Applied Mathematical Sciences, Vol. 14, 2020, no. 16, 755 - 763
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ams.2020.914263

On the Mixed Fractional Brownian Motion Time Changed by Inverse α-Stable Subordinator

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Abstract

A time-changed mixed fractional Brownian motion by inverse α-stable subordinator with index $\alpha \in (0,1)$ is an iterated process $Y^H_{T^\alpha}(a,b)$ constructed as the superposition of mixed fractional Brownian motion $N^H(a,b)$ and an independent inverse α-stable subordinator T^α. In this paper we prove that the process $Y^H_{T^\alpha}(a,b)$ is of long range dependence property for every $H > \frac{1}{2}$. We deduce that the time-changed fractional Brownian motion by inverse α-stable subordinator has long range dependence for all $H \in (0,1)$.

Mathematics Subject Classification: 60G20; 60G18; 60G15; 60G10

Keywords: Mixed Fractional Brownian Motion; Long-range Dependence; Subordination; Tempered Stable Subordinator; Gamma Subordinator

1 Introduction

The fractional Brownian motion (fBm for short) $B^H = \{B_t^H,\, t \geq 0\}$ with parameter H, is a centered Gaussian process with covariance function

$$\mathrm{Cov}(B_t^H,\, B_s^H) = \frac{1}{2}[t^{2H} + s^{2H} - |t-s|^{2H}], \quad s, t \geq 0, \tag{1}$$

where H is a real number in $(0,1)$, called the Hurst index or Hurst exponent. The case $H = \frac{1}{2}$ corresponds to the Brownian motion (Bm for short).

An extension of the fBm was introduced by Cheridito [2] called the mixed fractional Brownian motion (mfBm for short) which is a linear combination between a Brownian motion and an independent fractional Brownian motion of Hurst exponent H, with stationary increments exhibit a long-range dependence for $H > \frac{1}{2}$. The mixed fractional Brownian motion has been discussed in [2] to present a stochastic model of the discounted stock price in some arbitrage-free and complete financial markets. This model is the process

$$X_t = X_0 \exp\{\mu t + \sigma(aB_t + bB_t^H)\},$$

where μ is the rate of the return and σ is the volatility.

A mfBm of parameters a, b and H is the process $N^H(a,b) = \{N_t^H(a,b),\ t \geq 0\}$, defined on the probability space $(\Omega, \mathcal{F}, P)$ by

$$N_t^H(a,b) = aB_t + bB_t^H, \quad t \geq 0$$

where $B = \{B_t, t \geq 0\}$ is a Brownian motion, $B^H = \left\{B_t^H, t \geq 0\right\}$ is an independent fractional Brownian motion of Hurst exponent $H \in (0,1)$ and a, b two real constants such that $(a,b) \neq (0,0)$. We refer also to [4, 12, 19, 23] for further information and applications on the mfBm.

The time-changed mixed fractional Brownian motion by inverse α-stable subordinator with index $\alpha \in (0,1)$ is defined as below

$$Y_{T^\alpha}^H(a,b) = \{N_{T_t^\alpha}^H(a,b),\ t \geq 0\},$$

where the parent process $N^H(a,b)$ is a mfBm with parameters a, b, $H \in (0,\, 1)$ and $T^\alpha = \{T_t^\alpha,\, t \geq 0\}$ is an inverse α-stable subordinator assumed to be independent of both Brownian and fractional Brownian motion. If $H = \frac{1}{2}$, the process $Y_{T^\alpha}^{\frac{1}{2}}(0,1)$ is called subordinated Brownian motion, it was investigated in [5, 14, 17]. When $a = 0$, $b = 1$ then $Y_{T^\alpha}^H(0,1)$ it is the process considered in [10, 11] called subordinated fractional Brownian motion. Time-changed process is constructed by taking superposition of tow independent stochastic systems. The evolution of time in external process is replaced by a non-decreasing stochastic process, called subordinator. The resulting time-changed process very often retain important properties of the external process, however certain characteristics might change. This idea of subordination was introduced by Bochner [1] and was explored in many papers (see [7, 8, 10, 16]).

The time-changed mixed fractional Brownian motion has been discussed in [6] to present a stochastic Black-Scholes model, whose price of the underlying stock is the process

$$S_t = S_0 \exp\{\mu T_t^\alpha + \sigma(aB_{T_t^\alpha} + bB^H_{T_t^\alpha})\},$$

where μ is the rate of the return, σ is the volatility and T^α is the α-inverse stable subordinator. Also the time-changed processes have found many interesting applications, for example in finance [6, 9, 18, 21, 24].

