On the Moments of the Subsistence Consumption Distribution under Habit Formation

Óscar Gutiérrez*

Universidad Autónoma de Barcelona, Spain

Abstract

The stationary distribution $Z$ whose density function is

$$p_Z(z; 
u, \delta, \sigma) \propto (1-z)^{\nu-1} z^{-\nu i} \exp(-2\delta i (\sigma^2 z))$$

is a monotone transformation of a distribution related to the class of forward equations studied by Wong [9]. We use the method of negative moments introduced by Chao and Strawderman [1] to calculate the moments of the distribution $Z$. The expectation is expressible in closed-form in terms of the upper incomplete gamma function. The rest of moments are obtained recursively. The distribution $Z$ is relevant in Macroeconomics (Constantinides [3]).

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1 Introduction

This paper studies the moments of the distribution whose density is (up to constants) \( p_z(z) \propto (1-z)^{n-1} z^{-\gamma - 1} \exp(-2\delta/\sigma^2 z) \). The distribution is relevant in Macroeconomics since it describes the stationary subsistence level of consumption in the Constantinides [3] model.\(^1\) In the solution of the Constantinides model, the unconditional mean and variance of the subsistence level of consumption, necessary for computing the mean and variance of the consumption growth rate, are calculated numerically. In this paper we provide closed-form solutions for the (positive and negative integer) moments of distribution \( Z \), expressible in terms of the upper incomplete gamma function. These expressions allow to test whether the theoretical model fits the data, and to perform exercises of comparative statics on the subsistence level of consumption and the consumption growth rate. Also, they can be used to simplify the characterization of the region of the parameter space consistent with the moment conditions in the endowment economy described in Chapman [2].

Let us introduce the problem at hand. In his Th. 2 Constantinides shows that the (unconditional) mean and variance of the consumption growth rate are respectively written in terms of the expectations \( E(Z) \) and \( E(1-Z)^2 \), where \( Z \) represents the stationary subsistence level of consumption. The subsistence level of consumption is a stochastic process with support on the interval \([0, 1)\), defined as the ratio of an (exponentially) averaged past consumption \( x(t) \) to current consumption \( c(t) \): \( z(t) = x(t)/c(t) \). Maintaining the notation of the original paper, the diffusion associated to process \( z(t) \) can be written in terms of five parameters \((m, n, a, b, \sigma)\):

\[
dz = \left[ b - (n + a - m^2 \sigma^2)z - m^2 \sigma^2 z^2 \right] (1-z) dt - z(1-z)m \sigma dW,
\]

(1)

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\(^1\) This model provides a solution to the so-called “equity premium puzzle” by introducing adjacent complementarity in consumption (habit formation). The equity premium puzzle is described in Mehra and Prescott [8]. Dynamical models where a representative agent has time-separable preferences find difficulty in explaining the (conditional and unconditional) moments of asset returns and consumption growth rates. Constantinides [3] relaxes the assumption of time-separable preferences by introducing complementary in consumption, thus solving the equity premium puzzle. Habit formation and “catching up with the Joneses” preferences are examples of non-standard preferences (time or state non-separable preferences). Grishchenko [7] documents that long-horizon aggregate returns are more consistent with habit formation than “catching up with the Joneses” preferences.
with $W_t$ a standard Wiener process (or standard Brownian motion). The process $z(t)$ attains a stationary distribution, henceforth noted by $Z$ (or simply $z$ if no confusion is possible). The density is:

$$p_z(z) = k \exp(2b l(m^2 \sigma^2))(1-z)^{2(n+a-b-m^2 \sigma^2)}(m^2 \sigma^2) z^{2(b-a-n)(m^2 \sigma^2)} \exp(-2b l(m^2 \sigma^2 z)), \quad 0 \leq z < 1$$ (2)

where

$$k^{-1} = \left( \frac{2b}{m^2 \sigma^2} \right)^{1-2(n+a-b)/m^2 \sigma^2} \Gamma \left( \frac{2(n+a-b)}{m^2 \sigma^2} - 1 \right),$$

with $\Gamma(\cdot)$ the gamma function and $m, n, a, b, \sigma$ positive model parameters. The derivation of Eqs. (1) and (2) can be found in Constantinides (op. cit., Th. 2 and App. B). The diffusion in Eq. (1) is related to the class of forward equations studied in Wong [9]. The density is unimodal and satisfies $p_z(0) = p_z(1) = 0$.

