Financial Crises and the Vicious Circle

Between Good and Bad Payers

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Abstract

This paper proposes a dynamic model based on continuous time Markov chains that allows to study the evolution of the number of bad and good payers in an open economy, known their number at the beginning of the period. An equilibrium analysis is performed and its implication on possible economic policy measures that may help to overcome financial crises are discussed.

Keywords: Financial crises, Contagion, Dynamic models, Markov chains, Equilibrium analysis

1 Introduction

The banking and financial crises that have taken place in recent years have been the subject of numerous studies aimed at identifying the causes and the main channels of propagation. In Masson (1999) it is proposed a classification of the spread of financial crises. The factors considered responsible for the occurrence of financial crises simultaneously or in rapid succession may be attributed to:
- symmetric shocks affecting simultaneously different geographical areas;
- real and financial spillovers due to financial or trade links between the different areas;
- irrational factors such as the panic that may cause the transition from one equilibrium to another within multiple equilibria.

From a macroeconomic point of view, until the beginning of the century the problem of contagion has been addressed considering:
- negative externalities that occur simultaneously in multiple areas;
- commercial or financial ties between several areas that allow a crisis that occurs in a given geographical area to move to others;
- irrational factors.

However, globalization has opened other possibilities of contagion due to the fact that financial market agents can interact across different countries. As reported in Zharikov (2019), globalization can amplify the transmission mechanisms of financial crises due to the defaults of a significant number of agents connected in financial networks.

A financial crisis occurs in the presence of problems of illiquidity and/or insolvency of the agents operating in financial markets or when a bail out is performed in support of intermediaries in crisis. In both cases, to determine the financial crisis is the existence of agents who fail to meet their financial commitments.

In recent years the attention paid by the literature to the interactions among agents as a cause of possible financial crises has been increasing and the topic has been analyzed both in economic and forecasting perspective.

As far as the first perspective is concerned, Glasserman and Young (2016) analyze which factors in a network of financial agents contribute to trigger chain bankruptcies; Pollak and Guan (2017) show that trends in capital buffers and the distribution and type of assets have a significant effect on the predictions of financial network contagion models and that the rising trend in ownership of banks by banks amplifies shocks to the financial system.

About the second approach, Wang and Wu (2017) propose probabilistic models based on neural networks to predict financial crises; the role of institutions in preventing infections triggered by the interconnection between the various agents has been recently studied in Zharikov (2019) and in Kosmidou et al. (2019).

In fact the presence on the market of many agents who fail to meet their financial commitments (bad payers) can affect the survival of the remainder of the agents who are considered reliable (good payers) because of trade and/or financial links that can be established between the two groups.

In this paper we analyze from a mathematical point of view how the default of some agents can imply the failure of others using a typical stochastic model of contagion in the epidemiological field. Such model, called the SIR model (Susceptible, Infected, Recovered), in the version of Kendall studies the spread of infectious diseases by contact between two groups of individuals, respectively, infected individuals and healthy individuals. The model assumes that there is the possibility of removal from both groups for healing, for the acquisition of immunity, for isolation or for death (for further details see Bailey, 1975).

The model can be adapted to the financial environment by dividing the set of the agents on the market at a given time into two subsets: the bad payers belong to the first subset, while the good payers belong to the second subset. Hence the model here proposed studies a contagion spread dynamics mainly due to interactions between these two subsets of agents and assumes that at the initial time $t = 0$ the number of agents belonging to each of the two subsets is known. At time $t > 0$ the
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Cardinality of the two subsets is random for the possible displacements of agents from one group to another, for the emergence of new agents or market exit (bankruptcy) of some agents.

The paper is organized as follows: in Section 2 the model is introduced and analyzed, in Section 3 the system equilibrium is studied and the results for some special cases are shown, in Section 4 the conclusions are drawn.

2 The model

We consider an open market where the number of the agents (individuals, banks, corporations etc.) varies over time and where the agents set is divided into two subsets: bad payers (subset 1) and good payers (subset 2). If the market were closed and \( n_1 \) (\( n_2 \)) is the number of bad (good) payers then the total number of the agents would be \( N = n_1 + n_2 \) constant over time. In our case \( n_1 + n_2 \) varies over time because new agents may enter the market and/or current agents may exit the market for various reasons (e.g. death, default).

