Applied Mathematical Sciences, Vol. 14, 2020, no. 13, 643 - 651
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ams.2020.914235

Limit Cycles of a Class of Non-Hamiltonian Systems

Weijiao Xu

School of Science
Tianjin University of Commerce
Tianjin 300134, P.R. China

Abstract

In this paper, we study limit cycle bifurcations of a class of non-Hamiltonian systems. The unperturbed system has a 2-polycycle and two homoclinic loops outside and a double homoclinic loop inside. With different perturbation, by the Melnikov functions and bifurcation theories, we found that $H(10, 7) \geq 15$.

Mathematics Subject Classification: 34C05, 37G15

Keywords: Limit cycle, Liénard system, Melnikov function, Bifurcation

1 Introduction

Consider the following system

$$\dot{x} = H_y(x, y) + \varepsilon p(x, y, \varepsilon, \delta),\ \ \dot{y} = -H_x(x, y) + \varepsilon q(x, y, \varepsilon, \delta), \tag{1}$$

where $H(x, y)$, $p(x, y, \varepsilon, \delta)$ and $q(x, y, \varepsilon, \delta)$ are analytic functions, $\varepsilon \geq 0$ is small enough and $\delta \in D \subset R^m$ is a vector parameter with D compact. When $\varepsilon = 0$, (1) becomes a Hamiltonian system

$$\dot{x} = H_y,\ \ \dot{y} = -H_x. \tag{2}$$

So, we will call (1) a near-Hamiltonian system.

Suppose the unperturbed system (2) has four families of periodic orbits $L_l(h), L(h), \overline{L}_l(h), \overline{L}(h)$ given by $H(x, y) = h$.(See Figure 1.)

If $\{L_l(h)\}, l=1,2$ is bounded, then the boundary can be a center $C_l(x_{C_l}, y_{C_l})$ or a homoclinic loop L_1 with a hyperbolic saddle $O(0,0)$, where $H(C_l)=\eta$ and $H(L_1)=0$. If $\{L(h)\}$ is bounded, then the boundary can be a double homoclinic L with a hyperbolic saddle $O(0,0)$ or 2-polycycle $\Gamma^2=(\widehat{L}_1\bigcup S_1)\bigcup(\widehat{L}_2\bigcup S_2)$, where $H(\Gamma^2)=\mu$ and $H(L)=\eta$. If $\{\overline{L}_l(h)\}, l=1,2$ is bounded, then the boundary can be a center $D_l(x_{D_l}, y_{D_l})$ or a homoclinic loop $\overline{L}_l$ with a hyperbolic saddle S_l, where $H(D_l)=\beta$ and $H(\overline{L}_l)=\mu$. If $\{\overline{L}_l(h)\}$ is bounded, then the boundary can be a compound loop $\Gamma^*=\Gamma^2\bigcup\overline{L}_1\bigcup\overline{L}_2$, where $H(\Gamma^*)=\mu$.

As we know, one can study the number of limit cycles if system (1) in a neighborhood of the boundary by using the asymptotic expansion of the first order Melnikov function or Abelian integral

$$\begin{aligned}
M_l(h,\delta) &= \oint_{L_l(h)}(qdx-pdy)|_{\varepsilon=0},\ h\in(\eta,0),\ l=1,2,\\
M(h,\delta) &= \oint_{L(h)}(qdx-pdy)|_{\varepsilon=0},\ h\in(0,\mu),\\
\overline{M}_l(h,\delta) &= \oint_{\overline{L}_l(h)}(qdx-pdy)|_{\varepsilon=0},\ h\in(\beta,\mu),\ l=1,2,\\
\overline{M}(h,\delta) &= \oint_{\Gamma^*}(qdx-pdy)|_{\varepsilon=0},\qquad h\in(\mu,+\infty).
\end{aligned} \tag{3}$$

By using simple zeros of $M_l(h,\delta), M(h,\delta), \overline{M}_l(h,\delta), \overline{M}(h,\delta)$ respectively, we can obtain the limit cycles of system (1). This kind of problem is the famous Hilbert's 16th problem [1] and there are many results about it, especially for $H(x,y)$ are polynomials of degree 1 to 6 in (1) (See [2-7]). In this paper, we let $H(x,y)$ is polynomial of degree 7, $p(x,y,\varepsilon,\delta)=0$ and $q(x,y,\varepsilon,\delta)$ is polynomial of degree 11, then (1) has the following form

$$\dot{x}=y,\ \dot{y}=-x(x^2-1)(x^2-\frac{1}{4})(x^2-2)+\varepsilon\sum_{i=0}^{5}a_ix^{2i}y. \tag{4}$$

According the system (4), we get the following result

Theorem 1 *Let $a_5\neq 0$, the system (4) has at least 15 limit cycles.*

2 Proof of Theorem 1

In this section, we use the expansion of the first order Melnikov functions to obtain the number of limit cycles of system (4).

