Applied Mathematical Sciences, Vol. 13, 2019, no. 11, 503 - 510
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ams.2019.9462

Equations for the Four-Point Rectangle

G. L. Silver

6439 Bethany Village Drive #302
Centerville OH 45359-3575, USA

Abstract

Four numbers in a rectangular array can be interpolated by the bilinear equation. If the numbers are positive, they can be interpolated by a new bi-cubic equation. The array can be interpolated by eight new fourth-degree equations. The positive square-roots of the fourth-degree equations are new bi-quadratic equations that are applicable to the analysis of the same four-point arrays. The new equations are suitable for the analysis of two-parameter laboratory experiments.

Mathematics Subject Classification: 65D05, 65D07, 65D17

Keywords: Interpolation, rectangle array, cubic equations, quartic equations

Introduction

The bilinear equation is the principal instrument for the interpolation of four numbers in a rectangular array. Several years ago, a new equation appeared for the interpolation of four positive, bilinear numbers in rectangular array. The new equation is also exact on the squares of the original bilinear numbers [1].

A cubic equation for four positive numbers in a rectangular array

Let the letters A, B, C, D define a four-point rectangle as in Fig. 1. The coordinates of vertices A, B, C, and D are (–1,–1), (1,–1), (–1,1), and (1,1) respectively. The distance between any vertex and its two nearest neighbors is two units. Randomly chosen points within the rectangle can have two positive coordinate numbers, or two negative coordinate numbers, or one positive and one negative coordinate number.

C D

A B

Fig. 1. A four-point rectangle

If numbers at vertices A, B, C, and D are 9^3, 7^3, 3^3, 1^3, respectively, the rectangle in Fig. 1 can be interpolated by the expression $z = (5–x–3y)^3$. The expansion of this expression has four positive terms and six negative terms. The proliferation of signs invites mistakes when applying polynomial equations of degree 3 or higher degrees.

For the sake of simplicity, let a new coordinate system be selected: 0 .. 2 in the x-direction, and 0 .. 2 in the y-direction. In the new coordinate system, vertices A, B, C, D in Fig. 1 have (x,y) coordinates (0,0), (2,0), (0,2), and (2,2), respectively. This choice of coordinates continues the tradition of two units of distance between nearest vertices in the four-point rectangle. The alternative system has the advantage that it eliminates the proliferation of sign changes in polynomial interpolation equations. It is also suggests new interpolation methods.

In the new coordinate system, the general form of a cubic equation for the four-point rectangle is Eq. (1). It has three unknowns: (xc), (yc), and (xyc).

$$z = (A + (xc)x + (yc)y + (xyc)xy)^3 \tag{1}$$

A second change can be made for the sake of simplicity. Suppose the original data at vertices [A,B,C,D] are [8,64,512,1000], respectively, as in Fig. 1. Divide each number by the number at vertex A of the four-point rectangle. The original data [A,B,C,D] are now [1,8,64,125], respectively.

The expression on the right-hand side of Eq. (1) can now be applied to generate four new simultaneous equations. For example, at the origin of the new coordinate system the coordinates of point A are (0,0) and the datum at vertex A is 1. The new set of simultaneous equations is Eqs. (3), (4), and (5). These equations derive from the transformed data and the new coordinate system: x = 0 .. 2, y = 0 .. 2. See Fig. 1. For computational purposes, the original set of four simultaneous equations has been reduced to a set of three simultaneous equations: Eqs. (3), (4), and (5).

$$1^3 – 1 = 0 \tag{2}$$

$$(1 + (2)xc)^3 – 8 = 0 \tag{3}$$

$$(1 + (2)yc)^3 – 64 = 0 \tag{4}$$

$$(1 + (2)xc + (2)yc + (4)xyc)^3 - 125 = 0 \quad (5)$$

The set of solutions to the simultaneous Eqs. (3), (4), and (5) has several members but only one member does not contain the imaginary unit ***I***. That member has xc = 0.50, yc = 1.50, and xyc = 0.0. To reproduce the original data, the former divisor (the number 8) now prefixes the new interpolation equation as in Eq. (6).

$$z = (8)(1 + (0.50)x + (1.50y))^3 \quad (6)$$

Equation (6) reproduces the original data in the coordinate system (x = 0 .. 2, y = 0 .. 2). This example illustrates how to obtain a third-degree, cubic equation for interpolating a four-point rectangle. The method first changes the coordinates: it takes the origin of the new coordinate system as the lower left vertex of rectangle (A) as (x,y) = (0,0). The dimensions of the rectangle remain two units horizontally and two units vertically. All interior points of rectangle ABCD have (x,y) coordinates that are zero or positive numbers.