Unlike in the SIR model we assume that for a given time interval \([0,T]\) the number of bad payers is represented by the stochastic process \( N_{1,t} \) that can take only positive integer values (0, 1, 2, \ldots). In particular \( N_{1,t} \) is a continuous time Markov chain, hence the future number of bad payers \( N_{1,t+\Delta t} \) (\( \Delta t > 0 \)) may depend only on the current number of bad payers \( N_{1,t} \) and not on the past numbers \( N_{1,s} \) (\( s < t \)). The same applies to the stochastic process \( N_{2,t} \) representing the number of good payers at time \( t \).

If \( n = (n_1, n_2) \) and \( p(n, t) = \Pr[ N_{1,t} = n_1, N_{2,t} = n_2 ] \) with \( n_1^0, n_2^0 \) known, under appropriate regularity conditions \( p(n, t) \) satisfies the Master Equation (for further details see Gardiner, 1985):

\[
\frac{\partial p(n, t)}{\partial t} = \sum_{n' \neq n} q(n', n)p(n', t) - \sum_{n' \neq n} q(n, n')p(n, t)
\]

where \( q(n', n^*) = \lim_{\Delta t \to 0} \frac{\Pr[ N_{1,t+\Delta t}, N_{2,t+\Delta t} = n^* | N_{1,t}, N_{2,t} = n' ]}{\Delta t} \), \( n' \neq n^* \), is the instantaneous probability of transition from state \( n' \) in \( t \) to state \( n^* \) in \( t + \Delta t \) (\( \Delta t > 0 \)).

In theory, given \( q(n', n^*) \), Equation 1 can be solved in order to infer \( p(n, t) \) and thence any characteristic function of the stochastic processes \( N_{i,t} \) such as the first and second order moments \( \mu_{i,t} = E[N_{i,t}] \) and \( k_{i,t} = E[N_{i,t}N_{j,t}] \). In practice, very seldom Equation 1 can be actually solved, however it is always possible to
determine a differential equations system that \( \mu_{i,t} \) and \( k_{i,t} \) must satisfy. In order to determine such dynamic system the method of moment-generating functions can be applied using the following generating functions:

\[
W(\Phi,t) = E_t \left[ \exp(\Phi, N_t) \right] \quad \text{and} \quad \Gamma(\Phi,t) = \ln E_t \left[ \exp(\Phi, N_t) \right]
\]

where \( N_t = (N_{1,t}, N_{2,t}) \) and \( \Phi = (\Phi_1, \Phi_2) \).

Hence in particular the \( \Gamma \) function generates the first and second order moments of \( N_{i,t} \) (means \( \mu_i(t) \), variances \( \sigma_i(t) \) and covariance \( \sigma_{12}(t) \)).

If at time \( t \) the state is \( n = (n_1, n_2) \), in order to explain the agents dynamics we refer to Figure 1 where six possible events at time \( t + \Delta t \) are depicted:

- (A) \( (n_1, n_2) \rightarrow (n_1 + 1, n_2) = n_A \) a bad payer enters the market
- (B) \( (n_1, n_2) \rightarrow (n_1 - 1, n_2) = n_B \) a bad payer leaves the market
- (C) \( (n_1, n_2) \rightarrow (n_1, n_2 + 1) = n_C \) a good payer enters the market
- (D) \( (n_1, n_2) \rightarrow (n_1, n_2 - 1) = n_D \) a good payer leaves the market
- (E) \( (n_1, n_2) \rightarrow (n_1 - 1, n_2 + 1) = n_E \) a bad payer becomes a good payer
- (F) \( (n_1, n_2) \rightarrow (n_1 + 1, n_2 - 1) = n_F \) a good payer becomes a bad payer

![Figure 1. Agent dynamics.](image-url)
We assume that the transition rates from state \( n \) to state \( n' \) are given by:

(A) \( q(n, n_A) = an_1 + b \)

(B) \( q(n, n_B) = en_1 \)

(C) \( q(n, n_C) = cn_2 + d \)

(D) \( q(n, n_D) = fn_2 \)

(E) \( q(n, n_E) = gn_1 \left( n_2 + h \right) \)

(F) \( q(n, n_F) = mn_1 \left( n_2 + r \right) \)

For events (A) and (C) the transition rates have a component which increases linearly with the number of bad \((a)\) and good \((c)\) payers already in the market and a component which is independent from their number. In particular, considering the bad payers, \( b \) reflects the number of the reliable agents who enter the market during the time interval \( \Delta t \) while \( an_1 \) represents the number of agents who can be considered reliable thanks to their relations with reliable agents and not for their intrinsic characteristics. The same applies for the good payers.