By Theorem 2.2 in [8], for $0<-h\ll 1$, we have

$$M_1(h,\delta)=c_{01}+c_1h\ln|h|+c_{21}h+c_3(O,\delta)h^2\ln|h|+O(h^2),$$

where $\lambda=\frac{\sqrt{2}}{2}$ denotes an eigenvalue of O. By (4), the homoclinic orbit L_1 has the expression:

$$L_1:\ y_\pm=\pm\frac{1}{12}x\sqrt{72-198x^2+156x^4-36x^6},\ 0\le x\le x_0,\ x_0=\frac{1}{2}\sqrt{6-2\sqrt{3}}. \tag{5}$$

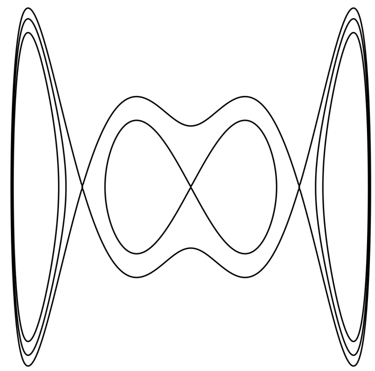

Figure 1: The portrait of $(1)|_{\varepsilon=0}$.

Then by Theorem 2.2 in [8] and (4) we have

$$c_{01} = \oint_{L_1} f(x,\delta)ydx = 2\int_0^{x_0}(a_0+a_1x^2+a_2x^4+a_3x^6+a_4x^8+a_5x^{10})y_+dx = 2\sum_{j=0}^{5} a_jI_j, \tag{6}$$

with

$$\begin{aligned} I_0 &= \int_0^{x_0} y_+dx = 0.1269637110, & I_1 &= \int_0^{x_0} x^2y_+dx = 0.02973711178, \\ I_2 &= \int_0^{x_0} x^4y_+dx = 0.01027524854, & I_3 &= \int_0^{x_0} x^6y_+dx = 0.004208444189, \\ I_4 &= \int_0^{x_0} x^8y_+dx = 0.001897364641, & I_5 &= \int_0^{x_0} x^{10}y_+dx = 0.0009098637648, \end{aligned}$$

$$c_1 = -\sqrt{2}a_0, \tag{7}$$

$$\begin{aligned} c_{21} &= \oint_{L_1} f(x,\delta)dt|_{c_1=0} = 2\int_0^{x_0}(a_1x^2+a_2x^4+a_3x^6+a_4x^8+a_5x^{10})dt \\ &= 2\int_0^{x_0}(a_1x^2+a_2x^4+a_3x^6+a_4x^8+a_5x^{10})\frac{1}{y_+}dx = 2\sum_{j=1}^{5} a_jJ_j, \end{aligned} \tag{8}$$

with

$$\begin{aligned} J_1 &= \int_0^{x_0} \frac{x^2}{y_+}dx = 1.225023372, & J_2 &= \int_0^{x_0} \frac{x^4}{y_+}dx = 0.5503773932, \\ J_3 &= \int_0^{x_0} \frac{x^6}{y_+}dx = 0.2869119956, & J_4 &= \int_0^{x_0} \frac{x^8}{y_+}dx = 0.1584015511, \\ J_5 &= \int_0^{x_0} \frac{x^{10}}{y_+}dx = 0.09019708532. \end{aligned}$$

In order to find $c_3(\delta)$ we make a change of variables of the form $x = u$, $y = \frac{\sqrt{2}}{2}v$. Then the system (4) becomes

$$\dot{u} = \frac{\sqrt{2}}{2}v,\ \dot{v} = \frac{\sqrt{2}}{2}u - \frac{11\sqrt{2}}{4}u^3 + \frac{13\sqrt{2}}{4}u^5 - \sqrt{2}u^7 + \varepsilon f(u,\delta)v. \tag{9}$$

