Applied Mathematical Sciences, Vol. 14, 2020, no. 13, 659 - 668
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ams.2020.914236

The Cox-Ingersol-Ross Model Parameters Estimation Based on Diffusion Bridges Processes[1]

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Abstract

This paper studies the parameters estimation of Cox-Ingersol-Ross process , in which data is a discrete time sample. The maximum likelihood estimation of the parameters is obtained by the approximate simulation of diffusion bridges and the simulated EM-algorithm.

Keywords: CIR model; EM-algorithm; likelihood estimation; diffusion bridges

1 Introduction

CIR model is widely used in the pricing of interest-sensitive undetermined equity, which is a one-dimensional diffusion process. Likelihood based estimation (including Bayesian) for discretely observed diffusion processes has been investigated by Ozaki (1985), Pedersen (1995), Poulsen (1999), Elerian, Chib and Shep-hard (2001), Eraker (2001), Roberts and Stramer (2001), Durham and Gal-lant (2002), AÏt-Sahalia and Mykland (2003), Beskos et al. Martingale estimating functions for discretely observed diffusions are reviewed in Sϕrensen (1997)and Sϕrensen (2010). Chib, Pitt and Shephard (2010) presented a general approach to simulation-based Bayesian inference for diffusion models when the data are discrete time observations of rather general, and possibly random,

[1]This work is supported by the Henan Science and Technology Tackling Key Project (No. 182102210364.

functionals of the continuous sample path, see also Golightly and Wilkinson (2008).

Kamil Kladivko (2007) used the maximum likelihood method to estimate the parameters of the US PRIBO3 data based on the CIR model, and analyzed and described the marginal distribution of the estimated parameters. Philip Gray (2005) study the bayesian estimation of the Vaseick model, the result of bayesian estimation is improved than the maximum likelihood estimation, and there is no obvious difference between the posterior mean estimation effect under the discretized asymptotic maximum likelihood and the posterior mean estimation effect under the accurate likelihood function. Guoshi Tong (2012) used an MCMC algorithm based on particle filter to simulate the likelihood function to investigate and analyze the CIR model estimation. Xiaoxia Feng and Dejun Xie (2012) used the interest rate data of the United States and Japan to estimate the parameters of the CIR model using Gibbs Sampling algorithm on the basis of Euler discretization, and pointed out that adding extended data can improve the estimation effect.

In this paper, we do this by simulating sample paths of the diffusion process given the data using ideas from Bladt, M. and Sϕrensen, M.(2014). However the main challenge to likelihood based inference for diffusion models is that the transition density, and hence the likelihood function, is not explicitly available and must therefore be approximated. We find maximum likelihood estimates by applying the Expectation Maximization (EM) algorithm to solve this problem.

2 Approximate Bridge Simulation based on CIR model

Bladt, M. and Sϕrensen, M.(2014) propose a simple method for the simulation of a one-dimensional diffusion bridge. In the following we present the theory as it was developed by Bladt and Sϕrensen.

Let $X = \{X_t\}_{t\geq 0}$ be a one-dimensional diffusion process given by the stochastic differential equation

$$dX_t = \alpha(X_t)dt + \sigma(X_t)dW_t \tag{2.1}$$

where W is a standard Wiener process, and where the coefficients α and σ are sufficiently regular to ensure that the equation has a unique weak solution that is a strong Markov process. Let a and b be given points in the state space of X, define X^1 and X^2 as the solutions

$$dX_t^i = \alpha(X_t^i)dt + \sigma(X_T^i)dW_t^i, i = 1, 2, X_0^i = a \text{and} X_0^2 = b, \tag{2.2}$$

where W^1 and W^2 be two independent standard Wiener processes. If the samples paths of the two processes intersect, they can be combined into a realization of a process that approximates $a(0, a, \Delta, b)$-bridge.

Theorem 2.1 [1] Let $\tau = \inf\{0 \leq t \leq t_n | X_t^1 = X_{\Delta-t}^2\}(\inf \varnothing = +\infty)$ and define

$$Z_t = \begin{cases} X_t^1, & \text{if} \quad 0 \leq t \leq \tau \\ X_{\Delta-t}^2, & \text{if} \quad \tau \leq t \leq t_n \end{cases} \tag{2.5}$$

Assume that

$$M = \int_l^x m(x)dx < \infty, \tag{2.6}$$

where

$$m(x) = \frac{1}{\sigma(x)^2} \exp(2 \int_z^x \frac{\alpha(y)}{\sigma(y)^2} dy), x \in (l, r)$$

is the density of the speed measure of the diffusion, z is an arbitrary point in the state space (l, r). Then the distribution of $\{Z_t\}_{0 \leq t \leq t_n}$, conditional on the event $\{\tau \leq t_n\}$, equals the distribution of a $(0, a, \Delta, b)$-bridge, conditional on the event that the bridge is hit by an independent diffusion with stochastic differential equation (2.1) and initial distribution with density $p_\Delta(b, \cdot)$. Furthermore, for ergodic diffusions, the probability that the process Z is a $(0, a, \Delta, b)$-bridge is close to one, that is, the simulated process is essentially a $(0, a, \Delta, b)$.

