singular time delay systems base their stability on the powerful and the dominant combination methodology Lyapunov Krasovski functional and the Jensen integral inequality [6][17][14]. Motivated by this idea and using the new Wirtinger based integral inequality, the present paper aims at establishing a new theorem that check the stability of neutral systems with additive time varying delays. Moreover, checking the stability of the singular systems. Noting that the use of the neutral approach and the Newton Leibniz formula has, as objective, the simplification of LKF derivatives. The paper is structured as follow, in the first section the definitions and lemmas will be defined, the second deals with the main results, in the last section a numerical examples will be illustrated.
Neutral approach for linear singular system stability

Notation:

Through this article, we denote by $\mathbb{R}^n$ the n dimensional real Euclidean space, $\mathbb{R}^{n\times n}$ is the set of $n\times n$ real matrices, the superscripts ‘T’ denotes the matrix transportation, for any square matrix $A \in \mathbb{R}^{n \times n}$. $P > 0$ ($P \geq 0$ means that $P$ is a real, symmetric and positive definitive (positive semi-definite) matrix, the symbol ‘*’ represents the symmetric elements in a symmetric matrix.

2. Problem formulation and preliminaries

Singular system with two additive time varying delays is described by the following representation:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + A_d(x(t - h_1(t) - h_2(t))) \\
x(t) &= \delta(t)
\end{align*}
\]

Where $x(t) \in \mathbb{R}^n$ the state vector, the matrices $E, A, A_d \in \mathbb{R}^{n \times n}$ are known real constant ones, noting that $E$ can be singular $h_1(t)$ and $h_2(t)$ are respectively the first time varying delay system, and second time varying delay system these delays satisfying (2), \( \forall t > 0 \)

\[
0 \leq h_1(t) \leq h_1^* \quad ; \quad 0 \leq h_2(t) \leq h_2^* \quad (2)
\]

Assumption 2.1: The delay $h(t)$ is assumed to satisfy the following conditions:

\[ h(t) = h_1(t) + h_2(t) \]

and

\[ 0 \leq \dot{h}_1(t) \leq d_1 \quad ; \quad 0 \leq \dot{h}_2(t) \leq d_2 \quad (3) \]

Marking too that:

\[ d = d_1 + d_2 \quad ; \quad \overline{h} = \overline{h}_1 + \overline{h}_2 \]

$\overline{h}_1, \overline{h}_2$ are a given bounded of delays, $d_1, d_2$ are respectively the derivatives of the given bounds delays depending on a varying time.

The following definitions and lemmas have been used to develop the main result.

Definition 2.1[7]: The singular time delay system is said to be regular, and impulse free, if the pair $(E,A)$ is regular and impulse free where

1. Pair $(E,A)$ is said to be regular if $\det(sE - A) \neq 0$
2. Pair $(E,A)$ is said to be impulse free if $\deg(\det(sE - A)) = \text{rank}(E)$.

Definition 2.2 [7]: System (1) is said to be admissible if it is regular, impulse free and asymptotically stable.

Definition 2.3 [7]: System (1) is asymptotically stable if, for any $\varepsilon > 0$ there exists a scalar $\delta(\varepsilon) > 0$, such that for any compatible initial condition $\varphi(t)$ with $\sup_{-h(t) \leq t \leq 0} \|\varphi(t)\| < \delta(\varepsilon)$, the solution $x(t)$ of (1) satisfies $\|x(t)\| < \varepsilon$ for $t > 0$ and $\lim_{t \to 0} x(t) = 0$. 
Lemma 2.1 [10]: (Leibniz-Newton formula)

\[
x(t) - x(t - d(t)) - \int_{t-d(t)}^{t} \dot{x}(\alpha) d\alpha = 0. \quad \Leftrightarrow \quad \int_{t-d(t)}^{t} \dot{x}(\alpha) d\alpha = x(t) - x(t - d(t)) \quad (4)
\]

Lemma 2.2 [2]: (Linear matrix inequality)

The LMI:

\[
\begin{bmatrix}
Q(y) & S(y) \\
S^T(y) & R(y)
\end{bmatrix} < 0.
\]

\[
\Leftrightarrow \quad \begin{cases}
R(y) < 0 \\
Q(y) - S(y)R^{-1}(y)S^T(y) < 0
\end{cases}
\]

Where \( Q(y) = Q^T(y) \), \( R(y) = R^T(y) \), \( S(y) \) depend affinely on \( y \).