Equation (6) can be rewritten for the standard coordinate system: x = –1 .. 1, y = –1 .. 1. To make this change, replace every (x) by (x+1) and every (y) by (y+1). Expand the resulting equation and rearrange the terms. See Eq. (7). That equation reproduces the original data in the original coordinate system.

$$z = 8(3.0 + 0.50(x) + 1.50(y))^3 \quad (7)$$

As a second example, let the numbers at vertices [A,B,C,D] be [8,27,343,125], respectively. Reducing the data (by division by the number at vertex A of the rectangle) renders a new set of numbers at vertices [A,B,C,D]. These numbers are [(1),(27/8), (343/8),(125/8)], respectively.

The new equations are Eqs. (8)–(11). They form a simultaneous set.

$$1^3 - 1 = 0 \quad (8)$$

$$(1 + 2(xc))^3 - 27/8 = 0 \quad (9)$$

$$(1 + 2(yc))^3 - 343/8 = 0 \quad (10)$$

$$(1 + 2(xc) + 2(yc) + 4(xyc))^3 - 125/8 = 0 \quad (11)$$

The solution of Eqs. (9)–(11), in real numbers, is {xyc = –3/8, xc = 1/4, yc = 5/4}. The interpolation equation for the rectangle in this example is Eq. (12).

$$z = (-1)(x + 3xy - 7y - 17)^3/64 \quad (12)$$

Fourth-power equations for four positive numbers in a rectangular array

Forth-power interpolation equations for the four-point rectangle are derived in the same manner as bi-cubic equations. Equation (13) now replaces Eq. (1).

$$z = (A + (xc)x + (yc)y + (xyc)xy)^4 \quad (13)$$

<u>First Example</u>

Let the data at vertices A, B, C, D, be [1^4, 3^4, 7^4, 9^4] or [1, 81, 2401, 6561], respectively. In this example, the ratios (A/A), (B/A), (C/A), (D/A) are similar to the ratios applied in the preceding section. In the new coordinate system (x = 0 .. 2, y = 0 .. 2) so Eq. (14) reduces to Eq. (15). Equations (16)–(18) follow as above.

$$(A/A + (xc)x + (yc)y + (xyc)xy)^4 - 1 = 0 \quad (14)$$
$$1^4 - 1 = 0 \quad (15)$$
$$(1 + 2(xc))^4 - 81 = 0 \quad (16)$$
$$(1 + 2(yc))^4 - 2401 = 0 \quad (17)$$
$$(1 + 2(xc) + 2(yc) + 4(xyc))^4 - 6561 = 0 \quad (18)$$

There are eight real solutions to Eqs. (16)–(18). They render eight potential interpolation equations that apply in the x = 0 .. 2 and y = 0 .. 2 coordinate system.

$$z = (1 + x - 4y - xy)^4 \quad (19)$$

$$z = (1 + x - 4y + 3.50xy)^4 \quad (20)$$

$$z = (1 + x + 3y)^4 \quad (21)$$

$$z = (1 + x + 3y - 4.50xy)^4 \quad (22)$$

$$z = (1 - 2x - 4y + 0.50xy)^4 \quad (23)$$

$$z = (1 - 2x - 4y + 5xy)^4 \quad (24)$$

$$z = (1 - 2x + 3y - 3xy)^4 \quad (25)$$

$$z = (1 - 2x + 3y + 1.50xy)^4 \quad (26)$$

The preceding equations are now multiplied by the number assigned to the vertex A in Fig. 1. In this case, that number is 1, so Eqs. (19)–(26) do not change. The next step is to convert Eqs. (19)–(26) to the original coordinate system: (x = –1 .. 1) and (y = – 1 .. 1). This transformation is accomplished by substituting (x+1) for every (x) and (y+1) for every (y) in Eqs. (19)–(26) and simplifying the result. The eight interpolation equations, for the original data in the original coordinate system, are Eqs. (27)–(34). Plot each equation and choose the most appropriate surface.

$$z = (xy + 5y + 3)^4 \tag{27}$$

$$z = (7xy + 9x - y + 3)^4/(16) \tag{28}$$

$$z = (5 + x + 3y)^4 \tag{29}$$

$$z = (9xy + 7x - 1 + 3y)^4/(16) \tag{30}$$

$$z = (xy - 3x - 7y - 9)^4/(16) \tag{31}$$

$$z = (3x + y + 5xy)^4 \tag{32}$$

$$z = (5x + 3xy + 1)^4 \tag{33}$$

$$z = (3xy - x + 9y + 7)^4/16 \tag{34}$$

<u>Second Example</u>

The data at vertices A, B, C, D in Fig. 1 are 16, 256, 625, 1296, respectively. The first step is to reduce the data by dividing them by A as above. The reduced data become 1, 16, 625/16, and 81, respectively. In the x = 0 .. 2, y= 0 .. 2 coordinate system (see above) the simultaneous equations to be solved are Eqs. (36)–(38).

