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The Fundamental Problems on Orthotropic Half-Plane with Holes

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Abstract

The first fundamental problem on orthotropic half-plane with holes is discussed. Based on the complex function method of anisotropic plate and integral equation theory, the integral equation expression of the solution is given. And the uniqueness of the solution is proved.

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1. Introduction

In real life, all kinds of materials are indispensable. Therefore, the properties and mechanical behavior of materials have been concerned by physicists, mathematicians and engineers, and become one of the main topics of their research. The research methods also from the initial analytical method, to a variety of numerical methods. With the advent of computer, numerical method has become the mainstream of research. However, the classical analysis method has its unique advantages, revealing that the mechanical behavior of materials is more accurate and reasonable. Because of the variety and complexity of materials, the classical analysis method becomes very difficult, especially for the complex materials.

In 1943, Green first gave the basic solution [1] for orthotropic materials expressed in real variables under plane stress, then Rizzo and Shippy further developed and perfected the work [2] , the basic elastic solution of the plane orthotropic material problem is gradually matured and widely used. There are two classical books related to anisotropic elasticity written by S. G. Lekhnitskii [3] and Chyanbin Hwu[4], respectively, which had a profound influence on later researchers. A great deal of research has been done in the past few decades, such as [5-7]. The aim of this paper is to study the steady-state problem of half-plane for orthotropic materials with holes. By employing the theory of anisotropic plates and the integral transformation, the problem is reduced to one Fredholm integral equation. The existence and uniqueness of solution is proved.

2. Formulation of Problems and Integral Equation

Let the orthotropic elastic material occupy the lower half-plane S^- of coordinate Z-plane, which have m holes with the boundary of $L_j(j = 1, 2, ..., m)$, the external boundary of S^- is X-axis, and mark it as L_0 , notation $L = \sum_{j=0}^{m} L_j$, $S^+ = Z - S^-$. We will discuss the first fundamental problem, that is finding the elastic equilibrium, under knowing the external stress $X_{nj}(t) + iY_{nj}(t)$ on $L_j(j = 0, 1, 2, ..., m)$. Besides, the stresses and the angle of rotation at infinity is also given. Obviously, the principal vectors of the external stresses on L_j is $X_j + iY_j = \int_{L_j}[X_{nj}(t) + iY_{nj}(t)]dt, (j = 0, 1, 2, ..., m)$.

Let $\varphi_1(z_1)$ and $\varphi_2(z_2)$ are the complex stress functions[3] for the elastic body. Because of the elastic body is in steady state, without loss of generality, we can assume the principle vectors of stress on $L_j(j = 0, 1, 2, ..., m)$. is zero, and no stresses or rotation at infinity. Thus, the stress functions $\varphi_1(z_1)$ and $\varphi_2(z_2)$ is sectionally holomorphic functions.

Suppose the affine transformation $z_k = x+\mu_k y(k = 1, 2)$ transforms the region $S^{\pm}$ into the regions $S_k^{\pm}$ of the Z_k plane $(k = 1, 2)$, and $L_j(j = 0, 1, 2, ..., m)$ change to $L_j^k(j = 0, 1, 2, ..., m; k = 1, 2)$. Notation $L^{(1)} = \sum_{j=0}^{m} L_j^{(1)}$, $L^{(2)} = \sum_{j=0}^{m} L_j^{(2)}$. According to the conditions known above and the theory of orthotropic plate [3], our problem boils down to the following boundary value problem with functions $\varphi_1(z_1)$ and $\varphi_2(z_2)$:

$$(1+i\mu_1)\varphi_1(t_1)+(1+i\overline{\mu_1})\overline{\varphi_1(t_1)}+(1+i\mu_2)\varphi_2(t_2)+(1+i\overline{\mu_2})\overline{\varphi_2(t_2)} = f(t)+C_j,$$

$$t \in L_j, t_1 \in L_j^{(1)}, t_2 \in L_j^{(2)}, \quad (j = 0, 1, 2, ..., m). \tag{2.1}$$

Where

$$f(t) = i \int_{t_0}^{t} [X_{nj}(t) + iY_{nj}(t)] ds,$$

t_0 is a certain point on L_j, $C_j (j = 0, 1, 2, ..., m)$ are undetermined constant. For definiteness, let $\overline{C_0 = 0}$.

