A Generalized Quaternion with Generalized Fibonacci Number Components

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Abstract

In this paper we introduce a generalized quaternion with generalized Fibonacci number components. For this quaternion we obtain the two types of Catalan’s identities and d’Ocagne’s identity.

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1 Introduction

A quaternion $p$ is defined by

$$p = p_0 + p_1i + p_2j + p_3k,$$

where $p_0, p_1, p_2, p_3 \in \mathbb{R}$, and $i$, $j$, $k$ are the standard basis in $\mathbb{R}^3$ such that

$$i^2 = j^2 = k^2 = -1,$$

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

On the other hand, if

$$i^2 = -1, \quad j^2 = k^2 = 1,$$

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j,$$
then $p$ is a split quaternion.

Many results have been reported in the literature concerning the properties
of quaternions or split quaternions related to diverse integer sequences. Hor-
ardam [8] defined the Fibonacci quaternion sequence $\{H_n\}$ and Lucas quaternion sequence $\{K_n\}$ as

$$H_n = F_n + F_{n+1} + F_{n+2} + F_{n+3},$$
$$K_n = L_n + L_{n+1} + L_{n+2} + L_{n+3},$$

where $F_n$ and $L_n$ are respectively the $n$th Fibonacci and Lucas numbers generated from the recurrence relations

$$F_0 = 0, \ F_1 = 1, \ F_n = F_{n-1} + F_{n-2} \ (n \geq 2),$$
$$L_0 = 2, \ L_1 = 1, \ L_n = L_{n-1} + L_{n-2} \ (n \geq 2).$$

Following the work of Horadam [8], a variety of results have appeared in
the literature. Halici [6] obtained the generating functions, Binet’s formulas

$$S_n = F_n + F_{n+1} + F_{n+2} + F_{n+3},$$
$$T_n = L_n + L_{n+1} + L_{n+2} + L_{n+3},$$

where $F_n$ is the $n$th Fibonacci number, $L_n$ is the $n$th Lucas number, and

$$i^2 = -\gamma, \ j^2 = -\delta, \ k^2 = -\gamma\delta,$$
$$ij = -ji = k, \ jk = -kj = \delta i, \ ki = -ik = \gamma j.$$


In this paper, motivated the works of [7], [2], we introduce a generalized quaternion with generalized Fibonacci number components. For this quaternion we obtain the two types of Catalan’s identities and d’Ocagne’s identity.
2 Main results

Throughout this section we use the notation \( \{G_n\} = S(g_0, g_1, a, b) \) to denote a generalized Fibonacci sequence \( \{G_n\} \) defined by the recurrence relation

\[
G_0 = g_0, \quad G_1 = g_1, \quad G_n = aG_{n-1} + bG_{n-2} \quad (n \geq 2).
\]

Hence \( \{G_n\} = S(0, 1, 1, 1) \) is the classical Fibonacci sequence, and \( \{G_n\} = S(2, 1, 1, 1) \) is the classical Lucas sequence, etc.

The Binet's formula for \( \{G_n\} = S(g_0, g_1, a, b) \) is given by [9]

\[
G_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta},
\]

(1)

where \( A = g_1 - g_0\beta, \quad B = g_1 - g_0\alpha, \quad \alpha \) and \( \beta \) are solutions of the equation \( x^2 - ax - b = 0 \). In particular, the Binet’s formulas for \( \{R_n\} = S(0, 1, a, b) \) and \( \{T_n\} = S(2, a, a, b) \) are respectively given by

\[
R_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad T_n = \alpha^n + \beta^n.
\]

**Definition 2.1** The generalized quaternion sequence \( \{Q_n\} \) with generalized Fibonacci number components is defined by

\[
Q_n = G_n + G_{n+1}i + G_{n+2}j + G_{n+3}k,
\]

(2)

where \( \{G_n\} = S(g_0, g_1, a, b) \), and

\[
i^2 = -\gamma, \quad j^2 = -\delta, \quad k^2 = -\gamma\delta,
\]

\[
i j = -ji = k, \quad jk = -kj = \delta i, \quad ki = -ik = \gamma j.
\]

According to [7], the Binet’s formula for the generalized quaternion sequence \( \{Q_n\} \) is given by

\[
Q_n = \frac{A\alpha^*\alpha^n - B\beta^*\beta^n}{\alpha - \beta},
\]

(3)

where \( A = g_1 - g_0\beta, \quad B = g_1 - g_0\alpha, \) and

\[
\alpha^* = 1 + \alpha i + \alpha^2 j + \alpha^3 k,
\]

\[
\beta^* = 1 + \beta i + \beta^2 j + \beta^3 k.
\]

