Numerical Solution of sine-Gordon Equation
by Spectral Method

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Abstract

Numerical methods are essential in solving nonlinear differential equations that do not have closed form solution. In this paper, we develop spectral function method that allows \( L_2 \) projection of an operator onto a finite dimensional Hilbert space to solve Sine-Gordon equation numerically. Orthogonal basis are used to establish computational algorithm. Numerical results are presented. The accuracy and efficiency of proposed method are discussed.

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1 Introduction

Numerical methods are crucial in solving nonlinear partial differential equations when closed form solutions are not possible. Finite difference, finite element, adomian decomposition, homotopy analysis and variational iterative methods have long been adopted as vital tools for efficient, accurate, and stable numerical solution of partial differential equations. In the past two decades however, spectral methods have emerged as a viable numerical scheme for the solution of partial differential equations because of its high accuracy.
Let $\Omega$ be an open bounded set of $\mathbb{R}$. Let us consider the following one-dimensional sine-Gordon equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} + \sin u(x,t) = f(x,t); \quad (x,t) \in Q$$

$$u(0,t) = u(L,t) = 0, \quad t \in (0,T)$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega$$

where $u$ is a function of $x$ and $t$, $T > 0$, $Q = (0,T) \times \Omega$, $f \in L^2(Q)$, $u_0 \in V = H^1(\Omega)$ and $u_1 \in H = L^2(\Omega)$. In this paper, we solve (1) by using spectral method with orthogonal basis of operator $A = -\Delta$. The paper is organized as follows. In section 2 we outline the Hilbert spaces and establish a self-adjoint operator. In section 3 we develop a spectral method for the solution of the state equations (1). In section 4 we introduce the computational algorithm and in section 5 we present numerical results.

## 2 Problem Setup

Let $H = L^2(\Omega)$ be a Hilbert space with following inner product and norm

$$(\phi, \psi) = \int_{\Omega} \phi(x)\psi(x)dx, \quad |\phi| = (\phi, \phi)^{\frac{1}{2}}$$

for all $\phi, \psi \in L^2(\Omega)$. Let $V = H^1_0(\Omega)$ be a Hilbert space with following inner product and norm

$$((\phi, \psi)) = (\nabla \phi, \nabla \psi), \quad \|\phi\| = ((\phi, \phi))^{\frac{1}{2}}$$

for all $\phi, \psi \in H^1_0(\Omega)$. The dual $H'$ is identified with $H$ leading to $V \subset H \subset V'$ with compact, continuous, and dense injections [7]. Hence there exists a constant $K_1 = K_1(\Omega)$ such that

$$|w| \leq K_1 \|w\| \quad \text{for any} \quad w \in V.$$  \hspace{1cm} (4)

Let $< u, v >_{V', V}$ denote the duality pairing between $V$ and $V'$. To use the variational formulation let us define the following bilinear form on $V \times V$

$$a(u, v) = \int_{\Omega} \nabla u \nabla v dx$$

for any $u, v \in H^1_0(\Omega)$. Clearly, $a(u, v)$ is bounded and coercive in $V$. Define a linear operator $A : D(A) = H^1_0(\Omega) \cap H^2(\Omega)$ into $H$ by $a(u, v) = (Au, v)$ for all $u \in D(A)$ and for all $v \in V$. Thapa [6] proved that $A$, an isomorphism between $D(A)$ and $H$, forms an orthonormal basis in $H$. 


3 Spectral Method

Let $\Omega = (0, L)$. To accommodate the boundary conditions in (1), we use non-normalized eigenfunctions $w_n = \sin(n \pi x)/L$, $n = 1, 2, 3, \ldots$, as a basis of $H = L_2(\Omega)$. Let $D_N = \text{span}(w_1, w_2, \ldots, w_N)$ be a finite dimensional subspace of $V$. Since $D_N$ is closed subspace of the Hilbert space, $D_N$ itself is a Hilbert space. Thus, we define a natural projection $P_N$ from $H$ onto $D_N \subseteq H$. Let $u_N(., t) \in D_N$ that satisfies

$$\left( \frac{\partial^2 u_N}{\partial t^2}, v \right) + ((u_N, v)) + (\sin u_N, v) = (f, v); \quad t \in (0, T)$$

$$((u_N(0) - u(0), v)) = 0, \quad \left( \frac{\partial u_N(0)}{\partial t} - u_1, v \right) \bar{0}, \quad v \in D_N \quad (6)$$

As in [4], for $r_1, r_2 \in \mathbb{R}$ with $0 \leq r_2 \leq r_1$ we establish

$$\|P_N u - u\|_{r_2} \leq C_2(1 + N^2)^{(r_2 - r_1)/2}\|u\|_{r_1} \quad \text{for } u \in H^r_0(\Omega) \quad (7)$$

Where $P_N$ is projection operator from $H$ onto $S_N$ and $C_2$ is a constant depends on $L$. Equation (6) can be written as

$$\frac{\partial^2 u_N}{\partial t^2} + Au_N + P_N u_N = P_N f(t), \quad t \in (0, T)$$

$$u_N(0) = P_N u_0, \quad \frac{\partial u_N(0)}{\partial t} = P_N u_1 \quad (8)$$