By removing $\overline{\varphi_2(t_2)}$ from Eq. (1.1), we get

$$a\varphi_1(t_1) + b\overline{\varphi_1(t_1)} + \varphi_2(t_2) = F(t) + E_j,$$
$$t \in L_j, t_1 \in L_j^{(1)}, t_2 \in L_j^{(2)}, \quad (j = 0, 1, 2, ..., m). \tag{2.2}$$

Where

$$F(t) = \frac{(1 - i\overline{\mu_2}) f(t) - (1 + i\overline{\mu_2}) \overline{f(t)}}{2i(\mu_2 - \overline{\mu_2})}, \quad E_j = \frac{(1 - i\overline{\mu_2}) C_j - (1 + i\overline{\mu_2}) \overline{C_j}}{2i(\mu_2 - \overline{\mu_2})},$$

$$a = \frac{\mu_1 - \overline{\mu_2}}{\mu_2 - \overline{\mu_2}}, \quad b = \frac{\overline{\mu_1} - \overline{\mu_2}}{\mu_2 - \overline{\mu_2}}.$$

For solving the above boundary value problem (2.2), we introduce the following integral transformation:

$$\varphi_1(z_1) = \frac{1}{2\pi i} \int_{L^{(1)}} \frac{\omega(t)}{t_1 - z_1} dt_1 - \frac{1}{2\pi i} \int_{L^{(1)}} \frac{\omega(t)}{t_1 - z_{00}^{(1)}} dt_1 + \sum_{j=1}^{m} \frac{b_j}{z_1 - z_{0j}^{(1)}}, z_1 \in Z_1^-, \tag{2.3}$$

$$\varphi_2(z_2) = \frac{1}{2\pi i} \int_{L^{(2)}} \frac{-a\omega(t) + b\overline{\omega(t)}}{t_2 - z_2} dt_2 - \frac{1}{2\pi i} \int_{L^{(2)}} \frac{-a\omega(t) + b\overline{\omega(t)}}{t_2 - z_{00}^{(2)}} dt_2, z_2 \in Z_2^-, \tag{2.4}$$

Hence $\omega(t) \in \hat{H}, t \in L_0^{(k)} (k = 1, 2)$, and $\omega(t) \in H, t \in L_j^{(k)} (j = 1, 2, ..., m, k = 1, 2)$, H is one class of Hölder continuous functions and $\hat{H}$ is one class of Hölder continuous functions on infinity line, $z_{00}^{(k)}$ is a fixed point on the upper half-Z_k plane $(k = 1, 2)$, $z_{0j}^{(1)}$ is a fixed point on the interior region bounded by $L_j^{(1)}$ in the Z_1 plane,

$$b_j = i \int_{L_j^{(1)}} [\omega(t) d\overline{t_1} - \overline{\omega(t)} dt_1], \quad j = 1, 2, ..., m. \tag{2.5}$$

It is easy to know that b_j is a real number, and C_j can be written as

$$C_j = \int_{L_j^{(1)}} \omega(t) ds, \quad j = 1, 2, ..., m. \tag{2.6}$$

When z tending to point t_0 on the curve L, z_1 tending to point t_{10} on the curve $L^{(1)}$ and z_2 tending to point t_{20} on the curve $L^{(2)}$. Using Plemelj's integral formula[8], from Eqs. (2.3) ,(2.4)and (2.2), we get

$$\overline{\omega(t_0)}+\frac{1}{2\pi i}\int_L \overline{\omega(t)}dlg\frac{\overline{t_1}-\overline{t_{10}}}{t_2-t_{20}}-\frac{a}{b}\frac{1}{2\pi i}\int_L \omega(t)dlg\frac{t_1-t_{10}}{t_2-t_{20}}-\frac{1}{2\pi i}\int_L \overline{\omega(t)}dlg\frac{\overline{t_1}-\overline{z_{00}^{(1)}}}{t_2-z_{20}^{(2)}}+$$

$$\frac{a}{b}\frac{1}{2\pi i}\int_L \omega(t)dlg\frac{t_1-z_{00}^{(1)}}{t_2-z_{20}^{(2)}}-\sum_{j=1}^{m}(\frac{a}{b}\frac{b_j}{t_0-z_{0j}^{(1)}}-\frac{\overline{b_j}}{\overline{t_0}-\overline{z_{0j}^{(1)}}})+\frac{(1-i\overline{\mu_2})C_j-(1+i\overline{\mu_2})\overline{C_j}}{2ib(\mu_2-\overline{\mu_2})}=-\frac{F(t_0)}{b}. \tag{2.7}$$

Obviously, Eq.(2.7) is a Fredholm integral equation. We will discuss this equation in the next section.

