On the Approximate Solutions of Linear Volterra Integral Equations of the First Kind

Christian Kasumo

Department of Science and Mathematics
School of Science, Engineering and Technology
Mulungushi University, P.O. Box 80415, Kabwe, Zambia

Abstract

In this paper we solve numerically Volterra integral equations of the first kind with separable kernels. This is done by first converting the first-kind Volterra integral equations into those of the second kind and then applying the fourth-order block-by-block method in conjunction with Simpson’s rule to find approximate solutions. Approximate solutions based on several numerical examples are given and confirm the convergence of the method. A comparison of numerical results from the block-by-block method with those from other existing methods available in the literature proves its superiority over those other methods.

Mathematics Subject Classification: 45A05, 45D05, 46N20

Keywords: First kind Volterra integral equation; Block-by-block method; Block-pulse functions; Repeated Simpson’s quadrature rule

1 Introduction

Many important problems in the physical sciences and engineering can be modelled by linear integral equations [1]. Some of the areas of application of integral equations include potential theory, acoustics, elasticity, fluid mechanics, heat transfer, radiative energy transfer, theory of population, etc. In many situations integral equations originate from the conversion of boundary-value or initial-value problems associated with ordinary or partial differential equations. However, many problems lead directly to integral equations and cannot
be modelled using differential equations. A wide range of numerical methods have been developed to solve integral equations by using various basis functions.

Saeedi et al. [2] solved a class of nonlinear VIEs of the first kind by converting them into linear VIEs of the second kind and then applying the Tau method which they found to be highly accurate. Biazar et al. [3] solved the nonlinear VIE of the first kind using the Adomian decomposition method. Babolian and Masouri [4] proposed a simple, efficient and direct method for solving Volterra integral equations of the first kind. They applied block-pulse functions and their operational matrix of integration to reduce the first kind integral equation to a linear lower triangular system. Linz [5] applied the rectangular, trapezoidal and midpoint methods for solving linear Volterra integral equations (LVIEs) of the first kind. Maleknejad et al. [6] selected wavelets as basis functions to solve LVIEs of the first kind and used the collocation method as a projection method for converting the integral equation into a system of linear equations. In a later study, Maleknejad and Rahimi [7] solved VIEs of the first kind by introducing and using a modification of block-pulse functions which were also later used by Masouri [1]. Biazar et al. [8] applied He’s homotopy perturbation method to solve systems of VIEs of the first kind. Masouri et al. [9] produced the approximate solution of VIEs of the first kind via a recurrence relation.

Sidorov et al. [10] proposed a novel approach to the construction of generalized solutions of VIEs of the first kind. These solutions were made up of a singular component constructed as a solution of special linear algebraic systems and a regular component constructed as a solution of special VIEs of the third kind. Mirzaee [11] converted linear VIEs of the first kind to a recurrence relation and then applied the repeated Simpson’s quadrature rule to the solution of these equations. The numerical results showed that the method had a high degree of accuracy for a sufficiently small step size $h$. Rahman et al. [12] used the Galerkin weighted residual method and Laguerre polynomials to find numerical solutions of VIEs of the first and second kinds with regular and singular kernels. Song and Kim [13] used the Elzaki transform to obtain numerical solutions of VIEs of the second kind. Mohamadi et al. [14] proposed a new computational technique for finding solutions for VIEs of the first kind by using Bernstein polynomials.

In this paper, we focus on VIEs of the first kind. In particular, we apply the method proposed by Wazwaz [15] to convert VIEs of the first kind into VIEs of the second kind and then solve them using the fourth-order block-by-block method of Paulsen et al. [16] in conjunction with Simpson’s rule. The block-by-block methods are essentially extrapolation procedures which produce a block
of values at a time. They are advantageous over linear multistep and step-by-step methods in that they can be of higher order and still be self-starting. In addition to not requiring special starting procedures or values, block-by-block methods have a simple structure, allow for easy switching of step-size and have the ability to compute several values of the unknown function at the same time [17]. Other numerical and analytical methods that have been used to solve VIEs include Neumann series, Nystrom, expansion, residual, Galerkin, homotopy, perturbation, variational iteration, Laplace transform, series solution and direct computation methods [18, 19, 20, 21, 22].