The error bounds for finite spectral approximations $u_N(t)$ can be established in the following result. Let $r_1 > 0$. If the solution of (1) satisfies $u \in H^r_0(\Omega)$, then there is a constant $C_3$ such that

$$|u(t) - u_N(t)| \leq C_3(1 + N^2)^{-\frac{r_1}{2}}, \quad t \in (0, T) \quad (9)$$

The proof is similar to the one in Gutman and Ha [8]. Equation (8) can be viewed as initial value problem in $H$.

$$u'' + Au + \sin u = f, \quad u(0) = u_0, \quad u'(0) = u_1, \quad (10)$$

We can write (10) as

$$\frac{d}{dt} \left[ u'(t) + A \int_0^t u(s) ds \right] = f(t) - \sin u(t) \quad (11)$$

Set $u_N = P_N u$ and applying the projection operator $P_N$ to (10), we have
\[ u_N'' + Au_N + \sin u_N = P_N f, \]
\[ u_N(0) = P_N u_0 \quad u_N'(0) = P_N u_1, \]
\quad (12)

Since \( \sin u(t) \in H \), and \( \left( \frac{d}{dt}, w_n \right) = \frac{d}{dt}(u, w_n) \) for \( u \in H \),
\[
\frac{d}{dt} P_N \left[ u'(t) + A \int_0^t u(s) ds \right] = P_N f - P_N \sin u(t)
\quad (13)
\]

Since \( P_N \) is linear, and the operators \( A, P_N \) are commutable on \( H \), (13) can be written as
\[
(P_N u)''(t) + A(P_N u)(t) = P_N f - P_N \sin u(t)
\quad (14)
\]

Subtracting (12) from (10), we get
\[
u_n'' - U_N'' + A(u_n - U_N) = -P_N \sin u + P_N \sin U_N
\quad (15)
\]

Set \( L_N = u_n - U_N \). We have,
\[
L_N'' + AL_N = P_N(\sin U_N - \sin u)
\quad (16)
\]

Multiplying (16) by \( 2L_N' \) and integrating over 0 to \( t \) we have,
\[
|L_N'|^2 + \|L_N\|^2 = 2 \int_0^t \left| (P_N(\sin U_N(s) - \sin u(s)), U_N'(s)) \right| ds
\quad (17)
\]
\[
\leq 2 \int_0^t \left| (P_N(\sin U_N(s) - \sin u(s)), U_N'(s)) \right| |U_N'(s)| ds
\quad (18)
\]
\[
\leq \int_0^t |U_N(s) - u(s)|^2 ds + \int_0^t c^2 \|U_N(s)\|^2 ds + \int_0^t |U_N'(s)|^2 ds
\quad (19)
\]
\[
\leq C_2(1 + N^2)^{-r} \|u\|_{L^2(0,T;H)}^2 + \int_0^t c^2 \|U_N(s)\|^2 ds + \int_0^t |U_N'(s)|^2 ds
\quad (20)
\]

using Gronwall lemma, we obtain the desired result.
4 Computational Algorithm

In this section we discuss the computational algorithm to find solutions $u_N$. Let $\{w_j\}_{j=1}^\infty = \{\sin j\pi x/L\}_{j=1}^\infty$ be eigenfunctions of $A$ that form a basis in $H$. Then $\{w_j\}_{j=1}^\infty$ is an basis on $V$. Fix $N \in \mathbb{N}$. Let $V_N = \text{span}\{w_1, w_2, \ldots, w_N\}$. Let $P_N : H \to V_N$ be the projection operator defined by $P_N v = \sum_{j=1}^N (v, w_j) w_j$ for any $v \in H$. Then the solution $u_N$ is given by

$$u_N(x, t) = \sum_{j=1}^N g_jN(t)w_j(x) \quad (21)$$

that satisfies

$$\frac{d^2}{dt^2}(u_N, w_j) + a((u_N, w_j)) + (\sin(u_N), w_j) = (P_N f(t), w_j)$$

$$u_N(0) = P_N u_0 \quad \text{and} \quad \frac{d}{dt}u_N(0) = P_N u_1 \quad \text{for any} \quad j \in \mathbb{N}. \quad (22)$$

Let $\bar{g}_N = \{g_{jN}\}_{j=1}^N \in \mathbb{R}^N$. We can rewrite (22) as the following vector differential equation

$$\bar{g}_N''(t) + \Lambda \bar{g}_N(t) = \bar{F}(t, \bar{g}_N) \quad (23)$$

with the initial data

$$\bar{g}_N(0) = \begin{bmatrix} (P_N u_0, w_1) \\ (P_N u_0, w_2) \\ \vdots \\ (P_N u_0, w_N) \end{bmatrix} = \begin{bmatrix} \int_0^1 u_0 \sin(\pi x)/L \, dx \\ \int_0^1 u_0 \sin(2\pi x)/L \, dx \\ \vdots \\ \int_0^1 u_0 \sin(N\pi x)/L \, dx \end{bmatrix}.$$

and

$$\bar{g}_N'(0) = \begin{bmatrix} (P_N u_1, w_1) \\ (P_N u_1, w_2) \\ \vdots \\ (P_N u_1, w_N) \end{bmatrix} = \begin{bmatrix} \int_0^1 u_1 \sin(\pi x)/L \, dx \\ \int_0^1 u_1 \sin(2\pi x)/L \, dx \\ \vdots \\ \int_0^1 u_1 \sin(N\pi x)/L \, dx \end{bmatrix}.$$

where $u_0 \in L^2(0, T; V)$ and $u_1 \in L^2(0, T; H)$. 

