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Fuzzy Foldness of BCI - Implicative Ideals in BCI - Algebras

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Abstract

The purpose of this paper to introduce new notions of (fuzzy) *n*-fold BCI - implicative ideals (fuzzy) *n*-fold weak BCI -implicative ideals in BCI -algebras and (fuzzy) n-fold weak BCI -implicative (weak) ideals in BCI-algebras. Moreover, investigate several properties of the foldness theory of BCI - implicative ideals in BCI - algebras. Finally, we construct a computer - a program for studying the foldness theory of BCI -implicative ideals in BCI - algebras.

Keywords: BCK/BCI algebras, BCI -implicative ideals of BCI-algebras, fuzzy BCI, the implicative ideal of BCI-algebra, Fuzzy point, (fuzzy) n-fold BCI-implicative ideals, (fuzzy) n-fold weak BCI-implicative ideals.

1. Introduction

The study of BCK/BCI-algebras was initiated by Iséki [3] as a generalization of the concept of set-theoretic difference, and propositional calculus has become the inspiration for many theoretical and applied studies. In (1965), Zadeh [12] was introduced the notion of a fuzzy subset of a set as a method for representing uncertainty. In 1991, Xi [11] defined fuzzy subsets in BCK/BCI-algebras.

Huang and Chen [1] introduced the notions of n-fold implicative ideal and n-fold (weak) commutative ideals. Jun [4] discussed the fuzzification of *n*-fold positive implicative, commutative, and implicative ideal of BCK-algebras. Muhiuddin, Kim, and Jun [7] discussed Implicative N–ideals of BCK–algebras based on Neutrosophic N–structures. Muhiuddin, Jun [8] gives other Results of Neutrosophic subalgebras in BCK/BCI -algebras based on Neutrosophic points. In this paper, we redefined BCI -implicative ideals of BCI -algebras and studied the foldness theory of fuzzy BCI -implicative ideals, BCI -implicative weak ideals, weak fuzzy BCI -implicative ideals, and weak BCI -implicative weak ideals in BCI -algebras. This theory can be considered as a natural generalization of BCI -implicative ideals. Indeed, given any BCI – algebras X, we use the concept of fuzzy point to characterize *n*-fold BCI -implicative ideals in X. Finally, we construct a computer - the program for studying foldness theory of BCI - implicative ideals in BCI - algebras.

2. Preliminaries

Definition 2.1[4]: Let X be an asset with binary be operation $*$ and a constant 0. Then $(X;*,0)$ is called a BCI – algebra if it satisfies the following conditions:

For any $x, y, z \in X$

BCI-1. $((x*y)*(x*z))*(z*y)=0$;

BCI-2. $(x*(x*y))*y=0$;

BCI-3. $x*x=0$;

BCI-4. $x*y=0$ and $y*x=0 \Rightarrow x=y$

A binary relation $\leq$ can be defined by

BCI-5. $x \leq y \Leftrightarrow x*y=0$,

then $(X,\leq)$ is a partially ordered set with least element 0.

The following properties also hold in any BCI-algebra ([6], [11]):

1. $x*0=x$;
2. $x*y=0$ and $y*z=0 \Rightarrow x*z=0$;
3. $x*y=0 \Rightarrow (x*z)*(y*z)=0$ and $(z*y)*(z*x)=0$;
4. $(x*y)*z=(x*z)*y$;
5. $(x*y)*x=0$;
6. $x*(x*(x*y))=x*y$; let $(X,*,0)$ be a BCI-algebra.

Definition 2.1 (Zadeh [14]). A fuzzy subset of a BCK/BCI - algebra X is a function $\mu: X \rightarrow [0,1]$.

Definition 2.2 (C. Lele, C. Wu, P. Weke, T. Mamadou, and C.E. Njock [7]). Let ξ be the family of all fuzzy sets in X . For $x \in X$ and $\lambda \in (0,1], x_\lambda \in \xi$ is a fuzzy point iff

$$x_\lambda(y) = \begin{cases} \lambda & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $\tilde{X} = \{x_\lambda : x \in X \ , \lambda \in (0,1]\}$ the set of all fuzzy points on X , and we define a binary operation on $\tilde{X}$ as follows

$$x_\lambda * y_\mu = (x * y)_{\min(\lambda,\mu)}$$

Remark 2.4 (C. Lele, C. Wu, P. Weke, T. Mamadou, and C. E. Njock [5]), the following conditions hold:

$\forall x_\lambda, y_\mu, z_\alpha \in \tilde{X}$

BCI-1'. $((x_\lambda * y_\mu) * (x_\lambda * z_\alpha)) * (z_\alpha * y_\mu) = 0_{\min(\lambda,\mu,\alpha)}$;

BCI-2'. $(x_\lambda * (x_\lambda * y_\mu)) * y_\mu = 0_{\min(\lambda,\mu)}$;

BCI-3'. $x_\lambda * x_\mu = 0_{\min(\lambda,\mu)}$;

BCK-5'. $0_\mu * x_\lambda = 0_{\min(\lambda,\mu)}$;

Remark 2.3 (C. Lele, C. Wu, P. Weke, T. Mamadou and C. E. Njock [5]). The condition BCI-4 is not valid $(\tilde{X}, *)$. So the partial order $\leq$ $(X, *)$ cannot be extended to $(\tilde{X}, *)$.