The main contribution of this paper consists in the following:

1. It revisits integral equations by way of showing their practical importance, occurrence and applications.

2. The paper compares solutions of LVIEs of the first kind using different numerical methods.

3. The study provides motivation to other researchers to undertake in-depth studies on integral equations using different numerical techniques.

4. It highlights and underscores the relationship between LVIEs of the first and second kinds.

The rest of this paper is organized as follows: Section 2 outlines the proposed methods of solution. Section 3 presents and discusses some results based on selected numerical examples and some conclusions are given in Section 4.

2 Description of the Method

Consider the general linear Volterra integral equation (LVIE) of the form:

\[ f(x) = cu(x) + \lambda \int_a^{b(x)} K(x,t)u(t)dt, \quad 0 \leq x \leq 1, \quad (1) \]

where \( K(x,t) \) is the kernel and \( f(x) \) the forcing function of the integral equation. Both \( K \) and \( f \) are known continuous functions. In this study, \( x \in [0, \infty) \), so that \( K : [0, \infty) \to \mathbb{R} \) and \( f : [0, \infty) \to \mathbb{R} \). Also, \( u : [0, \infty) \to \mathbb{R} \) is the unknown function to be determined. If \( c = 1 \) and \( b(x) = x \), then (1) becomes a LVIE of the second kind. If, however, \( c = 0 \) and \( b(x) = x \), then (1) is a LVIE of the first kind which is the focus of this paper. Note that if the kernel \( K(x,t)u(t) = K(x,t,u(t)) \), then the VIE is nonlinear. The parameter \( \lambda \) is usually omitted but is important in certain theoretical investigations (e.g., stability analysis). Without loss of generality, it is assumed that the interval
of integration in equation (1) is \([0, x]\) and \(0 \leq x \leq 1\), since any finite interval can be transformed to this interval by linear maps \([1]\). Thus, the LVIE of the first kind is given by

\[
f(x) = \lambda \int_0^x K(x, t)u(t)dt, \quad 0 \leq x \leq 1,
\]

(2)

where \(x\) and \(t\) are real numbers. By the method of Wazwaz \([15]\), we show that subject to suitable conditions the LVIE of the first kind (2) can easily be converted into a VIE of the second kind. This addresses the problem of the inherent ill-posedness of VIEs of the first kind. In an ill-posed system, small changes in the data result in very large changes in the results or solutions obtained \([23]\). The goal of converting a LVIE of the first kind into one of the second kind is accomplished by differentiation or by integration by parts. Differentiating both sides of (2) with respect to \(x\), we obtain

\[
f'(x) = \lambda K(x, x)u(x) + \lambda \int_0^x K_x(x, t)u(t)dt
\]

(3)

by using the Leibniz rule. If \(K(x, x) \neq 0\) for values of \(x\) in the interval of integration, then dividing both sides of (3) by \(\lambda K(x, x)\) yields

\[
u(x) = \frac{f'(x)}{\lambda K(x, x)} - \int_0^x \frac{K_x(x, t)}{K(x, x)}u(t)dt.
\]

Let \(g(x) = \frac{f'(x)}{\lambda K(x, x)}\) and \(H(x, t) = -\frac{K_x(x, t)}{K(x, x)}\). Then we have

\[
u(x) = g(x) + \int_0^x H(x, t)u(t)dt,
\]

(4)

which is a VIE of the second kind. Once in this form, the equation can now be solved using one of many available methods for solving second-kind VIEs. In this paper, we will use the block-by-block method to solve (4).

This method, briefly outlined in this section, is described fully in Linz \([5, 17]\) and Paulsen et al. \([16]\). In this regard, we use a fixed grid \(u = 0, h, 2h, \ldots\). The numerical solution of the general linear Volterra integral equation of the second kind (4), where the kernel \(H(x, t)\) and the forcing function \(g(x)\) are known functions and \(u(x)\) is the unknown function to be determined, is of the form

\[
u_n = g_n + h \sum_{i=1}^{n} w_i H_{n,i} u_i,
\]

(5)
where \( u_i \) is the numerical approximation to \( u(\text{ih}) \), \( H_{n,i} = H(\text{nh}, \text{ih}) \), \( u_n = u(\text{nh}) \) and \( g_n = g(\text{nh}) \). The \( w_i \) are the integration weights. Here, the block-by-block method will be used in conjunction with Simpson’s rule of integration to obtain solutions in blocks of two values.