Cox, Ingersoll and Ross[2,3] put forward a generalized equilibrium single factor model in 1985. The CIR model assumes that $R = \{R_t\}_{t \geq 0}$ be a one-dimensional diffusion given by the stochastic differential equation

$$dR_t = \alpha(\mu - R_t)dt + \sigma\sqrt{R_t}dW_t \tag{2.7}$$

where W_t is a Wiener process, μ and σ are the recovery speed, long-term mean and volatility, respectively, and are constants. The above formula shows that the interest rate R_t fluctuates up and down near the long-term mean μ, the parameter α indicates the rate at which the interest rate returns to the μ. However, unlike the fluctuation of interest rate in the Vasicek model, the CIR model sets the fluctuation of interest rate to be proportional to the square root of interest rate level, and the variance of volatility increases with the increase of interest rate itself. Under this model, for any time $s, t(s \leq t)$, when the short-term interest rate information at time s is known, the instantaneous short-term interest rate R_t at time t obeys the non-central distribution χ^2.

Due to

$$m(x) = \frac{1}{\sigma^2 x} \exp(2 \int_z^x \frac{\alpha(\mu - y)}{\sigma^2 y} dy) = \frac{1}{\sigma^2 z^{\frac{\mu\alpha}{\sigma^2}}} x^{\frac{\mu\alpha - \sigma^2}{\sigma^2}} \exp\{-\frac{\alpha}{\sigma^2} x + \frac{\alpha}{\sigma^2} z\},$$

we have
$$M=\int_0^\infty m(x)dx=\frac{1}{\alpha z^{\frac{\mu\alpha}{\sigma^2}}}e^{\frac{\alpha}{\sigma^2}z}\Gamma(\frac{\mu\alpha}{\sigma^2})<\infty.$$
This means that the above diffusion bridge simulations can be used CIR model.

Let W^1 and W^2 be two independent standard Wiener processes and define R^1 and R^2 as the solutions to
$$dR_t^i=\alpha(\mu-R_t^i)dt+\sigma\sqrt{R_t^i}dW_t^i \tag{2.8}$$
where $i=1,2$. $R_0^1=a$ and $R_{t_n}^2=b$, the process R^1 from a forward in time starting at time 0, and R^2 from b backward in time starting at time t_n. Let $Y_{\delta i}^1, i=1,2,\cdots,N$ and $Y_{\delta i}^2, i=1,2,\cdots,N$ be (independent) simulations of R^1 and R^2 in $[0,t_n]$ with step size $\delta=t_n/N$. Then a simulation of an approximation to a $a(0,a,t_n,b)$-bridge is obtained by the following rejection sampling scheme. Keep simulating Y^1 and Y^2 until the sample paths cross, that is, until there is an i such that either $Y_{\delta i}^1\geq Y_{\delta(N-i)}^2$ and $Y_{\delta(i+1)}^1\geq Y_{\delta(N-(i+1))}^2$ or $Y_{\delta i}^1\leq Y_{\delta(N-i)}^2$ and $Y_{\delta(i+1)}^1\leq Y_{\delta(N-(i+1))}^2$. Once a trajectory crossing has been obtained, define
$$B_{\delta i}=\begin{cases}Y_{\delta i}^1, & \text{for} \quad i=0,1,\cdots,\nu-1\\ Y_{\delta(N-i)}^2, & \text{for} \quad i=\nu,\cdots,N\end{cases} \tag{2.9}$$
where $\nu=\min\{i\in 1,\cdots,N|Y_{\delta i}^1\leq Y_{\delta(N-i)}^2\}$ if $Y_0^1\geq Y_{t_n}^2$, and $\nu=\min\{i\in 1,\cdots,N|Y_{\delta i}^1\geq Y_{\delta(N-i)}^2\}$ if $Y_0^1\leq Y_{t_n}^2$. Then B approximates a $(0,a,t_n,b)$-bridge.