3. Uniqueness of the Solution

The solution of equation (2.7) can be obtained by using the general integral equation theory[9]. We mainly discuss the uniqueness of the solution of the equation. Therefore, it is only necessary to prove that the corresponding homogeneous equation of equation (2.7) has only zero solution. Suppose that the solution of the corresponding homogeneous equation of equation (2.7) is $\omega_0(t)$. Making $\varphi_1^0(z_1), \varphi_2^0(z_2), b_j^0, C_j^0$ like Eqs.(2.3)-(2.6), in the same way that we did in the previous section, we get

$$(1+i\mu_1)\varphi_1^0(t_1)+(1+i\overline{\mu_1})\overline{\varphi_1^0(t_1)}+(1+i\mu_2)\varphi_2^0(t_2)+(1+i\overline{\mu_2})\overline{\varphi_2^0(t_2)}=C_j,$$
$$t\in L_j, t_1\in L_j^{(1)}, t_2\in L_j^{(2)}, \quad (j=0,1,2,...,m). \tag{3.1}$$

Equation (3.1) shows that $\varphi_1^0(z_1), \varphi_2^0(z_2)$ are the solutions of the first fundamental problem , which has no stress and no angle of rotation at the infinity, and no external stress on the boundary of S^-.

According to the conclusion in [3], we have

$$\varphi_1^0(z_1)=D_1; \quad \varphi_2^0(z_2)=D_2, \tag{3.2}$$

where D_1, D_2 are complex number.

Substituting Eq.(3.2) into Eq.(3.1), we get

$$(1+i\mu_1)D_1+(1+i\overline{\mu_1})\overline{D_1}+(1+i\mu_2)D_2+(1+i\overline{\mu_2})\overline{D_2}=C_0^0=0, t\in L_0, \tag{3.3}$$

and

$$C_j=(1+i\mu_1)D_1+(1+i\overline{\mu_1})\overline{D_1}+(1+i\mu_2)D_2+(1+i\overline{\mu_2})\overline{D_2}=0, t\in L_j, (j=1,2,...,m). \tag{3.4}$$

Such then, according to Eqs.(2.3), (2.4) and (3.2), we have

$$D_1 = \frac{1}{2\pi i}\int_{L^{(1)}} \frac{\omega_0(t)}{t_1 - z_1} dt_1 - \frac{1}{2\pi i}\int_{L^{(1)}} \frac{\omega_0(t)}{t_1 - z_{00}^{(1)}} dt_1 + \sum_{j=1}^{m} \frac{b_j^0}{z_1 - z_{0j}^{(1)}}, \qquad (3.5)$$

$$D_2 = \frac{1}{2\pi i}\int_{L^{(2)}} \frac{-a\omega_0(t) + b\overline{\omega_0(t)}}{t_2 - z_2} dt_2 - \frac{1}{2\pi i}\int_{L^{(2)}} \frac{-a\omega_0(t) + b\overline{\omega_0(t)}}{t_2 - z_{00}^{(2)}} dt_2. \qquad (3.6)$$

Now, we define two analytic functions $\varphi_k^*(z_k)$ in region S_k^+ of Z_k-plane $(k = 1, 2)$, so that

$$i\varphi_1^*(z_1) = \frac{1}{2\pi i}\int_{L^{(1)}} \frac{\omega_0(t)}{t_1 - z_1} dt_1 - \frac{1}{2\pi i}\int_{L^{(1)}} \frac{\omega_0(t)}{t_1 - z_{00}^{(1)}} dt_1, z_1 \in S_1^+, \qquad (3.7)$$

$$i\varphi_2^*(z_2) = \frac{1}{2\pi i}\int_{L^{(2)}} \frac{-a\omega_0(t) + b\overline{\omega_0(t)}}{t_2 - z_2} dt_2 - \frac{1}{2\pi i}\int_{L^{(2)}} \frac{-a\omega_0(t) + b\overline{\omega_0(t)}}{t_2 - z_{00}^{(2)}} dt_2, z_2 \in S_2^+. \qquad (3.8)$$