For $\varepsilon = 0$, the Hamiltonian function of the above system is

$$\widetilde{H}(u,v) = \frac{\sqrt{2}}{4}(v^2 - u^2) + \frac{11\sqrt{2}}{16}u^4 - \frac{13\sqrt{2}}{24}u^6 + \frac{\sqrt{2}}{8}u^8 = \sqrt{2}H(x,y). \quad (10)$$

Let

$$\widetilde{M}_1(h,\delta) = \oint_{\widetilde{H}(u,v)=h} f(u,\delta)vdu \quad (11)$$

denote the Melnikov function of the new system (9). Then by Theorem 2.2 in [8], we have

$$\widetilde{M}_1 = \widetilde{c}_{01} + \widetilde{c}_1 h \ln|h| + \widetilde{c}_{21}h + \widetilde{c}_3 h^2 \ln|h| + O(|h|^2),$$

where

$$\widetilde{c}_3(O,\delta) = a_1. \quad (12)$$

Note that

$$M_1(h,\delta) = \frac{1}{\sqrt{2}}\widetilde{M}_1(\sqrt{2}h,\delta),$$

we have

$$c_3(O,\delta) = \sqrt{2}\widetilde{c}_3(O,\delta) = \sqrt{2}a_1. \quad (13)$$

Next, we discuss the property of $M_1(h,\delta)$ for $0 < h + \frac{169}{6144} \ll 1$. We move the center $C_1(\frac{1}{2},0)$ into the origin by letting $x = x_1 + \frac{1}{2}$, $y = \frac{\sqrt{42}}{8}y_1$, and $t = \frac{8}{\sqrt{42}}\tau$, system (4) becomes

$$\begin{aligned}\frac{dx_1}{d\tau} &= y_1,\\ \frac{dy_1}{d\tau} &= -x_1 - \tfrac{23}{21}x_1^2 + \tfrac{34}{7}x_1^3 + \tfrac{40}{7}x_1^4 - \tfrac{64}{21}x_1^5 - \tfrac{16}{3}x_1^6 - \tfrac{32}{21}x_1^7 + \varepsilon\tfrac{8}{\sqrt{42}}f(x_1+\tfrac{1}{2},\delta)y_1.\end{aligned} \quad (14)$$

For $\varepsilon = 0$, the Hamiltonian function of (14) is

$$\begin{aligned}\widetilde{H}(x_1,y_1) &= -\tfrac{169}{4032} + \tfrac{1}{2}(x_1^2+y_1^2) + \tfrac{23}{63}x_1^3 - \tfrac{17}{14}x_1^4 - \tfrac{8}{7}x_1^5 + \tfrac{32}{63}x_1^6 + \tfrac{16}{21}x_1^7 + \tfrac{4}{21}x_1^8\\ &= \tfrac{32}{21}H(x,y).\end{aligned}$$

From Theorem 2 in [9], we have

$$\begin{aligned}\widetilde{b}_{01} &= \pi c_{00} = \tfrac{2\sqrt{42}\pi}{21}(4a_0 + a_1 + \tfrac{1}{4}a_2 + \tfrac{1}{16}a_3 + \tfrac{1}{64}a_4 + \tfrac{1}{256}a_5),\\ \widetilde{b}_{11} &= \pi(\tfrac{15}{2}h_{30}^2c_{00} + c_{20} - 3h_{30}c_{10}) = \sqrt{42}\pi\sum_{i=0}^{5}a_iN_i,\\ \widetilde{b}_{21} &= \pi[(\tfrac{1155}{8}h_{30}^4 + \tfrac{35}{2}h_{40}^2 - \tfrac{315}{2}h_{30}^2h_{40})c_{00} + (35h_{30}h_{40} - \tfrac{105}{2}h_{30}^3)c_{20} + (\tfrac{35}{2}h_{30}^2 - 5h_{40})c_{10}]\\ &= \sqrt{42}\pi\sum_{i=6}^{11}a_{i-6}N_i,\end{aligned}$$

where

$$\begin{array}{llll} N_0 = \frac{5290}{27783}, & N_1 = \frac{1637}{55566}, & N_2 = \frac{42965}{222264}, & N_3 = \frac{126629}{889056}, \\ N_4 = \frac{252629}{3556224}, & N_5 = \frac{420965}{14224896}, & N_6 = \frac{161606360}{15752961}, & N_7 = \frac{11396390}{15752961}, \\ N_8 = \frac{41628355}{31505922}, & N_9 = \frac{127991275}{126023688}, & N_{10} = \frac{264786955}{504094752}, & N_{11} = \frac{452015395}{2016379008}. \end{array}$$