We can also establish the following conditions $\forall x_\lambda, y_\mu, z_\alpha \in \tilde{X}$:

1'. $x_\lambda * 0_\mu = x_{\min(\lambda,\mu)}$;

2'. $x_\lambda * y_\mu = 0_{\min(\lambda,\mu)}$ and $y_\mu * z_\alpha = 0_{\min(\mu,\alpha)} \Rightarrow x_\lambda * z_\alpha = 0_{\min(\lambda,\alpha)}$;

3'. $x_\lambda * y_\mu = 0_{\min(\lambda,\mu)} \Rightarrow (x_\lambda * z_\alpha) * (y_\mu * z_\alpha) = 0_{\min(\lambda,\mu,\alpha)}$ and
$(z_\alpha * y_\mu) * (z_\alpha * x_\lambda) = 0_{\min(\lambda,\mu,\alpha)}$;

4'. $(x_\lambda * y_\mu) * z_\alpha = (x_\lambda * z_\alpha) * y_\mu$;

5'. $(x_\lambda * y_\mu) * x_\lambda = 0_{(\lambda,\mu)}$;

6'. $x_\lambda * (x_\lambda * (x_\lambda * y_\mu)) = x_\lambda * y_\mu$;

We recall that if A is a fuzzy subset of a BCK/BCI algebra X , then we have the following:

$\tilde{A} = \{x_\lambda \in \tilde{X} : A(x) \geq \lambda \ , \ \lambda \in (0,1]\}$. (i)

$\forall \lambda \in (0,1], \tilde{X}_\lambda = \{x_\lambda : x \in X\}$, and $\tilde{A}_\lambda = \{x_\lambda \in \tilde{X}_\lambda : A(x) \geq \lambda\}$ (ii)

One can easily check that $(\tilde{X}_\lambda ; * , 0_\lambda)$ it is a BCK-algebra.

Definition 2.4 (Isèki [2]). A nonempty subset of BCK/BCI -algebra X is called an ideal of X if it satisfies
1. $0 \in I$;
2. $\forall x, y \in X$, $(x * y \in I$ and $y \in I) \Rightarrow x \in I$

Definition 2.5 4 (Liu and Meng [6]). A nonempty subset I of BCI - algebra X is called a BCI -implicative ideal if it satisfies:
1. $0 \in I$;
2. $\forall x, y, z \in X$

$$((((x * y) * y) * (0 * y)) \in I \text{ and } z \in I) \Rightarrow x * ((y * (y * x))(0 * (0 * (x * y)))) \in I$$

Definition 2.6 (Xi [13]). A fuzzy subset A of a BCK/BCI algebra X is a fuzzy ideal iff
1. $\forall x \in X$, $A(0) \geq A(x)$;
2. $\forall x, y \in X$, $A(x) \geq \min(A(x * y), A(y))$.

Definition 2.7 (Xi [13]). A fuzzy subset A of a BCI-algebra X is called a fuzzy BCI -implicative ideal of X iff.
1. $\forall x \in X$, $A(0) \geq A(x)$;
2. $\forall x, y, z \in X$

$$A\left(x * \left((y * (x * y)) * \left(0 * \left(0 * (x * y)\right)\right)\right)\right) \geq \min\left\{A\left(((x * y) * y) * (0 * y)\right), A(z)\right\}$$

Definition 2.8 (C. Lele, C. Wu, P. Weke, T. Mamadou, and C.E. Njock [5]). $\tilde{A}$ is a weak ideal of $\tilde{X}$ iff
1. $\forall \nu \in \operatorname{Im}(A)$; $0_\nu \in \tilde{A}$;
2. $\forall x_\lambda, y_\mu \in X$. Such that $x_\lambda * y_\mu \in \tilde{A}$ and $y_\mu \in \tilde{A}$ we have $x_{\min(\lambda,\mu)} \in \tilde{A}$.

Theorem 2.9 (C. Lele, C. Wu, P. Weke, T. Mamadou and C.E. Njock [5]). Suppose that A is a fuzzy subset of a BCK-algebra X , then the following conditions are equivalent:

1. A is a fuzzy ideal ;
2. $\forall x_\lambda, y_\mu \in \tilde{A}$, $(z_\alpha * y_\mu) * x_\lambda = 0_{\min(\lambda,\mu,\alpha)} \Rightarrow z_{\min(\lambda,\mu,\alpha)} \in \tilde{A}$;

3. $\forall t \in (0,1]$, the t-level subset $A^t = \{x \in X : A(x) \geq t\}$ in an ideal when $A^t \neq \phi$
4. $\tilde{A}$ is a weak ideal.

3. Fuzzy *N*-fold BCI – implicative ideal in BCI- Algebras

Throughout this paper $\tilde{X}$ is the set of fuzzy points on BCI-algebra X and $n \in \mathrm{N}$ (where N the set of all the natural numbers).

Let us denote $(\cdots((x*y)*y)*\cdots)*y$ by $x*y^n$

Moreover, $(\cdots((x_{\min(\lambda,\mu)}*0_\mu)*0_\mu)*\cdots)*0_\mu$ by $x_\lambda * y_\mu^n$ (where y and y_μ occurs respectively n times) with $x, y \in X$, $x_\lambda, y_\lambda \in \tilde{X}$.