Lemma 2.3 [1]: (Wirtinger-based integral inequality)

For a symmetric matrix \( R > 0 \), scalars \( a \) and \( b \) with \( a < b \) and vector \( \omega \) such that the integration concerned is well defined [4], the following inequality holds:

\[
(b-a) \int_{a}^{b} \omega^T(s) R \omega(s) \geq \chi_1^T R \chi_1 + 3 \chi_2^T R \chi_2
\]

(5)

Where

\[
\begin{cases}
\chi_1 = \omega(b) - \omega(a) \\
\chi_2 = \omega(b) + \omega(a) - \frac{2}{b-a} \int_{a}^{b} \omega(s) ds
\end{cases}
\]

Lemma 2.4 [8]: If the pair \((E, A)\) is regular and impulse free, then there exist two invertible matrices \( M \in \mathbb{R}^{n \times n} \) and \( N \in \mathbb{R}^{n \times n} \) such that:

\[
MEN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad MAN = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}
\]

According to this lemma we can also write the following terms in order to make available the neutral approach.

\[
MEN = \bar{E} \quad ; \quad MAN = \bar{A} \quad \text{Where} \quad \bar{E} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad ; \quad \bar{A} = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}
\]

Let

\[
MA_d N = \bar{A}_d = \begin{bmatrix} A_{d_1} & A_{d_2} \\ A_{d_3} & A_{d_4} \end{bmatrix} \quad ; \quad N^{-1} x(t) = \mu(t) = \begin{bmatrix} \mu_1(t) \\ \mu_2(t) \end{bmatrix}
\]

Where the partitions are compatible with the structure of \( \bar{E} \), then the singular system is equivalent to:

\[
\bar{E} \bar{\mu}(t) = \bar{A} \mu(t) + \bar{A}_d \mu(t - h(t)) \quad (6)
\]

When \( \mu(t) \) take its over form, (2) can be written as (7):

\[
\begin{cases}
\dot{\mu}_1(t) = A_1 \mu_1(t) + A_{d_1} \mu_1(t - h(t)) + A_{d_2} \mu_2(t - h(t)) \\
0 = \mu_2(t) + A_{d_3} \mu_2(t - h(t)) + A_{d_4} \mu_2(t - h(t))
\end{cases}
\]

Inspired by (3) we can rewrite the second equations as (4):

\[
\frac{d}{dt} [\mu_2(t) + A_{d_3} \mu_2(t - h(t)) + A_{d_4} \mu_2(t - h(t))] = 0
\]
Neutral approach for linear singular system stability

By merging (7) and (8) we have:
\[
\mu(t) = -\left(1 - h(t)\right)A_d \mu_1(t - h(t)) - \left(1 - h(t)\right)A_d \mu_2(t - h(t)) - \mu_2(t) - A_d \mu_1(t - h(t)) - A_d \mu_2(t - h(t))
\]

So the new system is deducing as follow:
\[
\begin{bmatrix}
\dot{\mu}_1(t) \\
\dot{\mu}_2(t)
\end{bmatrix} =
\begin{bmatrix}
A \mu_1(t) + A_d \mu_1(t - h(t)) + A_d \mu_2(t - h(t)) \\
-\mu_2(t) - A_d \mu_1(t - h(t)) + A_d \mu_2(t - h(t))
\end{bmatrix} + (1 - h(t)) \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]
\[
\begin{bmatrix}
\dot{\mu}_1(t - h(t)) \\
\dot{\mu}_2(t - h(t))
\end{bmatrix}
\]

Let \( \dot{A} = \begin{bmatrix} A_1 & 0 \\ 0 & -I_{n-1} \end{bmatrix} \) ; \( \dot{A}_d = \begin{bmatrix} A_{d1} & A_{d2} \\ -A_{d3} & -A_{d4} \end{bmatrix} \) ; \( \dot{C}(t) = (1 - h(t)) \begin{bmatrix} 0 \\ 0 \end{bmatrix} \)

So the neutral system has as representation
\[
\begin{bmatrix}
\mu(t) - \dot{C}(t)\mu(t - h(t)) = \dot{A} \mu(t) + \dot{A}_d \mu(t - h(t)) \\
\mu(t) = \varphi(t) \quad t \in [-h(t), 0]
\end{bmatrix}
\]

Based on this approach we aim at established a new criteria that verifying the stability of the varying delay dependent singular system in the next part.