In the following theorem, we derive two types of Catalan’s identities for the generalized quaternion sequence \( \{Q_n\} \).
Theorem 2.2 Let \( n \) and \( r \) be non-negative integers such that \( n \geq r \). Then
\[
Q_n^2 - Q_{n-r}Q_{n+r} = (-b)^{n-r} AB(2\hat{Q}_r + CR_r + DT_r)R_r,
\]
(4)
\[
Q_n^2 - Q_{n+r}Q_{n-r} = (-b)^{n-r} AB(2\hat{Q}_r + CR_r - \hat{D}T_r)R_r,
\]
(5)
where \( \{R_n\} = S(0,1,a,b) \), \( \{T_n\} = S(2,a,a,b) \) and
\[
\hat{Q}_r = R_r + R_{r+1}i + R_{r+2}j + R_{r+3}k,
\]
\[
C = (\gamma b - 1)(\delta b^2 + 1),
\]
\[
D = (\delta b^2 - 1)i + a(\gamma b - 1)j - (a^2 + 2b)k,
\]
\[
\hat{D} = (\delta b^2 + 1)i + a(\gamma b + 1)j + a^2k.
\]

Proof. From (3), we have
\[
(\alpha - \beta)^2(Q_n^2 - Q_{n-r}Q_{n+r})
\]
\[
= (A\alpha^*\alpha^n - B\beta^*\beta^n)^2 - (A\alpha^*\alpha^{n-r} - B\beta^*\beta^{n-r})(A\alpha^*\alpha^{n+r} - B\beta^*\beta^{n+r})
\]
\[
= AB^*\beta^*\left((\alpha^{n-r}\beta^{n+r} - \alpha^n\beta^n) + AB\beta^*\alpha^*(\alpha^{n+r}\beta^{n-r} - \alpha^n\beta^n)\right)
\]
\[
= AB(\alpha\beta)^n\left(\frac{\alpha^*\beta^*(\beta^r - \alpha^r)}{\alpha^r} + \frac{\beta^*\alpha^*(\alpha^r - \beta^r)}{\beta^r}\right)
\]
\[
= AB(\alpha\beta)^{-n-r}(\alpha^r - \beta^r)(\beta^*\alpha^*\alpha^r - \alpha^*\beta^*\beta^r),
\]
and so
\[
Q_n^2 - Q_{n-r}Q_{n+r} = (-b)^{n-r} AB\left(\frac{\beta^*\alpha^*\alpha^r - \alpha^*\beta^*\beta^r}{\alpha - \beta}\right)R_r.
\]
Now
\[
\alpha^*\beta^* = (1 + \alpha i + \alpha^2 j + \alpha^3 k)(1 + \beta i + \beta^2 j + \beta^3 k)
\]
\[
= \lambda_0 + \lambda_1 i + \lambda_2 j + \lambda_3 k,
\]
where
\[
\lambda_0 = 1 + \gamma b - \delta b^2 + \gamma \delta b^3
\]
\[
= 2 + (\gamma b - 1)(\delta b^2 + 1),
\]
\[
\lambda_1 = \alpha + \beta + \delta \alpha^2 \beta^3 - \delta \alpha^3 \beta^2
\]
\[
= 2\beta - (\delta b^2 - 1)(\alpha - \beta),
\]
\[
\lambda_2 = \alpha^2 + \beta^2 - \gamma \alpha \beta^3 + \gamma \alpha^3 \beta
\]
\[
= 2\beta^2 - a(\gamma b - 1)(\alpha - \beta),
\]
\[
\lambda_3 = \alpha^3 + \beta^3 - \alpha \beta^2 - \alpha^2 \beta
\]
\[
= 2\beta^3 + (a^2 + 2b)(\alpha - \beta).
\]
Hence
\[
\alpha^*\beta^* = 2\beta^* + C - D(\alpha - \beta).
\]
Similarly we have
\[ \beta^* \alpha^* = 2\alpha^* + C + D(\alpha - \beta). \]
Then
\[
\begin{align*}
\beta^* \alpha^* \alpha^r - \alpha^* \beta^* \beta^r & = \left(2\alpha^* + C + D(\alpha - \beta)\right)\alpha^r - \left(2\beta^* + C - D(\alpha - \beta)\right)\beta^r \\
& = 2(\alpha^* \alpha^r - \beta^* \beta^r) + C(\alpha^r - \beta^r) + D(\alpha^r + \beta^r)(\alpha - \beta) \\
& = 2\left(\alpha^r - \beta^r + (\alpha^{r+1} - \beta^{r+1})i + (\alpha^{r+2} - \beta^{r+2})j + (\alpha^{r+3} - \beta^{r+3})k\right) \\
& \quad + C(\alpha^r - \beta^r) + D(\alpha^r + \beta^r)(\alpha - \beta),
\end{align*}
\]
and the proof of (4) is completed.