For events (B) and (D) \( e \) and \( f \) represent respectively the bad and good payers default probabilities rates.

For event (E) \( gh \) represents the probability rate that a bad payer becomes a good payer, regardless the number of good payers. The term \( gn_2 \) represents the probability rate that a bad payer becomes good thanks to the number of good payers (contagion effect).

For event (F) \( mr \) represents the probability rate that a good payer becomes a bad payer, regardless the number of bad payers. The term \( mn_1 \) represents the probability rate that a good payer becomes bad thanks to the number of bad payers (contagion effect).

### 3 Equilibrium analysis

By substituting conditions (3) into Master Equation (1) we can determine the following equation that must be satisfied by \( \Gamma(\Phi, t) \) given in Equation (2):

\[
\frac{\partial}{\partial t} \Gamma(\Phi, t) = \frac{\partial}{\partial t} \ln E_t \left[ \exp(\Phi, N_t) \right] = \frac{\partial}{\partial t} E_t \left[ \exp(\Phi, N_t) \right] =
\]

\[
= \frac{\partial}{\partial t} \sum_n \exp(\Phi, N_t) p(n, t) = \sum_n \exp(\Phi, N_t) \frac{\partial}{\partial t} p(n, t) = \sum_n \exp(\Phi, N_t) p(n, t) = \left[ b \left( \exp(\Phi_1) - 1 \right) + d \left( \exp(\Phi_2) - 1 \right) \right]
\]
\begin{align*}
+ & \left[ a \left( \exp(\Phi_1) - 1 \right) + e \left( \exp(-\Phi_1) - 1 \right) + gh \left( \exp(-\Phi_1 + \Phi_2) - 1 \right) \right] \frac{\partial}{\partial \Phi_1} \Gamma + \\
+ & \left[ c \left( \exp(\Phi_2) - 1 \right) + f \left( \exp(-\Phi_2) - 1 \right) + mr \left( \exp(\Phi_1 - \Phi_2) - 1 \right) \right] \frac{\partial}{\partial \Phi_2} \Gamma + \\
+ & \left[ g \left( \exp(-\Phi_1 + \Phi_2) - 1 \right) + m \left( \exp(\Phi_1 - \Phi_2) - 1 \right) \right] \left[ \frac{\partial}{\partial \Phi_1} \Gamma - \frac{\partial}{\partial \Phi_2} \Gamma + \frac{\partial^2}{\partial \Phi_1 \partial \Phi_2} \Gamma \right].
\end{align*}

Neglecting derivatives of order higher than two, we can pose:
\[ \Gamma(\Phi, t) = \mu_1 \Phi_1 + \mu_2 \Phi_2 + \frac{1}{2} \nu_1 \Phi_1^2 + \frac{1}{2} \nu_2 \Phi_2^2 + \nu_1 \Phi_1 \Phi_2. \]