Let

$$\widetilde{M}_1(h,\delta) = \oint_{\widetilde{H}(x_1,y_1)=h} \frac{8}{\sqrt{42}} f(x_1+\frac{1}{2},\delta) y_1 dx_1$$

denote the Melnikov function of the new system (14). Then by Theorem 2 in [9], we have

$$\widetilde{M}_1 = \widetilde{b}_{01}(h+\frac{169}{4032}) + \widetilde{b}_{11}(h+\frac{169}{4032})^2 + \widetilde{b}_{21}(h+\frac{169}{4032})^3 + O((h+\frac{169}{4032})^4).$$

Note that

$$M_1(h,\delta) = \frac{21}{32}\widetilde{M}_1(\frac{32}{21}h,\delta),$$

we have

$$\begin{aligned} b_{01}(\delta) &= \widetilde{b}_{01}(\delta) = \frac{2\sqrt{42}\pi}{21}(4a_0 + a_1 + \frac{1}{4}a_2 + \frac{1}{16}a_3 + \frac{1}{64}a_4 + \frac{1}{256}a_5), \\ b_{11}(\delta) &= \frac{32}{21}\widetilde{b}_{11}(\delta) = \frac{32}{21}\sum_{i=0}^{5} a_i N_i, \\ b_{21}(\delta) &= (\frac{32}{21})^2\widetilde{b}_{21}(\delta) = (\frac{32}{21})^2 \sum_{i=6}^{11} a_i N_i. \end{aligned} \tag{15}$$

Then by Theorem 2.2 in [8], for $0 < \frac{1}{48} - h \ll 1$, we have

$$M(h,\delta) = \widehat{c}_0 + \widehat{c}_1(h-\frac{1}{48})\ln|h-\frac{1}{48}| + \widehat{c}_2(h-\frac{1}{48}) + \widehat{c}_3(h-\frac{1}{48})^2\ln|h-\frac{1}{48}| + O((h-\frac{1}{48})^2),$$

where $\widehat{\lambda} = \frac{\sqrt{6}}{2}$ denotes an eigenvalue of S_1. By (4), the homoclinic orbits $\widehat{L}_1$ and $\widehat{L}_2$ have the expressions:

$$\begin{aligned} \widehat{L}_1 : \ y_+ &= -\tfrac{1}{12}(x^2-1)\sqrt{-36x^4+84x^2+6}, \ -1 \le x \le 1, \\ \widehat{L}_2 : \ y_- &= \tfrac{1}{12}(x^2-1)\sqrt{-36x^4+84x^2+6}, \ -1 \le x \le 1. \end{aligned}$$

By (4), we have

$$\widehat{c}_0 = \oint_{\widehat{L}_1+\widehat{L}_2} f(x,\delta) y dx = 2\int_{-1}^{1} (a_0 + a_1x^2 + a_2x^4 + a_3x^6 + a_4x^8 + a_5x^{10}) y_+ dx = 2\sum_{j=0}^{5} a_j K_j, \tag{16}$$

with

$$\begin{aligned} K_0 &= \int_{-1}^{1} y_+ dx = 0.4675070270, & K_1 &= \int_{-1}^{1} x^2 y_+ dx = 0.1259831529, \\ K_2 &= \int_{-1}^{1} x^4 y_+ dx = 0.0596594218, & K_3 &= \int_{-1}^{1} x^6 y_+ dx = 0.034771258, \\ K_4 &= \int_{-1}^{1} x^8 y_+ dx = 0.022742298, & K_5 &= \int_{-1}^{1} x^{10} y_+ dx = 0.016021790. \end{aligned}$$