Definition 3.1. A nonempty subset I of a BCI - algebra X is an *n*-fold BCI - implicative ideal of X if it satisfies :

1. $0 \in I$;
2. $\forall x, y, z \in X$,.

$$((((x*y)*y)*(0*y)) \in I \text{ and } z \in I) \Rightarrow x*\Big((y*(y*x))*\big(0*\big(0*(x*y^n)\big)\big)\Big) \in I$$

Definition 3.2 A fuzzy subset A of X is called a fuzzy n-fold BCI -implicative ideal of X if it satisfies :

1. $\forall x \in X$, $A(0) \geq A(x)$;
2. $\forall x, y, z \in X$,

$$A\Big(x*\Big((y*(x*y))*\big(0*\big(0*(x*y^n)\big)\big)\Big)\Big) \geq \min\Big\{A\big(((x*y)*y)*(0*y)\big), A(z)\Big\}$$

Definition 3.3. $\tilde{A}$ is BCI -the implicative weak ideal of $\tilde{X}$ iff

1. $\forall \nu \in \operatorname{Im}(A)$, $0_\nu \in \tilde{A}$;
2. $\forall x_\lambda, y_\mu, z_\alpha \in \tilde{X}$

$$((((x_\lambda * y_\mu)*y_\mu)*(0_\alpha * y_\mu)) \in \tilde{A} \text{ and } z_\alpha \in \tilde{A})$$

$$\Rightarrow x_\lambda * \Big((y_\mu * (y_\mu * x_\lambda)) * \big(0_\alpha * \big(0_\alpha * (x_\lambda * y_\mu)\big)\big)\Big) \in \tilde{A}$$

Definition 3.4. $\tilde{A}$ is an n-fold BCI -implicative weak ideal of $\tilde{X}$ iff

1. $\forall \nu \in \operatorname{Im}(A)$, $0_\nu \in \tilde{A}$;

2. $\forall x_\lambda, y_\mu, z_\alpha \in \tilde{X}$

$$(\left(\left(\left(x_\lambda * y_\mu\right) * y_\mu\right) * \left(0_\alpha * y_\mu\right)\right) \in \tilde{A} \text{ and } z_\alpha \in \tilde{A})$$

$$\Rightarrow x_\lambda * \left(\left(y_\mu * \left(y_\mu * x_\lambda\right)\right) * \left(0_\alpha * \left(0_\alpha * \left(x_\lambda * y^n{}_\mu\right)\right)\right)\right) \in \tilde{A}$$

Example 3.5. Example 3.17. Let $X = \{0, a, b, c\}$ be a BCI-algebra with Cayley table as follows:

*	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	c	c	0

Let A be a fuzzy set in X defined by $A(0) = A(a) = A(b) = 1$ and $A(c) = t$, where $t = [0,1)$. One can quickly check that for $n > 2$

$$\tilde{A} = \{0_\lambda : \lambda \in (0,1]\} \bigcup \{a_\lambda : \lambda \in (0,1]\} \bigcup \{b_\lambda : \lambda \in (0,1]\} \bigcup \{c_\lambda : \lambda \in [0,1)\}$$

It is an n-fold BCI - implicative weak ideal.

Remark 3.6. $\tilde{A}$ It is a 1-fold BCI - the implicative weak ideal of a BCK-algebra $\tilde{X}$ iff $\tilde{A}$ is BCI - the implicative weak ideal of $\tilde{X}$.

Theorem 3.7. If A it is a fuzzy subset of X , then A is a fuzzy n-fold BCI - implicative ideal iff $\tilde{A}$ is an n-fold BCI - implicative weak ideal.

Proof. $\Rightarrow$ - Let $\lambda \in \operatorname{Im}(A)$, it is easy to prove that $0_\lambda \in \tilde{A}$;

- Let $(\left(\left(\left(x_\lambda * y_\mu\right) * y_\mu\right) * \left(0_\alpha * y_\mu\right)\right) \in \tilde{A}$ and $z_\alpha \in \tilde{A})$, thus

$$A\ (\left(\left(\left(x_\lambda * y_\mu\right) * y_\mu\right) * \left(0_\alpha * y_\mu\right)\right) \geq \min\left(\lambda, \mu, \alpha\right) \text{ and } A\left(z_\alpha\right) \geq \alpha \quad .$$

Since A it is a fuzzy n-fold BCI - implicative ideal, we have

$$A\left(x * \left(\left(y * \left(x * y\right)\right) * \left(0 * \left(0 * \left(x * y^n\right)\right)\right)\right)\right) \geq \min\left\{A\left(\left(\left(x * y\right) * y\right) * \left(0 * y\right)\right), A\left(z\right)\right\}$$

$$\geq \min\left(\min\left(\lambda, \mu, \alpha\right), \alpha\right) = \min(\lambda, \mu, \alpha) \ .$$

Therefore $\left(x * \left(\left(y * \left(y * x\right)\right) * \left(0 * \left(0 * \left(x * y^n\right)\right)\right)\right)\right)_{\min(\lambda,\mu,\alpha)} =$

$$x_\lambda * \left(\left(y_\mu * \left(y_\mu * x_\lambda\right)\right) * \left(0_\alpha * \left(0_\alpha * \left(x_\lambda * y^n{}_\mu\right)\right)\right)\right) \in \tilde{A}$$

$\Leftarrow$ - Let $x \in X$, it is easy to prove that $A(0) \geq A(x)$;