3 The Likelihood Function and the EM-Algorithm

In this section, we present an EM-algorithm for finding the maximum likelihood estimator for discretely observed CIR processes. Suppose that the only data available from a realization of CIR model are observations a times $t_1<t_2<\cdots<t_n$, $R_i=R_{t_i}$, $i=1,2,\cdots,n$. Discrete time observation of continous time process can be viewed as an incomplete observation problem, so it is not straightforward to implement the EM-algorithm. We summarize a modification of an approach in the spirit of Roberts and Stramer[4] using our diffusion simulation technique.

3.1 Likelihood function with full diffusion observation

In order to obtain a likelihood function, we use the standard h-transformation [5] to the stochastic differential equation (2.7)
$$h(x;\psi)=\int_0^x\frac{1}{\sigma u^{1/2}}du=\frac{2}{\sigma}x^{1/2}, \tag{3.1}$$

where $\psi = (\alpha, \mu, \sigma)$.Define $U_t = h(R_t; \psi)$. By Itô formula, U_t satisfies the stochastic differential equaiton

$$dU_t = m(U_t; \psi)dt + dW_t \tag{3.2}$$

with

$$m(U_t; \psi) = \frac{2\alpha\mu}{\sigma^2}U_t^{-1} - \frac{\alpha}{2}U_t - \frac{1}{2}U_t^{-1}.$$

Since $R_i = \frac{4U_{t_i}}{\sigma^2}$, we will think of the full dataset as $U_t, t \in [0, t_n 0$ and $R = (R_1, R_2, \cdots, R_n)$. To get the likelihood function of the sample path of U_n in $[0, t_n]$, first we calculate the distribution of formula (3.2) on the interval $[t_{i-1}, t_i]$, we find that

$$U_t \sim N(\int_{t_{i-1}}^{t_i} [\frac{2\alpha\mu}{\sigma^2}U_s^{-1} - \frac{\alpha}{2}U_s - \frac{1}{2}U_s^{-1}]ds, t_i - t_{i-1})$$

So we have the likelihood function of the R_n sample path is

$$\begin{aligned} &L(R_1, R_2, \cdots, R_n | U_t, t \in [0, t_n]) \\ &= \prod_{i=1}^{n} \phi(\frac{2}{\sigma}\sqrt{R_i}; \int_{t_{i-1}}^{t_i} [\frac{2\alpha\mu}{\sigma^2}U_s^{-1} - \frac{\alpha}{2}U_s - \frac{1}{2}U_s^{-1}]ds, t_i - t_{i-1}), \end{aligned} \tag{3.4}$$

where $\phi(u; a_1, a_2)$ denotes the density of the normal distribution with mean a_1 and variance a_2 evaluated at u.

Let P_ψ be the probability measure induced by $U = \{U_t\}_{t\in[0,t_n]}$ on (C, C), i.e. the probability measure with respect to which the coordinate process has the same distribution as U, and let Q be the Wiener measure on (C, C). We assume that the coefficient $m(U_t; \psi)$ satisfies conditions ensuring that the Girsanov theorem holds so that we have the Radon-Nykodym derivative

$$\frac{dP_\psi}{dQ}(B) = \exp\{\int_0^{t_n} m(B_t; \psi)dB_t - \frac{1}{2}\int_0^{t_n} m^2(B_t; \psi)dt\}. \tag{3.5}$$

The evaluation of $\frac{dP_\psi}{dQ}(B)$ is difficult because of the Ito integral term. To simplify the likelihood function, we apply the transformation

$$\begin{aligned} a(U_t; \psi) &= \int_0^{U_t} m(u; \psi)du = \int_0^{U_t} (\frac{2\alpha\mu}{\sigma^2}u^{-1} - \frac{\alpha}{2}u - \frac{1}{2}u^{-1})du \\ &= (\frac{2\alpha\mu}{\sigma^2} - \frac{1}{2})\log U_t - \frac{\alpha}{4}U_t^2. \end{aligned} \tag{3.7}$$

By Itô formula

$$\int_0^{t_n} m(B_t; \psi)dB_t = a(B_{t_n}; \psi) - a(B_0; \psi) - \frac{1}{2}\int_0^{t_n} m'(B_t; \psi)dt \tag{3.8}$$

where m' denotes the derivative of $m(u;\psi)$ w.r.t. u. Inserting equation (3.8) in the equation (3.5), we obtain that

$$\frac{dP_\psi}{dQ}(B) = \exp\{a(B_{t_n};\psi) - a(B_0;\psi) - \frac{1}{2}\int_0^{t_n}[m^2(B_t;\psi) + m'(B_t;\psi)]dt\} \quad (3.9)$$