3. Main results

The stability of singular systems was widely checked by adopting the existing stability conditions, and since the approach of the neutral systems the solutions of new stability criteria were found easily by using a strict LMI approach, in this section the development of this result will be shown:

Theorem 1:

For given scalar \( h > 0 \), The neutral system (9) and thus the singular one (1) are asymptotically stable, if there exist \( 3n \times 3n \) matrix \( P = P^T > 0 \), \( n \times n \) \( Q_i = Q_i^T > 0 \) with \( (i = 1, 2) \) and \( R = R^T > 0 \), such the following LMI holds: \( \Psi < 0 \)

where:
\[
\Psi = \begin{bmatrix}
\psi_{11} & \psi_{12} & \psi_{13} & \psi_{14} & \psi_{15} \\
\psi_{21} & \psi_{22} & \psi_{23} & \psi_{24} & \psi_{25} \\
\psi_{31} & \psi_{32} & \psi_{33} & \psi_{34} & \psi_{35} \\
\psi_{41} & \psi_{42} & \psi_{43} & \psi_{44} & \psi_{45} \\
\psi_{51} & \psi_{52} & \psi_{53} & \psi_{54} & \psi_{55}
\end{bmatrix}
\]

Thus:
\[
\psi_{11} = P_{11} A + A^T P_{11} + 2P_{13} + \tilde{h} A^T R \tilde{A} - \frac{2R}{\tilde{h}} + Q_1 + Q_2 + A^T Q_2 A
\]
\[
\psi_{12} = P_{11} \dot{A}_d + \dot{A} P_{12} + P_{23} - P_{13} + \tilde{h} A^T R \dot{A}_d - \frac{2R}{\tilde{h}} + \dot{A}^T Q_2 A
\]
\[
\psi_{13} = 0 \\
\psi_{14} = P_{13} \dot{C} + P_{12} + \tilde{h} A^T R \dot{C} + \dot{A}^T Q_2 \dot{C}
\]
\[
\psi_{15} = P_{33} + \frac{6R}{h^2} \\
\psi_{22} = \tilde{h} A^T R \tilde{A}_d - \frac{4R}{h} + A^T Q_2 \tilde{A}_d - (1 - d) Q_2 + P_{12} \dot{A}_d + \dot{A}^T P_{12} - 2P_{23}
\]
\[
\psi_{23} = 0 \\
\psi_{24} = P_{22} \dot{C} + P_{22} + \tilde{h} A^T R \dot{C} + \dot{A}^T Q_2 \dot{C}
\]
\[
\psi_{25} = -P_{33} + \frac{6R}{h^2} \\
\psi_{33} = -(1 - d) Q_1 \\
\psi_{34} = 0 \\
\psi_{35} = 0
\]
\[\Psi_{44} = \bar{h}\hat{C}^T R \hat{C} + \hat{C}^T Q_2 \hat{C} - (1 - d)Q_2; \quad \Psi_{45} = \hat{C}^T P_{13} + P_{23}; \quad \Psi_{55} = -\frac{12R}{h^2}\]

**Proof:**

For the simplicity, we define:  
\[\varepsilon(t) = [\mu^T(t) \quad \mu^T(t - h(t)) \quad \int_{t-h(t)}^{t} \mu^T(s) ds]^T\]

so the LKF chosen for the new criteria is:  
\[V(\mu_t) = \sum_{i=0}^{4} V_i(\mu_t)\]

\[V_1(\mu_t) = \varepsilon^T(t) P \varepsilon(t) \quad ; \quad V_2(\mu_t) = \int_{-h(t)}^{t} \mu^T(s) R \hat{\mu}(s) ds d\theta\]

\[V_3(\mu_t) = \int_{-h(t)}^{t} \mu^T(s) Q_1 \mu(s) ds \quad ; \quad V_4(\mu_t) = \int_{-h(t)}^{t} \left[ \mu(s) \right]^T Q_2 \left[ \frac{\mu(s)}{\mu(s)} \right] ds\]