- Let $x, y, z \in X$ and let

$$A\left(\left(\left(\left(x_{\lambda} * y_{\mu}\right) * y_{\mu}\right) * \left(0_{\alpha} * y_{\mu}\right)\right)\right) = \beta \text{ and } A(z) = \alpha \text{, then}$$

$$\left(\left(\left(\left(x * y\right) * y\right) * \left(0 * y\right)\right)_{\min(\beta,\alpha)} = \left(\left(\left(x_{\lambda} * y_{\mu}\right) * y_{\mu}\right) * \left(0_{\alpha} * y_{\mu}\right)\right) \in \tilde{A}$$

and $z_{\alpha} \in \tilde{A}$ Since $\tilde{A}$ is BCI - implicative weak ideal, we have

$$x_{\beta} * \left(\left(y_{\beta} * \left(y_{\beta} * x_{\beta}\right)\right) * \left(0_{\alpha} * \left(0_{\alpha} * \left(x_{\beta} * y_{\beta}^{\ n}\right)\right)\right)\right) =$$

$$\left(x * \left(\left(y * \left(y * x\right)\right) * \left(0 * \left(0 * \left(x * y^{n}\right)\right)\right)\right)\right)_{\min(\beta,\alpha)} \in \tilde{A} \text{ .}$$

$$\text{Thus } A\left(x * \left(\left(y * \left(x * y\right)\right) * \left(0 * \left(0 * \left(x * y^{n}\right)\right)\right)\right)\right) \geq \min(\beta,\alpha)$$

$$= \min\left\{A\left(\left(\left(x * y\right) * y\right) * \left(0 * y\right)\right), A(z)\right\} \text{ .}$$

Proposition 3.8. An n-fold BCI - the implicative weak ideal is a weak ideal.

Proof. Let $y_{\mu} = 0_{\alpha}$, $x_{\lambda}, y_{\mu} \in \tilde{A}$, and let

$x_{\lambda} * z_{\alpha} = \left(\left(\left(\left(x_{\lambda} * 0_{\alpha}\right) * 0_{\alpha}\right) * \left(0_{\alpha} * 0_{\alpha}\right)\right) * z_{\alpha} \in \tilde{A} \text{ and } z_{\alpha} \in \tilde{A}\right)$, since $\tilde{A}$ n-fold BCI - implicative weak ideal, we have

$$x_{\lambda} * \left(\left(0_{\alpha} * \left(0_{\alpha} * x_{\lambda}\right)\right) * \left(0_{\alpha} * \left(0_{\alpha} * \left(x_{\lambda} * 0_{\alpha}^{\ n}\right)\right)\right)\right) = \quad x_{\min(\lambda,\alpha)}$$

Thus $\tilde{A}$ is a weak ideal.

Corollary 3.9. A fuzzy n-fold BCI - implicative ideal is a fuzzy ideal.

Theorem 3.10. Let $\{\tilde{A}_{i \in I}\}$ be a family of *n*-fold BCI - implicative weak ideals and $\{A_{i \in I}\}$ be a family of fuzzy *n*-fold BCI - implicative ideals. Then: $(1) \bigcap_{i \in I} \tilde{A}_i$ is an *n*-fold BCI - implicative weak ideal.

(2))$\bigcup_{i \in I} \tilde{A}_i$ is an *n*-fold BCI - implicative weak ideal.

(3))$\bigcap_{i \in I} A_i$ is a fuzzy *n*-fold BCI - implicative ideal.

(4))$\bigcup_{i \in I} A_i$ is a fuzzy *n*-fold BCI - implicative ideal.

Proof. (1) $\forall \lambda \in \operatorname{Im}\left(\bigcap_{i\in I} \tilde{A}_i\right)$, then $\lambda \in \operatorname{Im}(\tilde{A}_i), \forall i$, so, $0_\lambda \in \tilde{A}_i, \forall i$, i.e. $0_\lambda \in \bigcap_{i\in I} \tilde{A}_i$. For every $x_\mu, y_\lambda, z_\alpha \in \tilde{X}$, if

$$\left(\left(\left(\left(x_\lambda * y_\mu\right) * y_\mu\right) * \left(0_\alpha * y_\mu\right)\right) \in \bigcap_{i\in I} \tilde{A}_i \text{ and } z_\alpha \in \bigcap_{i\in I} \tilde{A}_i\right) \text{, then}$$

$$\left(\left(\left(\left(x_\lambda * y_\mu\right) * y_\mu\right) * \left(0_\alpha * y_\mu\right)\right) \in \tilde{A}_i \text{ and } z_\alpha \in \tilde{A}_i\right) \ \forall i \text{, thus}$$

$$\Rightarrow x_\lambda * \left(\left(y_\mu * \left(y_\mu * x_\lambda\right)\right) * \left(0_\alpha * \left(0_\alpha * \left(x_\lambda * y^n{}_\mu\right)\right)\right)\right) \in \tilde{A}_i \ \forall i$$

So $\Rightarrow x_\lambda * \left(\left(y_\mu * \left(y_\mu * x_\lambda\right)\right) * \left(0_\alpha * \left(0_\alpha * \left(x_\lambda * y^n{}_\mu\right)\right)\right)\right) \in \bigcap_{i\in I} \tilde{A}_i$. Thus $\bigcap_{i\in I} \tilde{A}_i$ is an *n*-fold BCI - implicative weak ideals.