Our goal in this paper is to study the long-range dependence property of the time-changed mixed fractional Brownian motion model by inverse α-stable with index $\alpha \in (0,1)$. Also we show that the fractional Brownian motion time-changed by inverse α-stable subordinator is of long dependence for all $H \in (0,1)$. Finally we deduce some results of the mixed fractional Brownian motion of parameters a, b and H.

2 Main results and proofs

We begin by defining the inverse α-stable subordinator.

Definition 2.1. *The inverse α-stable subordinator $T^\alpha = \{T_t^\alpha,\ t \geq 0\}$ is defined in the following way*

$$T_t^\alpha = inf\{r > 0,\ \eta_r^\alpha \geq t\}, \tag{2}$$

where $\eta^\alpha = \{\eta_r^\alpha,\ r \geq 0\}$ is the α-stable subordinator [20, 22] with Laplace transform

$$E(e^{-u\eta_r^\alpha}) = e^{-ru^\alpha}, \quad \alpha \in (0,1).$$

The inverse α-stable subordinator is a non-decreasing Lévy process, starting from zero, has a stationary and independent increments with α-self similar. Specially, when $\alpha \uparrow 1$, T_t^α reduces to the physical time t.

Let T^α be an inverse α-stable subordinator with index $\alpha \in (0,1)$. From [13, 15], we know that

$$E(T_t^\alpha) = \frac{t^\alpha}{\Gamma(\alpha+1)} \quad and \quad E((T_t^\alpha)^n) = \frac{t^{n\alpha} n!}{\Gamma(n\alpha+1)}.$$

Lemma 2.2. *Let T^α be an inverse α-stable subordinator with index $\alpha \in (0,1)$ and B^H be a fBm. Then, by α-self-similar and non-decreasing sample path of T_t^α, we have*

$$E(B_{T_t^\alpha})^2 = \frac{t^\alpha}{\Gamma(\alpha+1)} \quad and \quad E(B^H_{T_t^\alpha})^2 = \left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right)^{2H}.$$

Proof. See [9, 15]. □

Definition 2.3. *Let $N^H(a,b) = \{N_t^H(a,b),\ t \geq 0\}$ be a mfBm and let T^α be an inverse α-stable subordinator with index $\alpha \in (0,1)$. The subordinated of $N^H(a,b)$ by means of T^α is the process $Y_{T^\alpha}^H(a,b) = \{Y_{T_t^\alpha}^H,\ t \geq 0\}$ defined by:*

$$Y_{T_t^\alpha}^H = N_{T_t^\alpha}^H(a,b) = aB_{T_t^\alpha} + bB_{T_t^\alpha}^H, \quad (a,b) \in R \times R \backslash \{0\}, \tag{3}$$

where the subordinator T_t^α is assumed to be independent of both the Bm and the fBm.

Remark 2.4. *When $\alpha \uparrow 1$, the processes $B_{T_t^\alpha}$ and $B_{T_t^\alpha}^H$ degenerate to B_t and B_t^H.*

Notation 2.5. *Let U and V be two centered random variables defined on the same probability space. Let*

$$Corr(U,V) = \frac{Cov(U,V)}{\sqrt{E(U^2)E(V^2)}}, \tag{4}$$

denote the correlation coefficient between U and V.

Now we discuss the long range dependent behavior of $Y_{T^\alpha}^H(a,b)$.

Definition 2.6. *A finite variance stationary process $\{X_t,\ t \geq 0\}$ is said to have long range dependence property [3], if $\sum_{k=0}^{\infty} \gamma_k = \infty$, where*

$$\gamma_k = Cov(X_k, X_{k+1}).$$

In the following definition we give the equivalent definition for a non-stationary process $\{X_t,\ t \geq 0\}$.

Definition 2.7. *Let $s > 0$ be fixed and $t > s$. Then process $\{X_t,\ t \geq 0\}$ is said to have long range dependence property property if*

$$Corr(X_t, X_s) \sim c(s)t^{-d}, \quad as \quad t \to \infty,$$

where $c(s)$ is a constant depending on s and $d \in (0,1)$.

The main result can be stated as follows.