The calculation of  \(\dot{V}_4(\mu_t)\) along the solution of neutral system (5) by introducing (lemma 2.1) leads to:

\[\dot{V}_4(\mu_t) = \chi(t)^T \begin{bmatrix} p_{11}A + A^T P_{13} + 2P_{13} & p_{13}A_d + A^T dP_{13} + P_{13} - P_{13} & 0 & p_{13}C + p_{12} & 0 & p_{33} \\ * & p_{12}A_d + A^T dP_{13} + P_{13} - 2P_{13} & 0 & P_{23}C + P_{22} & 0 & -P_{33} \\ * & * & 0 & 0 & 0 & \hat{C}^T P_{13} + P_{23} \\ * & * & * & 0 & \end{bmatrix} \chi(t)\]

Calculating The derivative of  \(V_2(\mu_t)\) along the solution of neutral system (5) leads to:

\[\dot{V}_2 = h(t) [\hat{\mu}^T(t) R \hat{\mu}(t); \int_{t-h(t)}^{t} \mu^T(s) R \hat{\mu}(s) ds\]

Here the expression of  \(\hat{\mu}(t)\) will take place, The derivative will be divided into two parts to develop, where:

\[\beta = h(t) \left[ \hat{A}_d \mu(t) + \hat{A}_d \mu(t - h(t)) + \hat{C} \hat{\mu}(t - h(t)) \right] R \left[ \hat{A}_d \mu(t) + \hat{A}_d \mu(t - h(t)) + \hat{C} \hat{\mu}(t - h(t)) \right] \]

\[\theta = \int_{t-h(t)}^{t} \hat{\mu}^T(s) R \hat{\mu}(s) ds\]

We denote that:  \(h(t) \leq \bar{h}\)  
The first expression  \(\beta\) lead to (12):

\[\beta \leq \bar{h} \chi(t)^T [A^T A_d 0 \hat{C} 0]^T R [A^T A_d 0 \hat{C} 0] \chi(t)\]

By adopting (lemma3) we can infer that the second expression  \(\theta\) will be calculated as follow:

\[\theta \leq \frac{1}{h(t)} \left[ (\mu(t) - \hat{\mu}(t - h(t)))^T R (\mu(t) - \hat{\mu}(t - h(t))) \right] R \left[ \mu(t) + \mu(t - h(t)) \right] - \frac{2}{h(t)} \int_{t-h(t)}^{t} \mu(s) ds \left[ \mu(t) + \mu(t - h(t)) \right] - \frac{2}{h(t)} \int_{t-h(t)}^{t} \mu(s) ds\]

Furthermore, we arrive at
Neutral approach for linear singular system stability

\[ \dot{V}_2 \leq \chi(t)^T \begin{bmatrix} \dot{\bar{h}}^T R \dot{\bar{A}} - \frac{2h}{h} & \bar{h} \dot{\bar{A}} R \dot{\bar{A}}_d - \frac{2h}{h} & 0 & \bar{h} \dot{\bar{A}} R \dot{C} \left( \frac{6h}{h^2} - \frac{8}{h} \right) \\ \dot{\bar{h}} \dot{\bar{A}} R \dot{\bar{A}}_d - \frac{4h}{h} & 0 & 0 & 0 \\ \dot{\bar{h}} \dot{\bar{A}} R \dot{C} & 0 & 0 & 0 \\ 0 & \dot{\bar{h}} \dot{\bar{C}} R \dot{C} & 0 & -\frac{12h}{h^2} \end{bmatrix} \chi(t) \]  

(14)

Calculating the derivative of \( V_3(\mu_t) \) and \( V_4(\mu_t) \) along the solution of neutral system (9) we get respectively

\[ \dot{V}_3 = \chi(t)^T \begin{bmatrix} Q_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (1 - d)Q_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \chi(t) \]  

(15)

\[ \dot{V}_4 = \chi(t)^T \begin{bmatrix} Q_2 + \dot{A}^T Q_2 \dot{\bar{A}} & \dot{A}^T Q_2 \dot{\bar{A}}_d - (1 - d)Q_2 \\ \dot{A}^T Q_2 \dot{\bar{A}}_d & 0 \\ \dot{A}^T Q_2 \dot{C} & 0 \\ 0 & 0 \end{bmatrix} \chi(t) \]  