According to the Plemelj formula, by Eq. (3.7) and (3.5) , let z_1 from S_1^+ and S_1^- tend to point t_{10} of $L^{(1)}$, respectively, we have

$$i\varphi_1^*(t_{10}) = \frac{1}{2}\omega_0(t_0) + \frac{1}{2\pi i}\int_{L^{(1)}} \frac{\omega_0(t)}{t_1 - t_{10}} dt_1 - \frac{1}{2\pi i}\int_{L^{(1)}} \frac{\omega_0(t)}{t_1 - z_{00}^{(1)}} dt_1, \qquad (3.9)$$

$$D_1 = -\frac{1}{2}\omega_0(t_0) + \frac{1}{2\pi i}\int_{L^{(1)}} \frac{\omega_0(t)}{t_1 - t_{10}} dt_1 - \frac{1}{2\pi i}\int_{L^{(1)}} \frac{\omega_0(t)}{t_1 - z_{00}^{(1)}} dt_1 + \sum_{j=1}^{m} \frac{b_j^0}{t_{10} - z_{0j}^{(1)}}, \qquad (3.10)$$

From Eqs. (3.9) and (3.10), we get $i\varphi_1^*(t_{10}) = \omega_0(t_0) - \sum_{j=1}^{m} \frac{b_j^0}{t_{10} - z_{0j}^{(1)}} + D_1$. It can also be written as

$$i\varphi_1^*(t_1) = \omega_0(t) - \sum_{j=1}^{m} \frac{b_j^0}{t_1 - z_{0j}^{(1)}} + D_1, \qquad (3.11)$$

In the same way, it can be obtained from Eqs. (3.10) and (3.8) with

$$i\varphi_2^*(t_2) = -a\omega_0(t) + b\overline{\omega_0(t)} + D_2. \qquad (2.12)$$

Subtracting $\omega_0(t)$ from Eqs. (3.11) and (3.12) , we get

$$a(\varphi_1^*(t_1) - i\sum_{j=1}^{m}\frac{b_j^0}{t_1 - z_{0j}^{(1)}}) + b(\overline{\varphi_1^*(t_1)} + i\sum_{j=1}^{m}\frac{\overline{b_j^0}}{\overline{t_1} - \overline{z_{0j}^{(1)}}}) + \varphi_2^*(t_2) = -i(aD_1 - b\overline{D_1} + D_2),$$

$$t \in L_j, (j = 0, 1, 2, ..., m), \tag{3.13}$$

By the values of notation a and b, Eq. (3.13) can also be written as

$$(1+i\mu_1)\varphi_1^*(t_1) + (1+i\overline{\mu_1})\overline{\varphi_1^*(t_1)} + (1+i\mu_2)\varphi_2^*(t_2) + (1+i\overline{\mu_2})\overline{\varphi_2^*(t_2)} - i(1+i\mu_1)\sum_{j=1}^{m}\frac{b_j^0}{t_1 - z_{0j}^{(1)}}$$

$$+i(1+i\overline{\mu_1})\sum_{j=1}^{m}\frac{\overline{b_j^0}}{\overline{t_1} - \overline{z_{0j}^{(1)}}}) = -i[(1+i\mu_1)D_1 - (1+i\overline{\mu_1})\overline{D_1} + (1+i\mu_2)D_2 - (1+i\overline{\mu_2})\overline{D_2}]. \tag{3.14}$$

Owing to $t_j = x + \mu_j y (j = 1, 2), t = x + iy$, it's easy to deduce that

$$\bar{t} = \frac{(1 + i\mu_j)\overline{t_j} - (1 - i\overline{\mu_j})t_j}{i(\overline{\mu_j} - \mu_j)}, j = 1, 2. \tag{3.15}$$