(2) . $\forall \lambda \in \operatorname{Im}\left(\bigcup_{i\in I} \tilde{A}_i\right)$, then $\exists i_0 \in I, such, that\ \lambda \in \tilde{A}_{i_0}$, so, $0_\lambda \in \tilde{A}_{i_0}$, i.e. $0_\lambda \in \bigcup_{i\in I} \tilde{A}_i$. For every $x_\mu, y_\lambda, z_\alpha \in \tilde{X}$, if

$$\left(\left(\left(\left(x_\lambda * y_\mu\right) * y_\mu\right) * \left(0_\alpha * y_\mu\right)\right) \in \bigcup_{i\in I} \tilde{A}_i \text{ and } z_\alpha \in \bigcup_{i\in I} \tilde{A}_i\right) \text{, then}$$

$\exists i_0 \in I$ such that

$$\left(\left(\left(\left(x_\lambda * y_\mu\right) * y_\mu\right) * \left(0_\alpha * y_\mu\right)\right) \in \tilde{A}_{i_0} \text{ and } z_\alpha \in \tilde{A}_{i_0}\right) \text{, thus}$$

$$\Rightarrow x_\lambda * \left(\left(y_\mu * \left(y_\mu * x_\lambda\right)\right) * \left(0_\alpha * \left(0_\alpha * \left(x_\lambda * y^n{}_\mu\right)\right)\right)\right) \in \tilde{A}_{i_0}$$

So $\Rightarrow x_\lambda * \left(\left(y_\mu * \left(y_\mu * x_\lambda\right)\right) * \left(0_\alpha * \left(0_\alpha * \left(x_\lambda * y^n{}_\mu\right)\right)\right)\right) \in \bigcup_{i\in I} \tilde{A}_i$. Thus $\bigcup_{i\in I} \tilde{A}_i$ is an *n*-fold BCI - implicative weak ideals.

(3) Follows from (1) and Theorem 3.7.
(4) Follows from (2) and Theorem 3.7.

4. Fuzzy N-fold weak BCI – implicative ideals in BCI – Algebras

In this section, we define and give some characterizations of (fuzzy) n-fold weak BCI - implicative (weak) ideals in BCI-algebras.

Definition 4.1. A nonempty subset I of X is called an n-fold weak BCI - implicative ideal of X if it satisfies

1. $0 \in I$;
2. $\forall x, y, z \in \tilde{X}$

$$\left(\left(\left(\left(x * y\right) * y\right) * \left(0 * y^n\right)\right) \in I \text{ and } z \in I\right) \Rightarrow x * \left(\left(y * \left(y * x\right)\right) * \left(0 * \left(0 * \left(x * y\right)\right)\right)\right) \in I$$

Definition 4.2. A fuzzy subset A of X is called a fuzzy n-fold weak BCI - implicative ideal of X if it satisfies

1. $\forall x \in X$, $A(0) \geq A(x)$;
2. $\forall x, y, z \in \tilde{X}$

$$A\Big(x * \big((y * (x * y)) * (0 * (0 * (x * y)))\big)\Big) \geq \min\Big\{A\big(((x * y) * y) * (0 * y^n)\big), A(z)\Big\}$$

Definition 4.3. $\tilde{A}$ is a weak BCI - implicative weak ideal of $\tilde{X}$ iff

1. $\forall \nu \in \text{Im}(A)$, $0_\nu \in \tilde{A}$;
2. $\forall x_\lambda, y_\mu, z_\alpha \in \tilde{X}$ $\Big(\big(((x_\lambda * y_\mu) * y_\mu) * (0_\alpha * y_\mu)\big) \in \tilde{A}$ and $z_\alpha \in \tilde{A}\Big)$

$$\Rightarrow x_\lambda * \Big((y_\mu * (y_\mu * x_\lambda)) * \big(0_\alpha * (0_\alpha * (x_\lambda * y_\mu))\big)\Big) \in \tilde{A}$$

Definition 4.4. $\tilde{A}$ is an n-fold weak BCI - implicative weak ideal of $\tilde{X}$ iff

1. $\forall \nu \in \text{Im}(A)$, $0_\nu \in \tilde{A}$;
2. $\forall x_\lambda, y_\mu, z_\alpha \in \tilde{X}$,

$$\Big(\big(((x_\lambda * y_\mu) * y_\mu) * (0_\alpha * y^n_\mu)\big) \in \tilde{A} \text{ and } z_\alpha \in \tilde{A}\Big)$$

$$\Rightarrow x_\lambda * \Big((y_\mu * (y_\mu * x_\lambda)) * \big(0_\alpha * (0_\alpha * (x_\lambda * y_\mu))\big)\Big) \in \tilde{A} .$$

Example 4.5. Let $X = \{0,1\}$ in which $*$ is given by

$$1 * 0 = 1 \text{ and } 0 * 0 = 0 * 1 = 1 * 1 = 0$$

Then $(X ; *, 0)$ it is a BCK-algebra. Let $t_1, t_2 \in (0,1]$ and let us define a fuzzy subset $A : X \to [0,1]$ by

$$t_1 = A(0) > A(1) = t_2$$

It is easy to check that for any $n > 2$

$$\tilde{A} = \{0_\lambda : \lambda \in (0, t_1]\} \bigcup \{I_\lambda : \lambda \in (0, t_2]\}$$

It is an n-fold weak BCI - implicative weak ideal.

Remark 4.6. $\tilde{A}$ is a 1-fold weak BCI - the implicative weak ideal of a BCK-algebra X iff $\tilde{A}$ is a weak BCI - implicative weak ideal.

Theorem 4.7. If A it is a fuzzy subset of X , then A is a fuzzy n-fold weak BCI - implicative ideal iff $\tilde{A}$ is an n-fold weak BCI - implicative weak ideal.

Proof. $\Rightarrow$ - Let $\lambda \in \text{Im}(A)$, it is easy to prove that $0_\lambda \in \tilde{A}$;

- Let $\left(\left(\left(\left(x_\lambda * y_\mu\right) * y_\mu\right) * \left(0_\alpha * y^n{}_\mu\right)\right) \in \tilde{A}$ and $z_\alpha \in \tilde{A}\right)$, thus $A\left(\left(\left(\left(x_\lambda * y_\mu\right) * y_\mu\right) * \left(0_\alpha * y_\mu\right)\right) \geq \min(\lambda,\mu,\alpha)\right)$ and $A(z_\alpha) \geq \alpha$.