Theorem 2.8. *Let $N^H(a,b) = \{N_t^H(a,b),\ t \geq 0\}$ be the mixed fractional Brownian motion of parameters a, b and H. Let $T^\alpha = \{T_t^\alpha,\ t \geq 0\}$ be an inverse α-stable subordinator with index $\alpha \in (0,1)$ assumed to be independent of both the Bm and the fBm. Then the time-changed mixed fractional Brownian motion by means of T^α has long range dependence property for every $H > \frac{1}{2}$.*

Proof. Let $T^\alpha = \{T_t^\alpha,\ t \geq 0\}$ be an inverse α-stable subordinator with index $\alpha \in (0,1)$ assumed to be independent of both the Bm and the fBm. Let $Y_{T^\alpha}^H(a,b)$ be the time-changed mixed fractional Brownian motion by means of the inverse α-stable subordinator T^α with index $\alpha \in (0,1)$. The process $Y_{T^\alpha}^H(a,b)$ is not stationary hence Definition 2.7 will be used to establish the long range dependence property.

Step 1: Let $s \leq t$. Since B^H has stationary increments, then we have

$$\begin{aligned}
Cov(Y_{T_t^\alpha}^H, Y_{T_s^\alpha}^H) &= E(Y_{T_t^\alpha}^H Y_{T_s^\alpha}^H) = \frac{1}{2}E\left[(Y_{T_t^\alpha}^H)^2 + (Y_{T_s^\alpha}^H)^2 - (Y_{T_t^\alpha}^H - Y_{T_s^\alpha}^H)^2\right] \\
&= \frac{1}{2}E\left[(N_{T_t^\alpha}^H(a,b))^2 + (N_{T_s^\alpha}^H(a,b))^2 - (N_{T_t^\alpha}^H(a,b) - N_{T_s^\alpha}^H(a,b))^2\right] \\
&= \frac{1}{2}E\left[(aB_{T_t^\alpha} + bB_{T_t^\alpha}^H)^2 + (aB_{T_s^\alpha} + bB_{T_s^\alpha}^H)^2\right] \\
&- \frac{1}{2}E\left[\left(a(B_{T_t^\alpha} - B_{T_s^\alpha}) + b(B_{T_t^\alpha}^H - B_{T_s^\alpha}^H)\right)^2\right] \\
&= \frac{1}{2}E\left[(aB_{T_t^\alpha} + bB_{T_t^\alpha}^H)^2 + (aB_{T_s^\alpha} + bB_{T_s^\alpha}^H)^2\right] \\
&- \frac{1}{2}E\left[(aB_{T_{t-s}^\alpha} + bB_{T_{t-s}^\alpha}^H)^2\right] \\
&= \frac{1}{2}E\left[(aB_{T_t^\alpha})^2 + (bB_{T_t^\alpha}^H)^2 + 2(aB_{T_t^\alpha} bB_{T_t^\alpha}^H)\right] \\
&+ \frac{1}{2}E\left[(aB_{T_s^\alpha})^2 + (bB_{T_s^\alpha}^H)^2 + 2(aB_{T_s^\alpha} bB_{T_s^\alpha}^H)\right] \\
&- \frac{1}{2}E\left[(aB_{S_{t-s}^{\lambda,\alpha}})^2 + (bB_{S_{t-s}^{\lambda,\alpha}}^H)^2 + 2(aB_{T_{t-s}^\alpha} bB_{T_{t-s}^\alpha}^H)\right].
\end{aligned}$$

Since B_t and B_t^H are independent and using Lemma 2.2 we get

$$\begin{aligned}
E(Y_{T_t^\alpha}^H Y_{T_s^\alpha}^H) &= \frac{a^2}{2}\left[E(B_{T_t^\alpha})^2 + E(B_{T_s^\alpha})^2 - E(B_{T_{t-s}^\alpha})^2\right] \\
&+ \frac{b^2}{2}\left[E(B_{T_t^\alpha}^H)^2 + E(B_{T_s^\alpha}^H)^2 - E(B_{T_{t-s}^\alpha}^H)^2\right] \\
&= \frac{a^2}{2}\left[\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{s^\alpha}{\Gamma(\alpha+1)} - \frac{(t-s)^\alpha}{\Gamma(\alpha+1)}\right] \\
&+ \frac{b^2}{2}\left[\left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right)^{2H} + \left(\frac{s^\alpha}{\Gamma(\alpha+1)}\right)^{2H} - \left(\frac{(t-s)^\alpha}{\Gamma(\alpha+1)}\right)^{2H}\right] \\
&= \frac{a^2\left[t^\alpha + s^\alpha - (t-s)^\alpha\right]}{2\Gamma(\alpha+1)} + \frac{b^2\left[t^{2\alpha H} + s^{2\alpha H} - (t-s)^{2\alpha H}\right]}{2[\Gamma(\alpha+1)]^{2H}}.
\end{aligned}$$