So, we have

$$d\bar{t} = \frac{(1 + i\mu_j)d\overline{t_j} - (1 - i\overline{\mu_j})dt_j}{i(\overline{\mu_j} - \mu_j)}, j = 1, 2. \tag{3.16}$$

Multiplying both sides of Eq.(3.14) by $d\bar{t}$, and then integrating along the $L_j (j = 1, 2, ..., m)$. we get

$$\int_{L_j}[(1+i\mu_1)\varphi_1^*(t_1) + (1+i\overline{\mu_1})\overline{\varphi_1^*(t_1)}]d\bar{t} + \int_{L_j}[(1+i\mu_2)\varphi_2^*(t_2) + (1+i\overline{\mu_2})\overline{\varphi_2^*(t_2)}]d\bar{t}$$

$$+\int_{L_j}[-i(1+i\mu_1)\sum_{j=1}^{m}\frac{b_j^0}{t_1 - z_{0j}^{(1)}} + i(1+i\overline{\mu_1})\sum_{j=1}^{m}\frac{\overline{b_j^0}}{\overline{t_1} - \overline{z_{0j}^{(1)}}}]d\bar{t} = 0, \quad (j = 1, 2, ..., m). \tag{3.17}$$

Notice that $\int_{L^{(1)}}\varphi_1^*(t_1)dt_1 = 0, \quad \int_{L^{(2)}}\varphi_2^*(t_2)dt_2 = 0$, from Eqs. (3.16) and (3.17), we have

$$\frac{1}{i(\overline{\mu_1} - \mu_1)}\int_{L_j^{(1)}}[(1 + \mu_1^2)\varphi_1^*(t_1)d\overline{t_1} - (1 + \overline{\mu_1^2})\overline{\varphi_1^*(t_1)}dt_1] + \frac{1}{i(\overline{\mu_2} - \mu_2)}\int_{L_j^{(2)}}[(1 + \mu_2^2)\varphi_2^*(t_2)d\overline{t_2}$$

$$-(1+\overline{\mu_2^2})\overline{\varphi_2^*(t_2)}dt_2] - \frac{1}{i(\overline{\mu_1} - \mu_1)}\sum_{j=1}^{m}b_j^0\int_{L_j^{(1)}}[\frac{(1 + \mu_1^2)d\overline{t_1}}{t_1 - z_{0j}^{(1)}} + \frac{(1 + \overline{\mu_1^2})dt_1}{\overline{t_1} - \overline{z_{0j}^{(1)}}}] + 4\pi b_j^0 = 0, (j = 1, 2, ..., m). \tag{3.18}$$

Owing that the first three terms on the left side of Eq. (3.18) are all pure imaginary number, there is

$$b_j^0 = 0, \quad (j = 1, 2, ..., m), \tag{3.19}$$

and then, Eq.(3.14) change to

$$(1 + i\mu_1)\varphi_1^*(t_1) + (1 + i\overline{\mu_1})\overline{\varphi_1^*(t_1)} + (1 + i\mu_2)\varphi_2^*(t_2) + (1 + i\overline{\mu_2})\overline{\varphi_2^*(t_2)}$$

$$= -i[(1+i\mu_1)D_1 - (1+i\overline{\mu_1})\overline{D_1} + (1+i\mu_2)D_2 - (1+i\overline{\mu_2})\overline{D_2}], t \in \sum_{j=0}^{m} L_j. \tag{3.20}$$

Equation (3.20) shows that $\varphi_1^*(z_1), \varphi_2^*(z_2)$ are the solutions of the first fundamental problem which has no stress and no angle of rotation at the infinity, and no external stress on the boundary of S^+. According to the conclusion in [3], when z be in the upper half-plane , we have

$$\varphi_1^*(z_1) = D_1^*; \quad \varphi_2^*(z_2) = D_2^*. \tag{3.21}$$

Notice that $\varphi_1^*(z_{00}^{(1)}) = 0, \varphi_2^*(z_{00}^{(2)}) = 0$, where z_{00} belongs to the upper half plane. From Eqs. (3.7) and (3.8), we have

$$\varphi_1^*(z_1) = 0; \quad \varphi_2^*(z_2) = 0, \quad z \in Z^+, \tag{3.22}$$