Linz [5] has shown that the block-by-block method always converges and has an order of convergence of four. This method reduces the VIE of the second kind into a system of algebraic equations which are solved by matrix methods to obtain the blocks.

**Definition 2.1.** Let \( u_0(h), u_1(h), \ldots \) denote the approximation obtained by a given method using step-size \( h \). Then a method is said to be convergent if and only if

\[
|u_i(h) - u(x_i)| \to 0, \text{ for } i = 0, 1, 2, \ldots, N,
\]

as \( h \to 0 \), \( N \to \infty \), such that \( Nh = a \).

**Definition 2.2.** A method is said to be of order \( q \) if \( q \) is the largest number for which there exists a finite constant \( C \) such that

\[
|u_i(h) - u(x_i)| \leq Ch^q, \quad i = 0, 1, 2, \ldots, \text{ for all } h > 0.
\]

**Remark 2.3.** By Theorem 3.1 in Paulsen et al. [16] and from results in Chapter 7 of Linz [17], it follows that for a fixed \( x \) so that \( nh = x \), the solution satisfies

\[
|u_n - u(x)| = O(h^4),
\]

provided that \( u \) is four times continuously differentiable as is the case here by Theorem 2.4 in Paulsen et al. [16]. On the other hand, for the block-by-block method \( |u_{2m+2} - u_{2m+1}| = O(h^4) \) as well.

**Remark 2.4.** According to Linz [5], the LVIE of the first kind (2) has a unique and continuous solution if the following assumptions hold:

(a) \( f(0) = 0 \),

(b) \( K(x, x) \neq 0 \) for all \( x \) in the range of integration (i.e., it vanishes nowhere),

(c) \( K(x, t) \) and \( f(x) \) are bounded and sufficiently smooth so that their derivatives exist.

The first assumption is the necessary and sufficient condition for the solution of the VIE of the second kind (4) to satisfy the original VIE of the first kind (2). The case where \( K(x, x) = 0 \) complicates the problem in that it requires repeated differentiation under certain conditions, but consideration of this case is beyond the scope of this paper.
3 Numerical Examples

In this section we show the computational efficiency of the proposed numerical method by implementing it on a selection of VIEs of the first kind. For each example a table is used to show the exact solution, the numerical results and the absolute errors, accompanied by appropriate graphs. For some examples, results from the block-by-block method are compared with those used in some of the literature. All the computations associated with these examples were performed using a Samsung Series 3 PC with an Intel Celeron CPU 847 at 1.10GHz and 6.0GB internal memory. To reduce computing time, the numerical method was implemented using the FORTRAN programming language, taking advantage of its DOUBLE PRECISION feature which gives a high degree of accuracy. A grid size of $h = 0.01$ was used throughout. The figures were constructed using MATLAB R2016a.

Example 3.1. Consider the Volterra integral equation of the first kind [11]:

$$x = \int_0^x e^{-x+t} u(t) dt, \quad 0 \leq x \leq 1,$$

(6)

with exact solution $u(x) = 1 + x$. Comparing (6) with (2), we see that $\lambda = 1$, $K(x, t) = e^{-x+t}$ and $f(x) = x$. Thus, $K(x, x) = 1 \neq 0$ for all $x$, so $f'(x) = 1$, $K(x, t) = -e^{-x+t}$ and $-\frac{1}{K(x, x)} = -1$. This means that $g(x) = \frac{f'(x)}{\lambda K(x, x)} = 1$ and $H(x, t) = \frac{K_x(x, t)}{K(x, x)} = e^{-x+t}$, so equation (6) becomes

$$u(x) = 1 + \int_0^x e^{-x+t} u(t) dt$$

(7)

which is a LVIE of the second kind. But from (6), the integral on the RHS of (7) equals $x$. Hence, (7) reduces to

$$u(x) = 1 + x,$$

which is the exact solution of the VIE of the first kind. The results are given in Table 1 where a comparison is made between the block-by-block method and repeated Simpson’s quadrature method of [11]. Figure 1(a) shows the approximate solution to (7). Also shown are the absolute errors defined as

$$e_{B-by-B} = |u(x) - u_{B-by-B}|$$

and

$$e_{SQR} = |u(x) - u_{SQR}|.$$
Table 1: Comparison of approximate and exact solutions from the block-by-block and repeated Simpson’s quadrature methods for $h = 0.01$ for Example 3.1