Then by Theorem 2.2 in [8], for $0 < \dfrac{1}{48} - h \ll 1$, we have

$$\overline{M}_1(h,\delta) = \bar{c}_{01} + \bar{c}_1(h-\frac{1}{48})\ln|h-\frac{1}{48}| + \bar{c}_{21}(h-\frac{1}{48}) + \bar{c}_3(h-\frac{1}{48})^2\ln|h-\frac{1}{48}| + O((h-\frac{1}{48})^2),$$

where $\bar{\lambda} = \frac{\sqrt{6}}{2}$ denotes an eigenvalue of S_1. By (4), the homoclinic orbits $\overline{L}_1$ has the expression:

$$\overline{L}_1: \ y_\pm = \pm\frac{1}{12}(x^2-1)\sqrt{-36x^4+84x^2+6}, \ 1 \le x \le \bar{x}_0, \ \bar{x}_0 = \frac{1}{6}\sqrt{42+6\sqrt{55}}.$$

By (4), we have

$$\bar{c}_{01} = \oint_{\overline{L}_1} f(x,\delta) y dx = 2\int_1^{\bar{x}_0} (a_0 + a_1x^2 + a_2x^4 + a_3x^6 + a_4x^8 + a_5x^{10}) y_+ dx = 2\sum_{i=0}^{5} a_i \overline{I}_i, \tag{17}$$

with

$$\begin{aligned} \overline{I}_0 &= \int_1^{\bar{x}_0} y_+ dx = 0.1584238578, & \overline{I}_1 &= \int_1^{\bar{x}_0} x^2 y_+ dx = 0.2852110860, \\ \overline{I}_2 &= \int_1^{\bar{x}_0} x^4 y_+ dx = 0.5304582796, & \overline{I}_3 &= \int_1^{\bar{x}_0} x^6 y_+ dx = 1.014945240, \\ \overline{I}_4 &= \int_1^{\bar{x}_0} x^8 y_+ dx = 1.989723792, & \overline{I}_5 &= \int_1^{\bar{x}_0} x^{10} y_+ dx = 3.982263512. \end{aligned}$$

Next, we discuss the property of $\overline{M}_1(h,\delta)$ for $0 < h + \dfrac{1}{12} \ll 1$. The center $D_1(\sqrt{2},0)$ into the origin by letting $x = \bar{x}_1 + \sqrt{2}$, $y = \frac{1}{\sqrt{7}}\bar{y}_1$, and $t = \frac{1}{\sqrt{7}}\bar{\tau}$, system (4) becomes

$$\begin{aligned} \frac{d\bar{x}_1}{d\bar{\tau}} &= \bar{y}_1, \\ \frac{d\bar{y}_1}{d\bar{\tau}} &= -\bar{x}_1 - \tfrac{109}{28}\sqrt{2}\bar{x}_1^2 - \tfrac{311}{28}\bar{x}_1^3 - \tfrac{215}{28}\sqrt{2}\bar{x}_1^4 - \tfrac{155}{28}\bar{x}_1^5 - \sqrt{2}\bar{x}_1^6 - \tfrac{1}{7}x_1^7 + \varepsilon\tfrac{1}{\sqrt{7}}f(\bar{x}_1+\sqrt{2},\delta)\bar{y}_1. \end{aligned} \tag{18}$$

For $\varepsilon = 0$, the Hamiltonian function of (18) is

$$\overline{H}(\bar{x}_1,\bar{y}_1) = -\frac{1}{12} + \frac{1}{2}(\bar{x}_1^2+\bar{y}_1^2) + \frac{109}{84}\sqrt{2}\bar{x}_1^3 + \frac{311}{112}\bar{x}_1^4 + \frac{43}{28}\sqrt{2}\bar{x}_1^5 + \frac{155}{168}\bar{x}_1^6 + \frac{1}{7}\sqrt{2}\bar{x}_1^7 + \frac{1}{56}\bar{x}_1^8 = H(x,y).$$

Let

$$\widetilde{\widetilde{M}}_1(h,\delta) = \oint_{\widetilde{\widetilde{H}}(x_1,y_1)=h} \frac{1}{\sqrt{7}} f(\overline{x}_1+\sqrt{2},\delta)\overline{y}_1 d\overline{x}_1$$

denote the Melnikov function of the new system (18). Then by Theorem 2 in [9], we have

$$\widetilde{\widetilde{M}}_1 = \widetilde{\widetilde{b}}_{01}(h+\frac{1}{12}) + \widetilde{\widetilde{b}}_{11}(h+\frac{1}{12})^2 + \widetilde{\widetilde{b}}_{21}(h+\frac{1}{12})^3 + O((h+\frac{1}{12})^4).$$