and then, from Eq.(3.11), we get

$$\omega_0(t) = -D_1, \quad t \in L_0. \tag{3.23}$$

On the other hand, according to the uniqueness theorem[3], when $t \in L_j (j = 1, 2, ..., m)$ and z is belongs to the region bounded by L_j, we have

$$\varphi_1^*(z_1) = iA_j^{(0)}\beta_1 z_1 + D_{1j}; \quad \varphi_2^*(z_2) = iA_j^{(0)}\beta_2 z_2 + D_{2j}, \tag{3.24}$$

where

$$\beta_1 = \begin{cases} -i[1 + \frac{i}{2}(\frac{Re\mu_1 - Re\mu_2}{Im\mu_1} - \frac{(Im\mu_2)^2 - (Im\mu_1)^2}{Im\mu_1(Re\mu_1 - Re\mu_2)})], & Re\mu_1 \neq Re\mu_2, \\ 1, & Re\mu_1 = Re\mu_2, \end{cases}$$

$$\beta_2 = \begin{cases} i[1 - \frac{i}{2}(\frac{Re\mu_1 - Re\mu_2}{Im\mu_2} - \frac{(Im\mu_2)^2 - (Im\mu_1)^2}{Im\mu_2(Re\mu_1 - Re\mu_2)})], & Re\mu_1 \neq Re\mu_2, \\ -\frac{Im\mu_1}{Im\mu_2}, & Re\mu_1 = Re\mu_2, \end{cases}$$

$A_j^{(0)}$ is an unknown real number, D_{1j}, D_{2j} are unknown complex numbers, and satisfied

$$(1+i\mu_1)D_{1j}+(1+i\overline{\mu_1})\overline{D_{1j}}+(1+i\mu_2)D_{2j}+(1+i\overline{\mu_2})\overline{D_{2j}}=0. \tag{3.25}$$

When z tend to some point t on L_0, using Eqs. (3.22),(3.20), we have $D=0$, which is

$$(1+i\mu_1)D_1-(1+i\overline{\mu_1})\overline{D_1}+(1+i\mu_2)D_2-(1+i\overline{\mu_2})\overline{D_2}=0. \tag{3.26}$$

From Eqs. (3.3) and (3.26), we get

$$D_1=D_2=0, \tag{3.27}$$

and then, from (3.23), we have

$$\omega_0(t)=0, \quad t\in L_0. \tag{3.28}$$

From Eqs. (3.11) and (3.24), as $t\in L_j(j=1,2,...,m)$, we have

$$\omega_0(t)=i\varphi_1^*(t_1)=-A_j^{(0)}\beta_1 t_1+iD_{1j}, \quad (j=1,2,...,m). \tag{3.29}$$

Due to $0=b_j^0=i\int_{L_j^{(1)}}[\omega_0(t)d\overline{t_1}-\overline{\omega_0(t)}dt_1],(j=1,2,...,m)$, which put the $\omega_0(t)$ into by (3.29), we get $A_j^{(0)}=0,(j=1,2,...,m)$. So, we have

$$\omega_0(t)=iD_{1j},t\in L_j, \quad (j=1,2,...,m). \tag{3.30}$$

Further, as $0=C_j^0=\int_{L_j^{(1)}}\omega_0(t)ds,(j=1,2,...,m)$, once more, put the $\omega_0(t)$ into by (3.30), we get

$$D_{1j}=0, \quad (j=1,2,...,m). \tag{3.31}$$

So, from Eqs. (3.30) and (3.28), we get

$$\omega_0(t)=0,t\in L_j, \quad (j=0,1,2,...,m). \tag{3.32}$$

This shows that the homogeneous equation corresponding to equation (2.7) has only zero solution. Therefore, the uniqueness of the solution is proved.

4. Conclusions

The main aim of this paper is the application of anisotropic plate theory and integral equation. We discuss the first fundamental problem of an orthotropic half plane with holes. The integral equation expression of the solution is obtained by using the classical analysis method, it is helpful for further analysis and precise numerical solution. The problems discussed in this paper, as well as similar problems, are often encountered in engineering practice. Compared with the isotropy materials, the orthotropic materials are more complex and difficult.

The method used in this paper is also applicable to the second fundamental problem and the mixed boundary value problem. In the future, we will focus on the solution of integral equations and numerical calculations.

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