Hence
\[ \frac{\partial \Gamma}{\partial t} = \dot{\mu}_1 \Phi_1 + \dot{\mu}_2 \Phi_2 + \frac{1}{2} \dot{\nu}_1 \Phi_1^2 + \frac{1}{2} \dot{\nu}_2 \Phi_2^2 + \dot{\nu}_1 \Phi_1 \Phi_2 \]
By expanding Equation (4) in Taylor series around \( \Phi = 0 \) neglecting derivatives of \( \Gamma \) with respect to \( \Phi \) of order higher than two and comparing with Equation (5) we obtain:
\begin{align*}
\dot{\mu}_1 = b + (a - e - gh) \mu_1 + mr \mu_2 + (m - g) \mu_1 + (m - g) \nu_1 \\
\dot{\mu}_2 = d + (c - f - mr) \mu_2 + gh \mu_1 + (g - m) \mu_1 + (g - m) \nu_1 \\
\dot{\nu}_1 = b + (a + e + gh) \mu_1 + mr \mu_2 + (m + g) \mu_1 + 2 \left[ (a - e - gh) + (m - g) \mu_2 \right] \nu_1 + \\
+ \left[ g + m + 2mr + 2(m - g) \right] \\
\dot{\nu}_2 = d + (c + f + mr) \mu_2 + gh \mu_1 + (m + g) \mu_1 + 2 \left[ (c - f - mr) + (g - m) \mu_1 \right] \nu_2 + \\
+ \left[ g + m + 2gh + 2(g - m) \mu_2 \right] \\
\dot{\nu}_{12} = -gh \mu_1 - mr \mu_2 - (g + m) \mu_1 \mu_2 + \left[ gh + (g - m) \mu_2 \right] \nu_1 + \left[ mr + (m - g) \mu_1 \right] \nu_2 + \\
+ \left[ (a - e - gh) + (c - f - mr) - (g + m) + (g - m) \mu_1 + (m - g) \mu_2 \right] \nu_{12}
\end{align*}
Given the cumbersome nature of the above equations, we analyze a simplified case:
(A) \( q(n, n_A) = b \)
(C) \( q(n, n_C) = d \)
(D) \( q(n, n_D) = 0 \)
(E) \( q(n, n_E) = gn_1 n_2 \)
(F) \( q(n, n_F) = mn_1 n_2 \)
In other words we assume that the market entry transition rates (A) and (C) are constant, that default is possible only for bad payers (B), and that only the last two transition rates (E) and (F) describe the interaction level (i.e. the contagion) between bad and good payers.
Posing \( \alpha = g + m \) and \( \beta = m - g \) we obtain
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\[ \mu_1 = b - e \mu_1 + \beta \mu_1 \mu_2 + \beta \mu_1 \]
\[ \mu_2 = b - \beta \mu_1 \mu_2 - \beta \mu_1 \]
\[ \nu_1 = b + e \mu_1 + \alpha \mu_1 \mu_2 + 2(e + \beta \mu_2) \nu_1 + (\alpha + 2 \beta \mu_1) \nu_1 \]
\[ \nu_2 = b + \alpha \mu_1 \mu_2 + 2 \beta \mu_1 \nu_2 + (\alpha - 2 \beta \mu_2) \nu_1 \]
\[ \nu_{12} = -\alpha \mu_1 \mu_2 - \beta \mu_2 \nu_1 + \beta \mu_1 \nu_2 - (e + \alpha + \beta \mu_1 - \beta \mu_2) \nu_1 \]

By posing the derivatives equal to zero we find the following equilibrium values:

\[ \mu_1 = \frac{2b}{e} \]
\[ \mu_2 = \frac{e}{2b} \frac{e}{2b} \nu_{12} \]
\[ \nu_1 = \frac{3b^2 \beta e + \alpha b^2 e + 4b^2 \beta^2 \nu_{12}}{\beta e^2 (b + \beta \nu_{12})} \]
\[ \nu_2 = \frac{b^2 \beta e + \alpha b^2 e - \beta e^2 (b + \beta \nu_{12}) \nu_{12}}{4b^2 \beta^2} \]
\[ \nu_{12} = \frac{-b \left(4b \beta^2 - 2e \beta^2 + \beta e^2\right) \pm \sqrt{b^2 \left(4b \beta^2 - 2e \beta^2 + \beta e^2\right)^2 - 4b^2 \beta^2 (\alpha + \beta) b^2}}{2b^2 e^2} \]

Since the latter equation is rather cumbersome, a fruitful approach is to analyze the problem when the parameters take some critical values.

Special case 1: very low bad payers default probabilities \((e \to 0)\).

\[ \nu_{12} = \frac{-b \left(4b \beta^2 - 2e \beta^2 + \beta e^2\right) \pm \sqrt{b^2 \left(4b \beta^2 - 2e \beta^2 + \beta e^2\right)^2 - 4b^2 \beta^2 (\alpha + \beta) e^3}}{2b^2 e^2} \]
\[ \nu_{12}^+ = O(e) \]
\[ \nu_{12}^- = \frac{4b^2}{e^2} + \frac{2b e}{\beta} + O(e) \]

If \( \nu_{12} = \nu_{12}^+ \) the first and second order moments are:

\[ \mu_1 = \frac{2b}{e} \]
\[ \mu_2 = \frac{e}{2b} \]
\[ \nu_{12}^+ = \frac{\alpha + \beta}{4b^2} e \]
\[ \nu_1 = \frac{2b}{e} \]
\[ \nu_2 = \frac{\alpha + \beta}{4b^2} e \]

The number of bad payers tends to infinity as its variance \((\nu_1 = \mu_1)\) like a Poisson
distribution), the number of good payers tends to zero as well as its variance and the correlation between the number of bad and good payers tends to 

\[
\rho = \frac{\mu_1}{\sqrt{\mu_1^2}} = -\sqrt{\alpha + \beta} / 2\beta \sqrt{2b}.
\]

**Remark.** Hence if \( e \to 0 \) and \( \mu_1 = \rho \mu_2 \), the market tends to be populated by bad payers only, with a variability which tends to infinity.