From Theorem 2 in [9], we have

$$\widetilde{\widetilde{b}}_{01} = 2\pi c_{00} = \frac{2\pi}{\sqrt{7}}(a_0+2a_1+4a_2+8a_3+16a_4+32a_5).$$

Note that

$$\overline{M}_1(h,\delta) = \widetilde{\widetilde{M}}_1(h,\delta),$$

we have

$$\overline{b}_{01}(\delta) = \widetilde{\widetilde{b}}_{01}(\delta) = 2\pi c_{00} = \frac{2\pi}{\sqrt{7}}(a_0+2a_1+4a_2+8a_3+16a_4+32a_5). \tag{19}$$

By (6)-(8) and (15), letting $c_{01}=c_1=c_{21}=b_{01}=b_{11}=0$, we can obtain

$$\begin{aligned} a_0 = 0, \quad & a_1 = 0.1241361361a_5,\ a_2 = -0.8868720077a_5, \\ & a_3 = 2.030053153a_5,\ \ a_4 = -2.124973201a_5. \end{aligned} \tag{20}$$

Then substituting (20) into (13), (15)-(17) and (19), we have

$$\begin{array}{rcll} c_3 &=& 0.1755550072a_5, & b_{21} = 4.147191809a_5, \\ \widehat{c}_0 &=& 0.0010101130878a_5, & \overline{b}_{01} = 25.98443034a_5, \\ \overline{c}_0 &=& 1.379502965a_5. & \end{array} \tag{21}$$

By (6)-(8) and (15), it is obvious that

$$\begin{aligned} \det\frac{\partial(c_{01},c_1,c_{21},b_{01},b_{11})}{\partial(a_0,a_1,a_2,a_3,a_4)} &= \begin{vmatrix} 2I_0 & 2I_1 & 2I_2 & 2I_3 & 2I_4 \\ -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 2J_1 & 2J_2 & 2J_3 & 2J_4 \\ \frac{8\sqrt{42}\pi}{21} & \frac{2\sqrt{42}\pi}{21} & \frac{\sqrt{42}\pi}{42} & \frac{\sqrt{42}\pi}{168} & \frac{\sqrt{42}\pi}{672} \\ \frac{32}{21}N_0 & \frac{32}{21}N_1 & \frac{32}{21}N_2 & \frac{32}{21}N_3 & \frac{32}{21}N_4 \end{vmatrix} \\ &\approx -0.00001134791019 \neq 0. \end{aligned}$$

Then by (20), we have $a_0 \equiv a_0^\star$, $a_1 \equiv a_1^\star$, $a_2 \equiv a_2^\star$, $a_3 \equiv a_3^\star$ and $a_4 \equiv a_4^\star$. Let us take $\delta_0 = (a_0^\star, a_1^\star, a_2^\star, a_3^\star, a_4^\star)$. Thus, with $a_5 \neq 0$, we have $c_3(\delta_0)b_{21}(\delta_0) > 0$, which indicates that there exists a root $h_1^\star \in (-\frac{169}{6144}, 0)$ by the mean value theorem. Similarly, we have $c_3(\delta_0)\widehat{c}_0(\delta_0) > 0$, which indicates that there exists a root $h^\star \in (0, \frac{1}{48})$ such that $M(h^\star, \delta_0) = 0$ under (20). Moreover, we also have $\overline{c}_{01}(\delta_0)\overline{b}_{01}(\delta_0) > 0$, which indicates that there exist no roots $\overline{h}^\star \in (-\frac{1}{12}, \frac{1}{48})$ such that $M(h^\star, \delta_0) = 0$ under (20). So we altogether get 14 limit cycles.

Then we can get one more limit cycle for $h > \frac{1}{48}$. Let $G(x) = \int_0^x g(x)dx,\ F(x, \delta_0) = \int_0^x f(x, \delta_0)dx$ hold and $x^\star > \frac{\sqrt{10}}{2}$ satisfy $G(x^\star(h)) = h$ for $h > \frac{1}{48}$.