Since A it is a fuzzy n-fold BCI - implicative ideal, we have

$$A\left(x * \left(\left(y * (x * y)\right) * \left(0 * \left(0 * (x * y)\right)\right)\right)\right) \geq \min\left\{A\left(\left(\left(x * y\right) * y\right) * \left(0 * y^n\right)\right), A(z)\right\}$$

$$\geq \min\left(\min(\lambda,\mu,\alpha),\alpha\right) = \min(\lambda,\mu,\alpha) .$$

Therefore $\left(x * \left(\left(y * (y * x)\right) * \left(0 * \left(0 * (x * y)\right)\right)\right)\right)_{\min(\lambda,\mu,\alpha)} =$

$x_\lambda * \left(\left(y_\mu * \left(y_\mu * x_\lambda\right)\right) * \left(0_\alpha * \left(0_\alpha * \left(x_\lambda * y_\mu\right)\right)\right)\right) \in \tilde{A}$

$\Leftarrow$ - Let $x \in X$, it is easy to prove that $A(0) \geq A(x)$;

- Let $x, y, z \in X$ and let

$A\left(\left(\left(\left(x_\lambda * y_\mu\right) * y^n{}_\mu\right) * \left(0_\alpha * y_\mu\right)\right) = \beta\right.$ and $A(z) = \alpha$, then

$\left(\left(\left(\left(x * y\right) * y\right) * \left(0 * y^n\right)\right)\right)_{\min(\beta,\alpha)} =$

$\left(\left(\left(x_\lambda * y_\mu\right) * y_\mu\right) * \left(0_\alpha * y^n{}_\mu\right)\right) \in \tilde{A}$ and $z_\alpha \in \tilde{A}$ Since $\tilde{A}$ is BCI - implicative weak ideal, we have

$$x_\beta * \left(\left(y_\beta * \left(y_\beta * x_\beta\right)\right) * \left(0_\alpha * \left(0_\alpha * \left(x_\beta * y_\beta\right)\right)\right)\right) =$$

$$\left(x * \left(\left(y * (y * x)\right) * \left(0 * \left(0 * (x * y)\right)\right)\right)\right)_{\min(\beta,\alpha)} \in \tilde{A} .$$

Thus $A\left(x * \left(\left(y * (x * y)\right) * \left(0 * \left(0 * (x * y)\right)\right)\right)\right) \geq \min(\beta,\alpha)$

$$= \min\left\{A\left(\left(\left(x * y\right) * y\right) * \left(0 * y^n\right)\right), A(z)\right\}$$

Proposition4.8. An n-fold weak BCI - implicative weak ideal is a weak ideal

Proof. Let $y_\mu = 0_\alpha$, $x_\lambda, y_\mu \in \tilde{A}$, and let

$x_\lambda * z_\alpha = \left(\left(\left(\left(x_\lambda * 0_\alpha\right) * 0_\alpha\right) * \left(0_\alpha * 0^n{}_\alpha\right)\right) * z_\alpha \in \tilde{A}\right.$ and $z_\alpha \in \tilde{A})$, since $\tilde{A}$ n-fold BCI - implicative weak ideal, we have

$$x_\lambda * \left(\left(0_\alpha * \left(0_\alpha * x_\lambda\right)\right) * \left(0_\alpha * \left(0_\alpha * \left(x_\lambda * 0_\alpha\right)\right)\right)\right) = x_{\min(\lambda,\alpha)}$$

Thus $\tilde{A}$ is a weak ideal

Corollary 4.9. A fuzzy n-fold weak BCI - implicative ideal is a fuzzy ideal.

Theorem 4.10. Let $\{\tilde{A}_{i\in I}\}$ be a family of *n*-fold weak BCI - implicative weak ideals and $\{A_{i\in I}\}$ be a family of fuzzy *n*-fold weak BCI - implicative ideals. Then (1) $\bigcap\limits_{i\in I}\tilde{A}_i$ is an *n*-fold weak BCI - implicative weak ideal.

(2)) $\bigcup_{i\in I}\tilde{A}_i$ is an *n*-fold weak BCI - implicative weak ideal.

(3)) $\bigcap\limits_{i\in I}A_i$ is a fuzzy *n*-fold weak BCI - implicative ideal.

(4)) $\bigcup_{i\in I}A_i$ is a fuzzy *n*-fold weak BCI - implicative ideal.

Proof. (1) $\forall\lambda\in\operatorname{Im}\left(\bigcap\limits_{i\in I}\tilde{A}_i\right)$, then $\lambda\in\operatorname{Im}(\tilde{A}_i),\forall i$, so, $0_\lambda\in\tilde{A}_i,\forall i$, i.e. $0_\lambda\in\bigcap\limits_{i\in I}\tilde{A}_i$. For every $x_\mu, y_\lambda, z_\alpha\in\tilde{X}$, if

$$(\left(\left(\left(x_\lambda * y_\mu\right) * y_\mu\right) * \left(0_\alpha * y^n{}_\mu\right)\right)\in\bigcap\nolimits_{i\in I}\tilde{A}_i \text{ and } z_\alpha\in\bigcap\nolimits_{i\in I}\tilde{A}_i) \text{ , then}$$

$$(\left(\left(\left(x_\lambda * y_\mu\right) * y_\mu\right) * \left(0_\alpha * y^n{}_\mu\right)\right)\in\tilde{A}_i \text{ and } z_\alpha\in\tilde{A}_i) \quad \forall i \text{ , thus}$$

$$\Rightarrow x_\lambda * \left(\left(y_\mu * \left(y_\mu * x_\lambda\right)\right) * \left(0_\alpha * \left(0_\alpha * \left(x_\lambda * y_\mu\right)\right)\right)\right)\in\tilde{A}_i \ \forall i$$