The parametrization used is redundant. For the sake of compactness we could redefine the model parameters, but we prefer to maintain the notation of the original paper where the distribution of $Z$ appeared for the first time. The convergence condition $n + a - b > m^2 \sigma^2$ is imposed.\(^2\)

In Section 2 we calculate $E(Z)$ in terms of the upper incomplete gamma function. In Section 3 we compute $E(z^N)$, with $N$ any integer number, by means of a recurrence relationship. Section 4 concludes.

2. **Expected value**

The following result gives $E(Z) = \int_0^1 z p_z(z) dz$.

**Theorem 1**: The expected value of $Z$ is given by:

$$E(Z) = \theta^\alpha \exp(\theta \Gamma(1 - \alpha; \theta))$$ (3)

with $\alpha = 2(n+a-b)/(m^2 \sigma^2) - 1$, $\theta = 2b l(m^2 \sigma^2)$ and $\Gamma(\cdot;\cdot)$ the upper incomplete gamma function.

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\(^2\) In particular, we can substitute the parameters $(m, n, a, b, \sigma)$ by $(\delta, \nu, \sigma)$, with $\delta = blm^2$ and $\nu = 2(n+a-b)/(m^2 \sigma^2) - 1$. The convergence condition is then $\nu > 1$. The density of $Z$ is rewritten as $p_z(z) = k e^{2\delta l / \sigma^2} (1-z)^{\nu-1} z^{-(\nu+1)} \exp(-2\delta l(\sigma^2 z))$ with $k \equiv (2\delta / \sigma^2)^{\nu} / \Gamma(\nu)$ as in Chapman [2]. The three parameters $(\delta, \nu, \sigma)$ do not suffice to re-parametrize Eq. (1).
**Proof:** Define \( X = Z^{-1} \). The moment generating function (mgf) of \( X \) is simple to calculate:

\[
\phi_X(t) := E(\exp(tX)) = E(\exp(tZ^{-1})) = \int_0^1 \exp(t/z)p_z(z)dz = \]

\[
= k \exp(2b/l(m^2\sigma^2)) \int_0^1 (1-z)^{(\alpha+a-b-m^2\sigma^2)/l(m^2\sigma^2)} z^{2(b-a-n)/l(m^2\sigma^2)} \exp[-(1/z)(-t+2b/l(m^2\sigma^2))]dz.
\]

Define \( \alpha := 2(n+a-b)/(m^2\sigma^2) - 1 \), \( \theta := 2b/l(m^2\sigma^2) \) and \( \hat{\theta} := -t + 2b/l(m^2\sigma^2) \). If we denote the density of \( Z \) by \( p_z(z;\alpha,\theta) \), the integrand in the last integral is proportional to \( p_z(z;\alpha,\hat{\theta}) \). Simple manipulations lead to:

\[
\phi_X(t) = (\hat{\theta}/\theta)^{-\alpha} \exp(t) \int_0^1 p_z(z;\alpha,\hat{\theta})dz \text{, so:}
\]

\[
\phi_X(t) = (1-t/\theta)^{-\alpha} \exp(t). \tag{4}
\]