Numerical solution of sine-Gordon equation
Let $\bar{Z}_1(t) = \bar{g}_N(t)$ and $\bar{Z}_2(t) = \bar{g}'_N(t)$. Then the initial value problem (23) can be reduced into the following system of first order ODEs

\[
\begin{align*}
\dot{\bar{Z}}_1(t) &= \bar{Z}_2(t) \\
\dot{\bar{Z}}_2(t) &= -\Lambda \bar{Z}_1(t) + \bar{F}(t, \bar{u}_N) \\
\bar{Z}_1(0) &= \bar{g}_N(0), \quad \bar{Z}_2(0) = \bar{g}'_N(0). 
\end{align*}
\]

Hence the approximate solution $u_N(x, t)$ is given by

\[
\begin{equation}
\tag{25}
\label{eq:approx_sol}
\end{equation}
\]

\[\begin{align*}
\bar{Z}_1(t) &= \sum_{j=1}^{N} g_j(t) w_j(x) \\
\bar{Z}_2(t) &= \sum_{j=1}^{N} g'_j(t) w_j(x)
\end{align*}\]

\section{5 Numerical Results}

In this section, we present examples to validate the scheme developed in section 4. The $L_2$ and $L_\infty$ errors defined below are used to test the accuracy of the scheme.

\[
\|u_{exact} - u_N\|_2 = \left( \sum_{i=1}^{N} |u_i - u_i^N|^2 \right)^{\frac{1}{2}}
\]

\[
\|u_{exact} - u_N\|_{\infty} = \max_{1 \leq i \leq N} \sum_{i=1}^{N} |u_i - u_i^N|
\]

Example 1

\[
\begin{align*}
\frac{\partial^2 u(x, t)}{\partial t^2} - \frac{\partial^2 u(x, t)}{\partial x^2} + \sin u(x, t) &= f(x, t); \quad (x, t) \in Q \\
u(0, t) &= u(L, t) = 0, \quad t \in (0, T) \\
u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega
\end{align*}
\]

with the following initial conditions.

\[
\begin{align*}
u_0 &= (1 - \cos \pi x), \quad u_1 = -(1 - \cos \pi x) \\
f(x, t) &= -e^{-t}(\pi^2 \cos \pi x) + e^{-t}(1 - \cos \pi x) + \sin(e^{-t}(1 - \cos \pi x))
\end{align*}
\]

With boundary conditions

\[
\begin{align*}
u(0, t) &= 0, \quad u(2, t) = 0 \quad (30)
\end{align*}
\]

Then the analytical solution of the given problem is

\[
\begin{equation}
\tag{31}
u(x, t) = e^{-t}(1 - \cos \pi x)
\end{equation}
\]

The $L_2$ and $L_\infty$ errors for different values of $N$ are presented in the following table.
Example 2:

\[
\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} + \sin u(x,t) = f(x,t); \quad (x,t) \in Q
\]
\[
u(0,t) = u(L,t) = 0, \quad t \in (0,T)
\]
\[
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega \quad (32)
\]

with the following initial conditions.

\[
u_0 = (1 - \sin \pi x), \quad u_1 = -(1 - \sin \pi x)
\]
\[
f(x,t) = e^{-t}(\pi^2 \sin \pi x) + e^{-t}(1 - \sin \pi x)) + \sin(e^{-t}(1 - \sin \pi x)) \quad (33)
\]

With boundary conditions

\[
u(0,t) = 0, \quad u(2,t) = 0 \quad (34)
\]

Then the analytical solution of the given problem is

\[
u(x,t) = e^{-t}(1 - \sin \pi x) \quad (35)
\]

The \(L_2\) and \(L_\infty\) errors for different values of \(N\) are presented in the following table.

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<th>(L_2) Error</th>
<th>(L_\infty) Error</th>
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<td>0.038771256</td>
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<tr>
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<td>$N$</td>
<td>$L_2$ Error</td>
<td>$L_\infty$ Error</td>
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### 6 Conclusion

In this article we have purposed spectral method for solving sine-Gordon equation with source function arising in the field of science and technology. This method makes the computation easy and efficient as it transforms differential equations into algebraic equations. To validate the accuracy and efficiency of the method, two examples are included and discussed. Numerical results confirm that the proposed spectral method has higher order of accuracy for small values of $N$.

### References


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