Combining (3.9) and (3.4), we see that the likelihood function for ψ based on the full data set U_t, $t \in [0, t_n]$ and $R = (R_1, \cdots, R_n)$ is given by

$$\begin{aligned} &L(\psi; R_1, R_2, \cdots, R_n; t \in [0, t_n]) \\ &= \prod_{i=1}^{n} \phi(\frac{2}{\sigma}\sqrt{R_i}; \int_{t_{i-1}}^{t_i} (\frac{2\alpha\mu}{\sigma^2}U_s^{-1} - \frac{\alpha}{2}U_s - \frac{1}{2}U_t^{-1})ds, t_i - t_{i-1}) \\ &\cdot \exp\{a(U_{t_n};\psi) - a(U_0;\psi) - \frac{1}{2}\int_0^{t_n}[m(U_t;\psi)^2 + m'(U_t;\psi)]ds\} \end{aligned}$$

Taking the logarithm of this equation, we obtain that

$$\begin{aligned} &\log L(\psi; R_1, R_2, \cdots, R_n; t \in [0, t_n]) \\ &= \sum_{i=1}^{n} -\frac{(\frac{2}{\sigma}\sqrt{R_i} - \int_{t_{i-1}}^{t_i} \frac{2\alpha\mu}{\sigma^2}U_s^{-1}ds + \frac{\alpha}{2}\int_{t_{i-1}}^{t_i} U_s ds + \frac{1}{2}\int_{t_{i-1}}^{t_i} U_s^{-1}ds)^2}{2(t_i - t_{i-1})} \\ &- \frac{n}{2}\log[2\pi(t_i - t_{i-1})] + \frac{4\alpha\mu - \sigma^2}{2\sigma^2}(\log U_{t_n} - \log U_0) + \frac{1}{4}(U_0^2 - U_{t_n}^2) \\ &- \frac{1}{2}\int_0^{t_n} \frac{16\alpha^2\mu^2 - 16\alpha\mu\sigma^2 + \sigma^4}{4\sigma^4}U_t^{-2}dt - \frac{2+\alpha^2}{8}\int_0^{t_n} U_t^2 dt + \frac{\alpha^2\mu}{\sigma^2}t_n. \quad (3.10) \end{aligned}$$

3.2 EM Algorithm

We can now apply the EM-algorithm to the full log-likelihood function (3.10) to obtain the maximum likelihood estimate of the parameter ψ. As the initial value for the algorithm, let $\hat{\psi}$ be any value of the parameter vector $\psi \in [\alpha, \mu, \sigma] \in \Psi \times (0, \infty)$. Then the EM-algorithm works as follow.

1. E-STEP

Generate M sample paths of the diffusion bridge process $\{R_t\}, R_t^{(k)}, k = 1, 2, \cdots, M$, here we modify method in Baltazar-Larios and Sϕrensen (2010). Substitute $U_t = \frac{2}{\sigma}R_t^{1/2}$ into equation (3.10), we get the objective function

$$g(\theta) = -\frac{1}{2(M-M_0)dt}\sum_{k=M_0+1}^{M}\sum_{i=1}^{n}(\frac{2}{\sigma}\sqrt{R_i} - \frac{\alpha\mu}{\sigma}\int_{t_{i-1}}^{t_i}(R_t^{(k)})^{-1/2}dt + \frac{\alpha}{\sigma}\int_{t_{i-1}}^{t_i}(R_t^{(k)})^{1/2}dt$$

$$+ \frac{\sigma}{4}\int_{t_{i-1}}^{t_i}(R_t^{(k)})^{-1/2}dt)^2 - \frac{n}{2}\log(2\pi dt) + \frac{1}{M-M_0}\sum_{k=M_0+1}^{M}\frac{4\alpha\mu - \sigma^2}{4\sigma^2}(\log R_{t_n}^{(k)} - \log R_0^{(k)})$$

$$+\frac{1}{M_0-M}\sum_{k=M_0+1}^{M}\frac{1}{\sigma^2}(R_0^{(k)}-R_{t_n}^{(k)})-\frac{1}{M_0-M}\sum_{k=M_0+1}^{M}\frac{16\alpha^2\mu^2-16\alpha\mu\sigma^2+\sigma^4}{32\sigma^2}\int_0^{t_n}(R_t^{(k)})^{-1}dt$$
$$-\frac{1}{M_0-M}\sum_{k=M_0+1}^{M}\frac{2+\alpha^2}{2\sigma^2}\int_0^{t_n}R_t^{(k)}dt+\frac{\alpha^2\mu}{\sigma^2}t_n, \tag{3.11}$$

where $dt = t_i - t_{i-1}$, and because the Markov chain is stable after a period of time, so M_0 represents to remove the first M_0 paths of M paths when estimating.