<table>
<thead>
<tr>
<th>$x$</th>
<th>$u(x)$</th>
<th>$u_{\text{B-by-B}}(x)$</th>
<th>$u_{\text{SQR}}(x)$</th>
<th>$\epsilon_{\text{B-by-B}}$</th>
<th>$\epsilon_{\text{SQR}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.2</td>
<td>1.2</td>
<td>1.20000005</td>
<td>1.19999999</td>
<td>$5 \times 10^{-8}$</td>
<td>$1 \times 10^{-8}$</td>
</tr>
<tr>
<td>0.4</td>
<td>1.4</td>
<td>1.3999998</td>
<td>1.39999999</td>
<td>$2 \times 10^{-8}$</td>
<td>$1 \times 10^{-8}$</td>
</tr>
<tr>
<td>0.6</td>
<td>1.6</td>
<td>1.60000002</td>
<td>1.59999999</td>
<td>$2 \times 10^{-8}$</td>
<td>$1 \times 10^{-8}$</td>
</tr>
<tr>
<td>0.8</td>
<td>1.8</td>
<td>1.80000007</td>
<td>1.79999999</td>
<td>$7 \times 10^{-8}$</td>
<td>$1 \times 10^{-8}$</td>
</tr>
<tr>
<td>1.0</td>
<td>2</td>
<td>2.00000000</td>
<td>1.99999999</td>
<td>$0$</td>
<td>$1 \times 10^{-8}$</td>
</tr>
</tbody>
</table>

Figure 1: Numerical solution for (a) Example 3.1 (b) Example 3.2

**Example 3.2.** Consider the following VIE of the first kind [11]:

$$(x^2 - x + 2) \sin x + 1 - \cos x = \int_0^x (x^2 - t + 2) u(t) dt, \quad 0 \leq x \leq 1. \quad (8)$$

The exact solution of this LVIE is $u(x) = \cos x$. Here, $\lambda = 1$, $f(x) = (x^2 - x + 2) \sin x + 1 - \cos x$, $K(x, t) = x^2 - t + 2 \Rightarrow K(x, x) = x^2 - x + 2 \neq 0$ in the range of integration $0 \leq x \leq 1$.

Thus, $f'(x) = (2x - 1) \sin x + (x^2 - x + 2) \cos x + \sin x$, $K_x(x, t) = 2x$ and $-\frac{1}{K(x, x)} = -\frac{1}{x^2 - x + 2}$, so that (8) can now be written as

$$u(x) = \frac{1}{x^2 - x + 2} [(2x-1) \sin x + (x^2 - x + 2) \cos x + \sin x] - \int_0^x \frac{2x}{x^2 - x + 2} u(t) dt \quad (9)$$

which is a second-kind LVIE. Solving (9) numerically gives a solution approximating the exact solution $u(x) = \cos x$ as shown in Table 2 and Figure 1(b). From Table 2, the absolute errors from the block-by-block method are much
Table 2: Comparison of approximate and exact solutions from the block-by-block and repeated Simpson’s quadrature methods for \( h = 0.01 \) for Example 3.2