Let us observe that \( E(Z) \) is equal to the moment of order -1 of \( X \): \( E(Z) = E(X^{-1}) \). The moments of negative order were introduced by Chao and Strawderman [1]. Negative moments are useful, for example, to compute the expectation of the maximum likelihood estimator of a negative binomial distribution. Appealing to Cressie et al. [4] and using our Eq. (4), we can write:

\[
E(X^{-1}) = \int \phi_X(-t)dt = \int_0^{\infty} \left( \frac{\theta+t}{\theta} \right)^{-\alpha} \exp(-t)dt = \exp(\theta) \int_0^{\infty} (s/\theta)^{-\alpha} \exp(-s)ds
\]

The last integral involves the upper incomplete gamma function, \( \Gamma(\alpha;z) := \int_0^z s^{-1} \exp(-s)ds \), which leads to Eq. (3). □

**Comments:** We must observe that \( 1-\alpha \) is necessarily negative since by assumption \( n + a - b > m^2\sigma^2 \). Eq. (3) can be alternatively expressed in terms of the exponential integrals. In Geller and Ng [6] the reader can find several alternative characterizations of the exponential integrals. The upper incomplete gamma function is well-defined for negative values of \( \alpha \), in contrast to the case of the

**lower incomplete gamma function**, \( \gamma(\alpha;z) := \int_0^z t^{\alpha-1} \exp(-t)dt \), which is only defined for positive values (the definition of the lower incomplete gamma function can be extended to encompass negative values by means of the

\[\text{With the re-parametrization introduced in footnote (2) we can write}
E(Z) = (2\delta/\sigma^2)^v \exp(2\delta/\sigma^2)\Gamma(1-v;2\delta/\sigma^2), \text{ with } v = \alpha \text{ and } 2\delta/\sigma^2 = \theta.\]
hypergeometric confluent function). The incomplete gamma functions are well-known functions with applications in many areas; see Gautschi [5] and references therein for approximating methods. Many software packages provide accurate approximations for the gamma functions. Eq. (3) avoids the numerical integrations required in Constantinides [3] or Chapman [2], where the computation of the complete gamma function $\Gamma(\cdot)$ is still required.

3. The moments of $Z$

Next we calculate $E(z^N)$, with $N$ any integer number. We use the Fokker-Plank equation to obtain a recursive relationship, which is also applicable to the negative moments. The procedure can be applied to other stationary processes in the class studied by Wong [9].

**Theorem 2:** The moments of $Z$ satisfy the following recursive relationship:

$$(m^2 \sigma^2 / 2)(N+1)E(z^{N+2}) = [(m^2 \sigma^2 / 2)(N+2) - (n+a)]E(z^{N+1}) + bE(z^N). \quad (5)$$

Proof: The Fokker-Plank equation associated to density $p_z(z)$ is obtained from the diffusion equation corresponding to process $z(t)$ in Eq. (1):

$$\frac{1}{2} \frac{\partial^2}{\partial z^2} \left[ z^2 (1-z)^2 m^2 \sigma^2 p_z(z,t) \right] - \frac{\partial}{\partial z} \left[ b - (n+a-m^2 \sigma^2) z - m^2 \sigma^2 z^2 \right](1-z)p_z(z,t) = \frac{\partial}{\partial t} p_z(z,t).$$

(Recall that $p_z$ denotes the density function and the subscripts do not indicate a derivative). We are interested in the stationary distribution $Z$, so the density must satisfy the following first-order ODE:

$$(1/2)(d/dz)[z^2(1-z)^2 m^2 \sigma^2 p_z(z)] - [b - (n+a-m^2 \sigma^2) z - m^2 \sigma^2 z^2](1-z)p_z(z) = 0$$

After operating it can be rewritten as:

$$(m^2 \sigma^2 / 2)(1-z)(d/dz)[z^2 p_z(z)] = (b - (n+a-m^2 \sigma^2) z)p_z(z). \quad (6)$$

We observe that $p_z(z)$ does not satisfy the Pearson equation. In order to calculate the moments of an arbitrary order, we multiply both sides of Eq. (6) by $z^N$ (with $N \geq 1$), and integrate by parts the left hand side. Making $u = z^N(1-z)$ and $dv = (d/dz)(z^2 p_z)dz$ we obtain:

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4 However, the density $p_z(y)$, associated to the distribution $Y=Z/(1-Z)$, does solve it. The stationary distribution $Y$ also plays a role in the habit formation model. The method proposed in the proof of Th. 1 can be used to calculate $E(Y)$ in an alternative manner to the Appendix B of Constantinides.
(m^2 \sigma^2 / 2)(N + 1) \int_0^1 z^{N+2} p_z(z)dz - (m^2 \sigma^2 / 2)N \int_0^1 z^{N+1} p_z(z)dz =

= b \int_0^1 z^N p_z(z)dz - (n + a - m^2 \sigma^2) \int_0^1 z^{N+1} p_z(z)dz.

This equality leads to Eq. (5). □

Comments: Taking N=0 in Eq. (5) we obtain the moment of order 2:

\[ E(z^2) = \frac{b-(n+a-m^2\sigma^2)E(z)}{m^2\sigma^2/2} \]

with E(z) given in Eq. (3). Then, any moment of order strictly higher than 2 can be calculated recursively by using Eqs. (5), (7) and (3). As an example, let us consider the parameter values used in Table 1 of Constantinides [3] with a=0.2 and b=0.172. We obtain for the first five moments: E(z) = 0.799, E(z^2) = 0.646, E(z^3) = 0.528, E(z^4) = 0.435 and E(z^5) = 0.362. The sequence is necessarily decreasing since Z is defined on [0, 1). We can also obtain:

\[ E(1-z)^2 = 1 + \frac{b-(n+a)E(z)}{m^2\sigma^2/2} \]

This quantity is required in the calculation of the unconditional variance of the consumption growth rate, which is necessary in the resolution of the equity premium puzzle based on habit formation. The recursive relationship also gives the negative moments of Z. Taking N = -1 in Eq. (5) we obtain:

\[ E(Z^{-1}) = \frac{n+a-m^2\sigma^2/2}{b} \]

The rest of negative moments are obtained recursively from Eq. (5). For the parameter configuration given above, the first negative moments are E(z^{-1}) = 1.266, E(z^{-2}) = 1.624, E(z^{-3}) = 2.114, E(z^{-4}) = 2.793 and E(z^{-5}) = 3.752. The integration of z^{-N} p_z(z) is not problematic since the presence of \( \exp(-2b/(m^2\sigma^2z)) \) in the density function \( p_z \) prevents from the appearance of singularities at \( z=0 \) (\( \lim_{z \downarrow 0} z^{-N} p_z(z) = 0 \) for any integer \( N \)).

4. Summary and Conclusions

In this paper we calculate the positive and negative moments of a distribution relevant in macroeconomics which is helpful to solve the equity-premium puzzle described by Mehra and Prescott [8]. This distribution (Z) is stationary and is a monotone transformation of another distribution (Y) whose density \( p_Y(y) \) satisfies the Pearson equation, which lies in the class of forward equations studied in Wong [9].
Our Eq. (3) expresses the expectation of Z in closed-form, thus avoiding the use of numerical integrations needed in Constantinides [3] or Chapman [2]. We have calculated \( E(Z) \) using the moment generating function of \( Z^{-1} \), making use of the negative moments introduced by Chao and Strawderman [1]. The analytical solution given in Eq. (3) involves the computation of the upper incomplete gamma function, which is computable by using commercial software packages. If instead the numerical integration is used, the computation of a gamma function is still required.

We also calculate the rest of moments, both positive and negative. They are obtained by means of Eq. (5). The positive moments depend on \( E(Z) \), which as has been mentioned involves the upper incomplete gamma function. The negative moments depend on \( E(Z^{-1}) \), written in terms of elementary functions (see Eq. (9)). The results obtained can be used to empirically test the theoretical model, and in other contexts where the model is applied (e.g., Chapman [2]).

References


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