(16)

Differentiating \( V_1(\mu_t) \), \( V_2(\mu_t) \), \( V_3(\mu_t) \) and \( V_4(\mu_t) \) with respect to \( \mu_t \) we arrive at:

\[ V(\mu_t) \leq \begin{bmatrix} \mu(t) \\ \mu(t - h(t)) \\ \mu(t - h_1(t)) \\ \dot{\mu}(t - h(t)) \\ \dot{\mu}(t - h_1(t)) \end{bmatrix}^T \Psi \begin{bmatrix} \mu(t) \\ \mu(t - h(t)) \\ \mu(t - h_1(t)) \\ \mu(t - h(t)) \end{bmatrix} \]  

(17)

Therefore \( \Psi < 0 \) implies \( \dot{V}(\mu_t) < 0 \) which means that the neutral system (9) hence the singular one (1) are asymptotically stable.

Remark 3.1: The presented theorem verifies the admissibility conditions in terms of LMI for singular systems (1) with additive time varying delays, through a neutral system approach. By using this approach a transformation of singular system with time varying delay is developed. In this work and motivated by the derivative of LYAPUNOV equation (10), the results of this new criteria will be compared with some existing one as in [18] and [15], and this comparison shows clearly that the gotten corollary allows us to get better performance.

Remark 3.2: At this paper the Wirtinger based integral inequality was introduced (lemme2.3) instead of Jensen integral inequality due to the evolution of the conservatism time varying delay systems [6][17].

Remark 3.3: We denote that the eigenvalues of matrix \( \dot{C}(t) \) are inside the unit circle [8], in order to guarantee this condition \( |\dot{h}(t)| < 1 \).
4. Numerical example

In this section a numerical example will be presented to illustrate the advantages of the proposed method, based on an academic example[8], considering the following singular system with:

\[ A = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad A_d = \begin{bmatrix} -1 & 1 \\ 0 & 0.5 \end{bmatrix} \]

The time varying delays satisfying (2) and (3), evidently the pair (E,A) is regular and impulse free and there exist two invertible matrices.

\[ M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad MEN = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad MAN = \begin{bmatrix} -0.5 & 0 \\ 0 & 1 \end{bmatrix} \quad MA_dN = \begin{bmatrix} -1 & 1 \\ 0 & -0.5 \end{bmatrix} \]

According to the (lemma 2.4) :

\[ \hat{C}(t) = (1 - h(t)) \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \end{bmatrix} \quad \hat{A} = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix} \quad \hat{A}_d = \begin{bmatrix} -1 & 1 \\ 0 & 0.5 \end{bmatrix} \]

By using the LMI toolbox of MATLAB ,with given \( d_1 \) and \( d_2 \) respectively the first and second derivatives of the time varying delays system ,the time delay can be founded such that the system is admissible through the new Theorem 1 as it is shown on the (Table 1).

<table>
<thead>
<tr>
<th>Method</th>
<th>d=0.1</th>
<th>d=0.3</th>
<th>d=0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corollary [15]</td>
<td>1.902</td>
<td>1.725</td>
<td>1.583</td>
</tr>
<tr>
<td>Theorem 1[14]</td>
<td>1.905</td>
<td>1.798</td>
<td>1.751</td>
</tr>
<tr>
<td>Theorem 1</td>
<td>2.062</td>
<td>1.8925</td>
<td>1.7875</td>
</tr>
</tbody>
</table>

**Table 1.** Maximal upper bound of the delay dependent

It is clear that our theorem gives more satisfying results than those in [18] and [15], as shown in figure 2.

**Figure 2.** Histogram showing comparison of results.
5. Conclusion

The present study deals with the stability of singular systems with additive time varying delays via the neutral system approach, motivated by an augmented Lyapunov KRASOVSKI function and the WIRTINGER based integral inequality method. The new criterion in term of LMIs was developed in this work. This paper was supported by a simulation in Matlab environment to prove the less conservatism of the proposed method.

References


**Received: May 19, 2020; Published: June 5, 2020**