2. M-STEP

The maximun $\hat{\sigma}, \hat{\mu}, \hat{\alpha}$ is obtained as the solution to the following equations

$$\frac{\partial g(\theta)}{\partial\mu}=\frac{2\alpha}{\sigma^2}a_1-\frac{\alpha^2\mu}{\sigma^2}a_2+\frac{\alpha^2}{\sigma^2}a_3+\frac{\alpha^2}{\sigma^2}a_3+\frac{\alpha}{4}a_2$$
$$+\frac{\alpha}{\sigma^2}b_1-\frac{2\mu\alpha^2-\alpha\sigma^2}{2\sigma^2}b_2+\frac{\alpha^2}{\sigma^2}t_n=0$$
$$\frac{\partial g(\theta)}{\partial\alpha}=-\frac{2}{\sigma^2}a_4+\frac{2\mu}{\sigma^2}a_1+\frac{2\alpha\mu}{\sigma^2}a_3-\frac{\alpha\mu^2}{\sigma^2}a_2-\frac{\alpha}{\sigma^2}a_5$$
$$-\frac{1}{4}a_3+\frac{\mu}{4}a_2+\frac{\mu}{\sigma^2}b_1-\frac{2\alpha\mu^2-\mu\sigma^2}{2\sigma^2}b_2-\frac{\alpha}{\sigma^2}b_3+\frac{2\alpha\mu}{\sigma^2}t_n=0$$
$$\frac{\partial g(\theta)}{\partial\sigma}=\frac{4}{\sigma^3}a_6-\frac{4\alpha\mu}{\sigma^3}a_1+\frac{4\alpha}{\sigma^3}a_4+\frac{\alpha^2\mu^2}{\sigma^3}a_2-\frac{2\alpha^2\mu}{\sigma^3}a_3$$
$$+\frac{\alpha^2}{\sigma^3}a_5-\frac{\sigma}{16}a_2-\frac{2\alpha\mu}{\sigma^3}b_1-\frac{2}{\sigma^3}b_4+(\frac{\alpha^2\mu^2}{\sigma^3}-\frac{\sigma^2}{16})b_2+\frac{2+\alpha^2}{\sigma^3}b_3$$
$$-\frac{2\alpha^2\mu}{\sigma^3}t_n=0. \tag{3.12}$$

where

$$a_1=\frac{1}{(M-M_0)dt}\sum_{k=M_0+1}^{M}\sum_{i=1}^{n}\sqrt{R_i}\int_{t_{i-1}}^{t_i}(R_t^{(k)})^{-1/2}dt$$
$$a_2=\frac{1}{(M-M_0)dt}\sum_{k=M_0+1}^{M}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(R_t^{(k)})^{-1/2}dt\int_{t_{i-1}}^{t_i}(R_t^{(k)})^{-1/2}dt$$
$$a_3=\frac{1}{(M-M_0)dt}\sum_{k=M_0+1}^{M}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(R_t^{(k)})^{1/2}dt\int_{t_{i-1}}^{t_i}(R_t^{(k)})^{-1/2}dt$$
$$a_4=\frac{1}{(M-M_0)dt}\sum_{k=M_0+1}^{M}\sum_{i=1}^{n}\sqrt{R_i}\int_{t_{i-1}}^{t_i}(R_t^{(k)})^{1/2}dt$$
$$a_5=\frac{1}{(M-M_0)dt}\sum_{k=M_0+1}^{M}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(R_t^{(k)})^{1/2}dt\int_{t_{i-1}}^{t_i}(R_t^{(k)})^{1/2}dt$$

$$a_6 = \frac{1}{(M-M_0)dt}\sum_{i=1}^{n} R_i$$

$$b_1 = \frac{1}{M-M_0}\sum_{k=M_0+1}^{M} [\log R_{t_n}^{(k)} - \log R_0^{(k)}]$$

$$b_2 = \frac{1}{M-M_0}\sum_{k=M_0+1}^{M} \int_0^{t_n} (R_t^{(k)})^{-1} dt$$

$$b_3 = \frac{1}{M-M_0}\sum_{k=M_0+1}^{M} \int_0^{t_n} R_t^{(k)} dt$$

$$b_4 = \frac{1}{M-M_0}\sum_{i=1}^{M} (R_0^{(k)} - R_{t_n}^{(k)}).$$

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Received: July 27, 2020; Published: August 17, 2020