Let $\overline{M}(h, \delta) = \oint_{\Gamma^\star} f(x, \delta)ydx$. By the symmetry and using (4) we have

$$\overline{M}(h, \delta) = \oint_{\Gamma^\star} f(x, \delta)ydx = -\oint_{\Gamma^\star} F(x, \delta)dy = -\oint_{\Gamma^\star} \frac{F(x, \delta)}{y}dG(x), \qquad (22)$$

then

$$\overline{M}(h, \delta_0) = -4\int_0^{x^\star(h)} \frac{F(x, \delta_0)}{\sqrt{2(h + G(x))}}dG(x),\ h > \frac{1}{48}.$$

Suppose $a_5 > 0$, let $x_0 > \frac{\sqrt{10}}{2}$ be such that $F(x, \delta) > a_5$ for $x > x_0$. Then we can write

$$\overline{M}(h, \delta_0) = m_1(h) + m_2(h),$$

where

$$m_1(h) = -4\int_0^{x_0} \frac{F(x, \delta_0)}{\sqrt{2(h + G(x))}}dG(x), m_2(h) = -4\int_{x_0}^{x_h^\star} \frac{F(x, \delta_0)}{\sqrt{2(h + G(x))}}dG(x),$$

From the mean value theorem for integral, it is obvious that $m_1(h) \to 0$ as $h \to \infty$. And we have $m_2(h) < -4\int_{G(x_0)}^{h} \frac{a_5}{\sqrt{2(h + z)}}dz \to -\infty$ as $h \to \infty$. It follows that $\overline{M}(h, \delta_0) \to -\infty$ as $h \to \infty$ for $a_5 > 0$. Moreover by Theorem 2.2 in [8] and under (21), we have $\overline{M}(h, \delta_0) = \widehat{c}_0(\delta_0) + 2\overline{c}_{01}(\delta_0) = 2.760017239a_5 > 0$ for $0 < h - \frac{1}{48} \ll 1$.

Then the claim follows. The system (4) can have 15 limit cycles for $(\varepsilon, a_0, a_1, a_2, a_3, a_4, a_5)$ near $(0, 0, 0.1241361361a_5, -0.8868720077a_5, 2.030053153a_5, -2.124973201a_5,\ a_5)$.

Acknowledgements. I am very grateful to the anonymous referees for their valuable comments and suggestions, which led to an improvement of my original manuscript.

References

[1] D. Hilbert, Mathematical problems (M. Newton, Transl.), *Bull. Amer. Math. Soc.,* **8** (1902), 437-479. https://doi.org/10.1090/s0002-9904-1902-00923-3

[2] T.R. Blows, N.G. Lloyd, The number of small-amplitude limit cycles of Liénard equations, *Math. Proc. Camb. Philos. Soc.,* **95** (2) (1984), 359-366. https://doi.org/10.1017/s0305004100061636

[3] M. Han, Liapunov constants and Hopf cyclicity of Liénard systems, *Ann. Differential Equations,* **15** (1999), 113-126.

[4] C.J. Christopher and S. Lynch, Small-amplitude limit cycle bifurcations for Lienard systems with quadratic or cubic damping or restoring forces, *Nonlinearity,* **12** (1999), 1099-1112. https://doi.org/10.1088/0951-7715/12/4/321

[5] M. Han, H. Yan, J. Yang and C. Lhotka, On the number of limit cycles of some liénard systems, *Can. Appl. Math. Q.,* **17** (2009), 61-83.

[6] Y. Xiong, Bifurcation of limit cycles by perturbing a class of hyper-elliptic Hamiltonian systems of degree five, *J. Math. Anal. Appl.,* **411** (2014), 559-573. https://doi.org/10.1016/j.jmaa.2013.06.073

[7] R. Asheghi and A. Bakhshalizadeh, Limit cycles in a Liénard system with a cusp and a nilpotent saddle of order 7, *Chaos Solitons Fractals,* **73** (2015), 120-128. https://doi.org/10.1016/j.chaos.2015.01.009

[8] M. Han, J. Yang, A. Tarta and Y. Gao, Limit cycles near homoclinic and heteroclinic loops, *J. Dyn. Diff. Eqn.,* **20** (2008), 923-944. https://doi.org/10.1007/s10884-008-9108-3

[9] Y. Hou and M. Han, Melnikov functions for planar near-Hamiltonian systems and Hopf bifurcations, *J. Shanghai Normal University (Natural Sciences),* **35** (2006), 1-10.

Received: July 25, 2020; Published: August 16, 2020