Hence for all $s \leq t$ and $H \in (0,1)$ we have

$$E(Y_{T_t^\alpha}^H Y_{T_s^\alpha}^H) = \frac{a^2\left[t^\alpha + s^\alpha - (t-s)^\alpha\right]}{2\Gamma(\alpha+1)} + \frac{b^2\left[t^{2\alpha H} + s^{2\alpha H} - (t-s)^{2\alpha H}\right]}{2[\Gamma(\alpha+1)]^{2H}}. \qquad (5)$$

Step 2: Let s be fixed. Then by Taylor's expansion we have for large t

$$\begin{aligned} E(Y^H_{T^\alpha_t} Y^H_{T^\alpha_s}) &\sim \frac{a^2}{2\Gamma(\alpha+1)} t^\alpha \left[\alpha \frac{s}{t} + s^\alpha t^{-\alpha} + o(t^{-2})\right] \\ &+ \frac{b^2}{2[\Gamma(\alpha+1)]^{2H}} t^{2\alpha H} \left[2\alpha H \frac{s}{t} + s^{2\alpha H} t^{-2\alpha H} + o(t^{-2})\right] \\ &\sim \frac{a^2 t^\alpha}{2\Gamma(\alpha+1)} \left[\alpha \frac{s}{t} + (\frac{s}{t})^\alpha + o(t^{-2})\right] \\ &+ \frac{b^2 t^{2\alpha H}}{2[\Gamma(\alpha+1)]^{2H}} \left[2\alpha H \frac{s}{t} + (\frac{s}{t})^{2\alpha H} + o(t^{-2})\right] \\ &\sim \frac{a^2 \alpha s}{2\Gamma(\alpha+1)} t^{\alpha-1} + \frac{b^2 \alpha s}{(\Gamma(\alpha+1))^{2H}} t^{2\alpha H-1}. \end{aligned}$$

Then for fixed s and large t, $Y^H_{T^\alpha_t}$ satisfies

$$E(Y^H_{T^\alpha_t} Y^H_{T^\alpha_s}) \sim \frac{a^2 \alpha s}{2\Gamma(\alpha+1)} t^{\alpha-1} + \frac{b^2 \alpha s}{(\Gamma(\alpha+1))^{2H}} t^{2\alpha H-1}. \tag{6}$$

Step 3: Let $\frac{1}{2} < H < 1$. Using Eqs. (4), (6) and by Taylor's expansion we get, as $t \to \infty$

$$\begin{aligned} Corr(Y^H_{T^\alpha_t}, Y^H_{T^\alpha_s}) &\sim \frac{\frac{a^2 \alpha s}{2\Gamma(\alpha+1)} t^{\alpha-1} + \frac{b^2 \alpha s}{(\Gamma(\alpha+1))^{2H}} t^{2\alpha H-1}}{\sqrt{\left[\frac{a^2\alpha}{2\Gamma(\alpha+1)} t^\alpha + \frac{b^2\alpha}{(\Gamma(\alpha+1))^{2H}} t^{2\alpha H}\right]} \sqrt{E(Y^{T^\alpha}_s)^2}} \\ &= \frac{\frac{a^2 \alpha s}{2\Gamma(\alpha+1)} t^{\alpha-1} + \frac{b^2 \alpha s}{(\Gamma(\alpha+1))^{2H}} t^{2\alpha H-1}}{\frac{|b|\alpha^{\frac{1}{2}} t^{\alpha H}}{(\Gamma(\alpha+1))^H} \sqrt{\left[\frac{a^2}{2b^2(\Gamma(\alpha+1))^{1-2H}} t^{\alpha(1-2H)} + 1\right]} \sqrt{E(Y^{T^\alpha}_s)^2}} \\ &\sim \frac{a^2 \alpha^{\frac{1}{2}} s t^{\alpha(1-H)-1}}{2|b|(\Gamma(\alpha+1))^{1-H} \sqrt{E(Y^{T^\alpha}_s)^2}} + \frac{|b|\alpha^{\frac{1}{2}} s t^{\alpha H-1}}{(\Gamma(\alpha+1))^H \sqrt{E(Y^{T^\alpha}_s)^2}}. \end{aligned}$$