<table>
<thead>
<tr>
<th>( x )</th>
<th>( u(x) )</th>
<th>( u_{\text{by-B}}(x) )</th>
<th>( u_{\text{SQR}}(x) )</th>
<th>( e_{\text{by-B}} )</th>
<th>( e_{\text{SQR}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.2</td>
<td>0.98006660</td>
<td>0.98006660</td>
<td>0.98006473</td>
<td>0.0187 \times 10^{-6}</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.92106098</td>
<td>0.92106098</td>
<td>0.92105587</td>
<td>0.511 \times 10^{-6}</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>0.82533562</td>
<td>0.82533556</td>
<td>0.82532591</td>
<td>6 \times 10^{-8}</td>
<td>9.71 \times 10^{-6}</td>
</tr>
<tr>
<td>0.8</td>
<td>0.69670677</td>
<td>0.60670671</td>
<td>0.69669131</td>
<td>6 \times 10^{-8}</td>
<td>1.546 \times 10^{-5}</td>
</tr>
<tr>
<td>1.0</td>
<td>0.54030210</td>
<td>0.54030210</td>
<td>0.54028032</td>
<td>0.2178 \times 10^{-5}</td>
<td></td>
</tr>
</tbody>
</table>

smaller than those from the Simpson’s quadrature method, underscoring the relative superiority of the block-by-block method.

**Example 3.3.** Given the first-kind VIE [12]:

\[
x = \int_{0}^{x} 3^{x-t}u(t)dt, \quad 0 \leq x \leq 1,
\]

having exact solution \( u(x) = 1 - x \ln 3 \), we note that \( \lambda = 1, K(x,t) = 3^{x-t}, K(x,x) = 1 \neq 0, f(x) = x \) and \( f'(x) = 1 \). To find \( K_{x}(x,t) \), let \( K(x,t) = 3^{x-t} \). Then, taking natural logarithms on both sides we have

\[
\ln K(x,t) = \ln 3^{x-t} \Rightarrow \ln K(x,t) = (x-t) \ln 3.
\]

Differentiating partially with respect to \( x \) gives

\[
\frac{\partial}{\partial x} \ln K(x,t) = \frac{\partial}{\partial x} (x-t) \ln 3
\]

\[
\frac{1}{K(x,t)} \frac{\partial}{\partial x} K(x,t) = \ln 3
\]

\[
\frac{\partial}{\partial x} K(x,t) = K_{x}(x,t) = K(x,t) \ln 3 = 3^{x-t} \ln 3
\]

Thus, \( g(x) = 1 \) and \( H(x,t) = -\frac{K_{x}(x,t)}{K(x,x)} = -3^{x-t} \ln 3 \). The equivalent VIE of the second kind is therefore

\[
u(x) = 1 - \ln 3 \int_{0}^{x} 3^{x-t}u(t)dt
\]

which reduces to \( u(x) = 1 - x \ln 3 \), the exact solution of the VIE of the first kind. The numerical solution of (11) is given in Table 3 and Figure 2(a). The high accuracy of the block-by-block method is evident from the fact that it produces negligible errors as shown in Table 3 and Figure 3(a).
Table 3: Comparison of approximate and exact solutions from the block-by-block method for \( h = 0.01 \) for Example 3.3

<table>
<thead>
<tr>
<th>( x )</th>
<th>( u(x) )</th>
<th>( u_{B-by-B}(x) )</th>
<th>( e_{B-by-B} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>0.89013880</td>
<td>0.89013875</td>
<td>( 5 \times 10^{-8} )</td>
</tr>
<tr>
<td>0.2</td>
<td>0.78027755</td>
<td>0.78027749</td>
<td>( 6 \times 10^{-8} )</td>
</tr>
<tr>
<td>0.3</td>
<td>0.67041636</td>
<td>0.67041636</td>
<td>0</td>
</tr>
<tr>
<td>0.4</td>
<td>0.56055510</td>
<td>0.56055510</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>0.45069385</td>
<td>0.45069373</td>
<td>( 1.2 \times 10^{-7} )</td>
</tr>
<tr>
<td>0.6</td>
<td>0.34083265</td>
<td>0.34083268</td>
<td>( 3 \times 10^{-8} )</td>
</tr>
<tr>
<td>0.7</td>
<td>0.23097140</td>
<td>0.23097129</td>
<td>( 1.1 \times 10^{-7} )</td>
</tr>
<tr>
<td>0.8</td>
<td>0.12111021</td>
<td>0.12111020</td>
<td>( 1 \times 10^{-8} )</td>
</tr>
<tr>
<td>0.9</td>
<td>0.01124895</td>
<td>0.01124887</td>
<td>( 8 \times 10^{-8} )</td>
</tr>
<tr>
<td>1.0</td>
<td>-0.09861231</td>
<td>-0.09861228</td>
<td>( 3 \times 10^{-8} )</td>
</tr>
</tbody>
</table>