So $\Rightarrow x_\lambda * \left(\left(y_\mu * \left(y_\mu * x_\lambda\right)\right) * \left(0_\alpha * \left(0_\alpha * \left(x_\lambda * y^n{}_\mu\right)\right)\right)\right)\in\bigcap\nolimits_{i\in I}\tilde{A}_i$. Thus $\bigcap\limits_{i\in I}\tilde{A}_i$ is an *n*-fold weak BCI - implicative weak ideal.

(2) . $\forall\lambda\in\operatorname{Im}\left(\bigcup\nolimits_{i\in I}\tilde{A}_i\right)$, then $\exists i_0\in I, such, that\ \lambda\in\tilde{A}_{i_0}$, so, $0_\lambda\in\tilde{A}_{i_0}$, i.e. $0_\lambda\in\bigcup\nolimits_{i\in I}\tilde{A}_i$. For every $x_\mu, y_\lambda, z_\alpha\in\tilde{X}$, if

$$(\left(\left(\left(x_\lambda * y_\mu\right) * y_\mu\right) * \left(0_\alpha * y^n{}_\mu\right)\right)\in\bigcup\nolimits_{i\in I}\tilde{A}_i \text{ and } z_\alpha\in\bigcup\nolimits_{i\in I}\tilde{A}_i) \text{ , then}$$

$\exists i_0\in I$ such that

$$(\left(\left(\left(x_\lambda * y_\mu\right) * y_\mu\right) * \left(0_\alpha * y^n{}_\mu\right)\right)\in\tilde{A}_{i_0} \text{ and } z_\alpha\in\tilde{A}_{i_0}) \text{ , thus}$$

$$\Rightarrow x_\lambda * \left(\left(y_\mu * \left(y_\mu * x_\lambda\right)\right) * \left(0_\alpha * \left(0_\alpha * \left(x_\lambda * y_\mu\right)\right)\right)\right)\in\tilde{A}_{i_0}$$

So $\Rightarrow x_\lambda * \left(\left(y_\mu * \left(y_\mu * x_\lambda\right)\right) * \left(0_\alpha * \left(0_\alpha * \left(x_\lambda * y_\mu\right)\right)\right)\right)\in\bigcup\nolimits_{i\in I}\tilde{A}_i$. Thus $\bigcup\limits_{i\in I}\tilde{A}_i$ is an *n*-fold weak BCI - implicative weak ideal.

(3) Follows from (1) and Theorem 4.7.

(4) Follows from (2) and Theorem 4.7.

5. Algorithms

Here We Give Some Algorithms For Studding The Structure Of The Foldness Of (Fuzzy) BCI - implicative Ideals In BCI-Algebras

Algorithms for BCI – implicative ideals of BCI-Algebra

```
Input (X : BCI-algebra, ∗ : binary operation, I : the subset of X);
Output("I is a BCI-implicative ideal of X or not");
Begin
 If I = ϕ then
   go to (1.);
 EndIf
 If 0 ∉ I then
   go to (1.);
 EndIf
 Stop:=false;
 i := 1;
 While i ≤ |X| and not (Stop) do
   j := 1;
  While j ≤ |X| and not (Stop) do
    k := 1;
    While k ≤ |X| and not (Stop) do
      If ((((x ∗ y) ∗ y) ∗ (0 ∗ y)) ∈ I and z ∈ I) then
        If x ∗ ((y ∗ (y ∗ x))(0 ∗ (0 ∗ (x ∗ y)))) ∉ I
          Stop:=true;
        EndIf
      EndIf
    Endwhile
   Endwhile
  Endwhile
  If Stop then
   Output ("I is a BCI-implicative ideal of X ")
  Else
   (1.) Output ("I is not a BCI-implicative ideal of X ")
 EndIf
End
```

Algorithms for N-fold BCI – implicative ideals of BCI-Algebra

```
Input (X : BCI - algebra, ∗ : binary operation, I : a subset of X);
Output("I is n-fold BCI-implicative ideal of X or not");
```

```
Begin
 If I = ϕ then
   go to (1.);
 EndIf
 If 0 ∉ I then
   go to (1.);
 EndIf
 Stop:=false;
 i := 1;
 While i ≤ |X| and not (Stop) do
   j := 1;
  While j ≤ |X| and not (Stop) do
     k := 1;
    While k ≤ |X| and not (Stop) do
     If ((((x * y) * y) * (0 * y)) ∈ I and z ∈ I) then
      If x * ((y * (y * x))(0 * (0 * (x * y^n)))) ∉ I
        Stop:=true;
      EndIf
     EndIf
    Endwhile
   Endwhile
  Endwhile
  If Stop then
   Output ("I is an n-fold BCI-implicative ideal of X")
  Else
   (1.) Output ("I is not an n-fold BCI-implicative ideal of X")
  EndIf
End
```

Algorithms for fuzzy BCI – implicative ideals of BCI-Algebra

Input (X : *BCI-algebra*, $*$: binary operation, A : a fuzzy subset of X);