Hence, for every $H \in (\frac{1}{2}, 1)$ we have

$$Corr(Y^H_{T^\alpha_t}, Y^H_{T^\alpha_s}) \sim \frac{a^2 \alpha^{\frac{1}{2}} s\, t^{\alpha(1-H)-1}}{2|b|(\Gamma(\alpha+1))^{1-H} \sqrt{E(Y^{T^\alpha}_s)^2}} + \frac{|b|\alpha^{\frac{1}{2}} s\, t^{\alpha H-1}}{(\Gamma(\alpha+1))^H \sqrt{E(Y^{T^\alpha}_s)^2}}. \tag{7}$$

The correlation function of $Y^H_{T^\alpha_t}$ decays like a mixture of power law $t^{-(1-\alpha(1-H))} + t^{-(1-\alpha H)}$. The non-stationarity time-changed process $Y^H_{T^\alpha}(a, b)$ exhibits long range dependence property for all $H > \frac{1}{2}$. □

Remark 2.9. *When $\alpha \uparrow 1$, in Eq. (5) we have*

$$\lim_{\alpha \to 1} E(Y^H_{T^\alpha_t} Y^H_{T^\alpha_s}) = a^2(s \wedge t) + \frac{b^2}{2}\left[t^{2H} + s^{2H} - (t-s)^{2H}\right].$$

Remark 2.10. *When $a = 0$ and $b = 1$ in Eqs. (6) and (7) we get*

$$E(Y^H_{T^\alpha_t} Y^H_{T^\alpha_s}) = E(B^H_{T^\alpha_t} B^H_{T^\alpha_s}) \sim \frac{\alpha s t^{2\alpha H-1}}{(\Gamma(\alpha+1))^{2H}}, \quad as \quad t \to \infty,$$

$$Corr(Y^H_{T^\alpha_t}, Y^H_{T^\alpha_s}) = Corr(B^H_{T^\alpha_t}, B^H_{T^\alpha_s}) \sim \frac{\alpha^{\frac{1}{2}} s t^{\alpha H-1}}{(\Gamma(\alpha+1))^{H}\sqrt{E(B^H_{T^\alpha_s})^2}}, \quad as \ t \to \infty.$$

Hence we obtain the following result.

Corollary 2.11. *The fractional Brownian motion time changed by inverse α-stable subordinator with index $\alpha \in (0,1)$ is of long range dependence for the Hurst exponent $H \in (0,1)$.*

Remark 2.12. *Similar result as Corollary 2.11 was obtained in [10] ([11]) in the case of fractional Brownian motion time changed by tempered stable subordinator (gamma subordinator).*

As application to the original process we obtain the following. .

Corollary 2.13. *Let $H > \frac{1}{2}$. When $\alpha \uparrow 1$, in Eqs. (6) and (7) we have*

$$\lim_{\alpha\to 1} E(Y^H_{T^\alpha_t} Y^H_{T^\alpha_s}) = \frac{a^2 s}{2} + b^2 s t^{2H-1}, \quad as \quad t \to \infty,$$

$$\lim_{\alpha\to 1} Corr(Y^H_{T^\alpha_t}, Y^H_{T^\alpha_s}) = \frac{a^2 s\, t^{-H}}{2|b|\sqrt{E(N^H_s(a,b))^2}} + \frac{|b| s\, t^{H-1}}{\sqrt{E(N^H_s(a,b))^2}}, \quad as \quad t \to \infty.$$

Hence using Remark 2.4 and corollary 2.13 we can see that the mixed fractional Brownian motion of parameters a, b and H has long range dependence property for all $H > \frac{1}{2}$ in sense of Definition 2.7.

Remark 2.14. *Let $H \in (0,1)$. Then*

$$Corr(B^H_t, B^H_s) \sim \frac{s t^{H-1}}{\sqrt{E(B^H_s)^2}}, \quad as \quad t \to \infty. \tag{8}$$

Indeed, we take $a = 0$ and $b = 1$ in Eq. (7). When $\alpha \uparrow 1$ and using Remark 2.4 we obtain Eq. (8).

The idea, used results for the time-changed process to obtain a results for the original one is already investigated in [7].

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Received: August 15, 2020; Published: October 8, 2020