Figure 2: Numerical solution for (a) Example 3.3 (b) Example 3.4

Example 3.4. Consider the first-kind LVIE [4]:

\[
x e^x = \int_0^x e^{x+t} u(t) dt
\]

having exact solution \( u(x) = e^{-x} \) for \( 0 \leq x \leq 1 \). Again, \( \lambda = 1, f(x) = xe^x, \) \( K(x, t) = e^{x+t} \) and \( K(x, x) = e^{2x} \neq 0 \) for values of \( x \) in the interval \( 0 \leq x \leq 1 \). Also, \( f'(x) = e^x + xe^x = (1 + x)e^x, \) \( K_x(x, t) = e^{x+t} \) and \( -\frac{1}{K_x(x, x)} = -\frac{1}{e^{2x}} = -e^{-2x} \). Thus, \( g(x) = \frac{f'(x)}{AK(x,t)} = \frac{(1+x)e^x}{e^{2x}} = (1 + x)e^{-x} \) and \( H(x,t) = \)
\[-\frac{K'(x,t)}{K(x,t)} = -e^{x+t} \cdot e^{-2x} = -e^{-x+t}\] and so the VIE of the second kind is

\[u(x) = (1 + x)e^{-x} - \int_0^x e^{-x+t} u(t)dt\] (13)

The results from the block-by-block method are compared with those from block-pulse functions used by Babolian and Masouri [4] in Table 4. The absolute errors are also shown and are consistent with the results obtained in Example 1 of [23]. Since the block-by-block method has smaller errors than the block-pulse function method, it performs better (see Figure 3(b)). Figure 2(b) shows a comparison of the numerical and exact solutions for the given VIE.

Table 4: Comparison of approximate and exact solutions from the block-by-block and block-pulse function methods for \(h = 0.01\) for Example 3.4

<table>
<thead>
<tr>
<th>(x)</th>
<th>(u(x))</th>
<th>(u_{B-by-B}(x))</th>
<th>(u_{BPF}(x)(m = 64))</th>
<th>(\epsilon_{B-by-B})</th>
<th>(\epsilon_{BPF})</th>
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</thead>
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<td>1</td>
<td>0.994792</td>
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<td>0.005208</td>
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<td>0.905768</td>
<td>0</td>
<td>0.000931</td>
</tr>
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<td>0.818731</td>
<td>0.824711</td>
<td>0</td>
<td>0.005980</td>
</tr>
<tr>
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<td>0.740818</td>
<td>0.740818</td>
<td>0.735426</td>
<td>0</td>
<td>0.005392</td>
</tr>
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<td>0.670320</td>
<td>0.669613</td>
<td>0</td>
<td>0.000707</td>
</tr>
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<td>0.5</td>
<td>0.606531</td>
<td>0.606531</td>
<td>0.603372</td>
<td>0</td>
<td>0.003159</td>
</tr>
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<td>0.6</td>
<td>0.548812</td>
<td>0.548812</td>
<td>0.549376</td>
<td>0</td>
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<tr>
<td>0.7</td>
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<td>0.496585</td>
<td>0.500213</td>
<td>0</td>
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<td>0.8</td>
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<td>0.449329</td>
<td>0.446058</td>
<td>0</td>
<td>0.003271</td>
</tr>
<tr>
<td>0.9</td>
<td>0.406570</td>
<td>0.406570</td>
<td>0.406141</td>
<td>0</td>
<td>0.000429</td>
</tr>
</tbody>
</table>

Figure 3: Absolute errors for (a) Example 3.3 (b) Example 3.4
4 Conclusion

In this paper we proposed the solution of linear Volterra integral equations of the first kind by first converting them into equations of the second kind and then applying the fourth-order block-by-block method which is known to perform extremely well for second-kind VIEs. Numerical examples have shown the relative superiority of the block-by-block method over such methods as the block-pulse function method and the repeated Simpson’s quadrature method that have been used in the literature.

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References


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