If \( \mu_1 = \mu_2 \) the first and second order moments are:

\[
\begin{align*}
\mu_1 &= \frac{2b}{e}, \\
\mu_2 &= \frac{2b}{e}, \\
\nu_1 &= \frac{4b^2}{e^2}, \\
\nu_2 &= \frac{4b^2}{e^2}.
\end{align*}
\]

On average the number of bad and good payers is equal ( \( \mu_1 = \mu_2 \) ), their variance tends to infinity and the correlation between the number of bad and good payers is perfectly negative ( \( \rho = \mu_1 / \sqrt{\mu_1 \mu_2} = -1 \)).

**Remark.** Hence if \( e \to 0 \) and \( \nu_1 = \nu_2 \), the market tends to be populated by the same number of bad and good payers with a variability which tends to infinity.

**Special case 2:** very high bad payers default probabilities \( (e \to \infty) \).

\[
\nu_{12} = \frac{-b(4b\beta^2 - 2e\beta^2 + \beta e^2) \pm \sqrt{1 - \frac{4b^2}{\beta^2} \frac{(\alpha + \beta)e^3}{2b^2(4b\beta^2 - 2e\beta^2 + \beta e^2)}}}{2\beta^2e^2}
\]

\[
\nu_{12}^+ = -\frac{b(\alpha + \beta)}{\beta e}, \\
\nu_{12}^- = -\frac{b}{\beta} + \frac{b(\alpha + 3\beta)}{\beta e}
\]

If \( \nu_1 = \nu_{12}^+ \) the first and second order moments are:

\[
\begin{align*}
\mu_1 &= \frac{2b}{e}, \\
\mu_2 &= \frac{e}{2b}, \\
\nu_1 &= \frac{b(\alpha + 3\beta)}{\beta e}, \\
\nu_2 &= \frac{(\alpha + \beta)}{2\beta^2} e
\end{align*}
\]

\( \mu_1, \nu_1, \nu_{12}, \to 0 \) and \( \mu_2, \nu_2 \to \infty \).

If \( \nu_{12} = \nu_{12}^- \) the first and second order moments are:
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\[ \mu_1 = \frac{2b}{e}, \quad \mu_2 = \frac{e}{\beta}, \quad \nu_{12} = -\frac{b}{\beta} \]
\[ \nu_1 = \frac{b}{\beta}, \quad \nu_2 = \frac{e^2}{2\beta^2} \]
\[ \mu_1 \to 0 \quad \nu_1 \to \text{cost} > 0 \quad \nu_{12} \to \text{cost} < 0 \quad \mu_2, \nu_2 \to \infty. \]

4 Conclusions

The model introduced in this paper is interesting from a theoretical point of view, but like all stochastic models for the study of contagion, it presents considerable difficulties in identifying both the explicit equations that describe the phenomenon and the conditions of equilibrium. However, we show how under some assumptions for transition rates it is possible to arrive at interesting considerations for equilibrium solutions.

In particular, from the cases examined it can be deduced that in the event that bad payers have a low probability of bankruptcy, on the market in equilibrium conditions, their average number tends to become very high and the same happens due to its variability which implies that bad payers infect good payers who will have an average number that tends to zero.

It has also been shown that when the probability of default of bad payers is high on equilibrium, the average number of bad payers equals the average number of good payers.

The results found are in agreement with those obtained in the case of epidemiological models in which the default rate can be considered a proxy of the mortality rate. In the first case when the mortality rate of infected people is low the epidemic spreads rapidly among healthy individuals, while when the mortality rate of infected individuals is much higher, the spread of the epidemic is much slower and the number of healthy individuals who become infected is much lower than in the previous case.

Possible developments of the model could concern the possibility of distinguishing agents in homogeneous groups (for instance according to liquidity availability or recovery rate in the event of default), in order to estimate the parameters that regulate the transition rates and understand which driver has the greatest impact on the creation of a financial crisis.

References


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