Output(" A is a fuzzy BCI-implicative ideal of X or not");

```
Begin
 Stop:=false;
 i := 1;
 While i ≤ |X| and not (Stop) do
   If A(0) < A(x_i) then
     Stop:=true;
   EndIf
    j := 1;
```

While $j \leq |X|$ and not (*Stop*) do
$k := 1$;
While $k \leq |X|$ and not (*Stop*) do
If
$A\left(x*\left(\left(y*\left(x*y\right)\right)*\left(0*\left(0*\left(x*y\right)\right)\right)\right)\right) \geq \min\left\{A\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right), A\left(z\right)\right\}$
then
Stop =true;
EndIf
Endwhile
Endwhile
Endwhile
If *Stop* then
Output ("A is not a fuzzy BCI-implicative ideal of X ")
Else
Output ("A is a fuzzy BCI-implicative ideal of X ")
EndIf
End

Algorithms for fuzzy N-fold BCI – implicative ideals of BCI-Algebra

Input (X : *BCI- algebra*, $*$: binary operation, A : a fuzzy subset of X);
Output ("A is a fuzzy n- fold BCI-implicative ideal of X or not");
Begin
Stop:=false;
$i := 1$;
While $i \leq |X|$ and not (*Stop*) do
If $A(0) < A(x_i)$ then
Stop:=true;
EndIf
$j := 1$;
While $j \leq |X|$ and not (*Stop*) do
$k := 1$;
While $k \leq |X|$ and not (*Stop*) do
$A\left(x*\left(\left(y*\left(x*y\right)\right)*\left(0*\left(0*\left(x*y^{n}\right)\right)\right)\right)\right) \geq \min\left\{A\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right), A\left(z\right)\right\}$
Stop =true;
EndIf
Endwhile
Endwhile
Endwhile
If *Stop* then

```
 Output ("A is not a fuzzy n- fold BCI-implicative ideal of X ")
  Else
    Output ("A is a fuzzy n- fold BCI-implicative ideal of X ")
  EndIf
End
```

Algorithms for N-fold weak BCI – implicative ideals of BCI-Algebra

```
Input( X : BCI-algebra, I : a subset of X, n ∈ N );
Output("I is an n-fold weak BCI-implicative ideal of X or not");
Begin
  If I = ϕ then
    go to (1.);
  EndIf
  If 0 ∉ I then
    go to (1.);
  EndIf
  Stop:=false;
  i := 1;
  While i ≤ |X| and not (Stop) do
    j := 1;
    While j ≤ |X| and not (Stop) do
      k := 1;
      While k ≤ |X| and not (Stop) do
        If ((((x_i * y_j) * y_j) * (0_k * y_j^n)) ∈ I and z_k ∈ I) then
          If x_i * ((y_j * (y_j * x_i)) * (0_k * (0_k * (x_i * y_j)))) ∉ I
            Stop:=true;
          EndIf
        EndIf
      Endwhile
    Endwhile
  Endwhile
  If Stop then
    Output ("I is an n-fold weak BCI-implicative ideal of X ")
  Else
   (1.) Output ("I is not an n-fold weak BCI-implicative ideal of X ")
  EndIf
End
```

Algorithms for fuzzy N-fold weak BCI – implicative ideals of BCI-Algebra

Input (X : *BCI-algebra*, $*$: binary operation, A fuzzy subset of X);
Output (" A is a fuzzy *n*-fold weak BCI-implicative ideal of X or not");
Begin
  Stop:=false;
  $i := 1$;
  While $i \le |X|$ and not (*Stop*) do
    If $A(0) < A(x_i)$ then
      Stop:=true;
    EndIf
      $j := 1$;
    While $j \le |X|$ and not (*Stop*) do
      $k := 1$;
      While $k \le |X|$ and not (*Stop*) do
        If

$$A\left(x_i * \left(\left(y_j * \left(x_i * y_j\right)\right) * \left(0_k * \left(0_k * \left(x_i * y_j\right)\right)\right)\right)\right) < \min\left\{A\left(\left(\left(\left(x_i * y_j\right) * y_j\right) * \left(0_k * y_j{}^n\right)\right)\right), A\left(z_k\right)\right\}$$

then
          Stop =true; ‹
        EndIf
      Endwhile
    Endwhile
  Endwhile
  If *Stop* then
    Output (" A is not a fuzzy *n*-fold weak BCI-implicative ideal of X ")
  Else
    Output (" A is a fuzzy *n*-fold weak BCI-implicative ideal of X ")
  EndIf
End

6. Conclusion and future research

In this paper, we introduce new notions of (fuzzy) *n*-fold BCI - implicative ideals, and (fuzzy) *n*-fold weak BCI - implicative ideals in BCI - algebras. Then we studied relationships between different types of n- fold BCI - implicative ideals and investigate several properties of the foldness theory of BCI - implicative - ideals in BCI - algebras. Finally, we construct some algorithms for studying the foldness theory of BCI - implicative ideals in BCI - algebras.

In our future study of foldness ideals in BCK/BCI algebras, maybe the following topics should be considered :

(1) developing the properties of foldness of implicative ideals of BCK/BCI algebras.

(2) They are finding useful results on other structures of the foldness theory of ideals of BCK/BCI algebras.

(3) We are constructing the related logical properties of such structures.

(4) One may also apply this concept to study some applications in many fields like decision making, knowledge base systems, medical diagnosis, data analysis, and graph theory. Y. Huang, and Z. Chen, On ideals in BCK-algebra, Math. Japonica, 50 (1999), pp. 211-226.

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