We can show that
\[
\begin{align*}
\alpha^* \beta^* & = 2\alpha^* + C - \hat{D}(\alpha - \beta), \\
\beta^* \alpha^* & = 2\beta^* + C + \hat{D}(\alpha - \beta).
\end{align*}
\]
Then (5) also proved similarly, and details are omitted.

The Catalan’s identities for the split $k$-Fibonacci and $k$-Lucas quaternions given in [10, Theorem 2.8] are special cases of (4).

The Catalan’s identities for the split Jacobsthal and Jacobsthal-Lucas quaternions given in [17, Theorem 6] are special cases of (5).

For $r = 1$, we obtain the Cassini’s identities for the generalized quaternion sequence \{\(Q_n\)\} as in the following corollary.

**Corollary 2.3** Let $n$ be a positive integer. Then
\[
\begin{align*}
Q^2_n - Q_{n-1}Q_{n+1} & = (-b)^{n-1}AB(2\hat{Q}_1 + C + a\hat{D}), \quad (6) \\
Q^2_n - Q_{n+1}Q_{n-1} & = (-b)^{n-1}AB(2\hat{Q}_1 + C - a\hat{D}), \quad (7)
\end{align*}
\]
where
\[
\begin{align*}
\hat{Q}_1 & = 1 + ai + (a^2 + b)j + (a^3 + 2ab)k, \\
C & = (\gamma b - 1)(\delta b^2 + 1), \\
D & = (\delta b^2 - 1)i + a(\gamma b - 1)j - (a^2 + 2b)k, \\
\hat{D} & = (\delta b^2 + 1)i + a(\gamma b + 1)j + a^2k.
\end{align*}
\]

The Cassini’s identities for the split Fibonacci and Lucas quaternions given in [1, Theorem 2.7] are special cases of (6).

The Cassini’s identities for the Fibonacci and Lucas generalized quaternions given in [2, Theorem 3.7] are special cases of (6).

In the next theorem, we derive the d’Ocagne’s identity for the generalized quaternion sequence \{\(Q_n\)\}.
Theorem 2.4 Let $m$ and $n$ be non-negative integers such that $m \geq n$. Then
\[ Q_m Q_{n+1} - Q_m Q_n = (-b)^n AB(2\hat{Q}_{m-n} + CR_{m-n} - \hat{D}T_{m-n}), \tag{8} \]
where \( \{R_n\} = S(0, 1, a, b) \) and \( \{T_n\} = S(2, a, a, b) \) and
\[
\begin{align*}
\hat{Q}_{m-n} &= R_{m-n} + R_{m-n+1}i + R_{m-n+2}j + R_{m-n+3}k, \\
C &= (\gamma b - 1)(\delta b^2 + 1), \\
\hat{D} &= (\delta b^2 + 1)i + a(\gamma b + 1)j + a^2k.
\end{align*}
\]

Proof. From (3), we have
\[
(\alpha - \beta)^2 (Q_m Q_{n+1} - Q_m Q_n)
= (A\alpha^m - B\beta^m)(A\alpha^{n+1} - B\beta^{n+1}) - (A\alpha^m - B\beta^m)(A\alpha^n - B\beta^n)
= AB(\alpha^m \beta^n - \beta^m \alpha^n)(\alpha - \beta)
= (-b)^n AB(\alpha^m \beta^n - \beta^m \alpha^n)(\alpha - \beta).
\]

Since
\[
\begin{align*}
\alpha^m \beta^n - \beta^m \alpha^n &= \left(2\alpha^* + C - \hat{D}(\alpha - \beta)\right)a^m - \left(2\beta^* + C + \hat{D}(\alpha - \beta)\right)b^m \\
&= 2(\alpha^m - \beta^m) + C(\alpha^m - \beta^m) - \hat{D}(\alpha^m + \beta^m)(\alpha - \beta) \\
&= 2(\alpha^m - \beta^m) + (\alpha^{m+1} - \beta^{m+1})i + (\alpha^{m+2} - \beta^{m+2})j + (\alpha^{m+3} - \beta^{m+3})k \\
&+ C(\alpha^m + \beta^m)(\alpha - \beta),
\end{align*}
\]
we obtain (8).

The d’Ocagne’s identities for the split $k$-Fibonacci and $k$-Lucas quaternions given in [10, Theorem 2.10] are special cases of (8).
The d’Ocagne’s identities for the split Jacobsthal and Jacobsthal-Lucas quaternions given in [17, Theorem 7] are special cases of (8).

References

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