$$1^4 - 1 = 0 \tag{35}$$

$$(1 + 2(xc))^4 - 16 = 0 \tag{36}$$
$$(1 + 2(yc))^4 - 625/16 = 0 \tag{37}$$
$$(1 + 2(xc) + 2(yc) + 4(xyc))^4 - 81 = 0 \tag{38}$$

Eight solutions are thereby obtained for xc, yc, and xyc. See Eqs. (39)–(46) below.

$$\{xc = -3/2,\ yc = -7/4,\ xyc = 5/8\} \tag{39}$$
$$\{xc = -3/2,\ yc = -7/4,\ xyc = 17/8\} \tag{40}$$
$$\{xc = -3/2,\ yc = 3/4,\ xyc = 7/8\} \tag{41}$$

$$\{xc = –3/2,\ yc = 3/4,\ xyc = –5/8\} \quad (42)$$
$$\{xc = 1/2,\ yc = –7/4,\ xyc = –3/8\} \quad (43)$$
$$\{xc = – 7/4,\ yc = 1/2,\ xyc = 9/8\} \quad (44)$$
$$\{xc = 1/2,\ yc = 3/4,\ xyc = –1/8\} \quad (45)$$
$$\{xc = 1/2,\ yc = 3/4,\ xyc = –13/8\} \quad (46)$$

The preceding eight solutions are substituted into Eq. (13). The former divisor 16 now prefixes each equation. These substitutions render intermediate, temporary forms (denoted by the letter T) of the eight interpolation equations based on Eqs. (19)–(26). See Eqs. (47)–(54).

$$T_1 = 16(1 – 3x/2 – 7y/4 + (5xy/8))^4 \quad (47)$$

$$T_2 = 16(1 – 3x/2 – 7y/4 + (17xy/8))^4 \quad (48)$$

$$T_3 = 16(1 –3x/2) + 3y/4 + 7xy/8)^4 \quad (49)$$

$$T_4 = 16(1 – 3x/2 + 3y/4 – 5xy/8)^4 \quad (50)$$

$$T_5 = 16(1 + x/2 – 7y/4 – 3xy/8)^4 \quad (51)$$

$$T_6 = 16(1 + x/2 – 7y/4 + 9xy/8)^4 \quad (52)$$

$$T_7 = 16(1 + x/2 + 3y/4 – xy/8)^4 \quad (53)$$

$$T_8 = 16(1 + x/2 + 3y/4 –13xy/8)^4 \quad (54)$$

The last step in the process of developing fourth-power interpolation equations for a four-point rectangle involves converting Eqs. (47)–(54) into equations that apply in the familiar coordinate system in which x = –1 .. 1, y = –1 .. 1. This transformation is accomplished by changing every (x) into (x+1) and changing every (y) into (y+1) followed by expanding and simplifying the results. See Eqs. (55)–(62).

$$z = (5xy – 7x – 9y – 13)^4/(256) \quad (55)$$
$$z = (17xy + 5x + 3y – 1)^4/(256) \quad (56)$$
$$z = (7xy – 5x + 13y + 9)^4/(256) \quad (57)$$
$$z = (5xy + 17x – y + 3)^4/(256) \quad (58)$$
$$z = (3xy – x + 17y + 5)^4/(256) \quad (59)$$
$$z = (9xy + 13x – 5y + 7)^4/(256) \quad (60)$$
$$z = (xy – 3x – 5y – 17)^4/(256) \quad (61)$$
$$z = (13xy + 9x + 7y – 5)^4/(256) \quad (62)$$

Each one of Eqs. (55)–(62) reproduces the original data at vertices A, B, C, D in Fig. 1. Equations that predict ridges or troughs may be questionable. Each equation estimates the center point of the rectangle where [x,y] = [0,0]. Equation (61) estimates center-point as ≈ 326.25391. The fourth root of this number is ≈ 4.25. This estimate is close to the average value of the fourth-roots of the original data: ≈ 4.25. That suggests Eq. (61) is a reasonable first choice from Eqs. (55)–(62).

Discussion

The positive square roots of Eqs. (27)–(34) are eight bi-quadratic interpolation equations for the four-point rectangle. One of these equations, derived from Eq. (29), is the same result as obtained by operational methods applied to the same data. See Eq. (7) in [1] and the discussion following that equation. This observation implies the preceding methods are potentially more fertile than methods that are based on applied operational calculus [2,3]. However, one example is not a satisfactory basis for generalizations.

The equations illustrated herein, including the positive square roots of equations containing the exponent 4, can be plotted and the resulting surfaces examined for their properties. Plots of response surfaces are useful for deciding which member of a group of interpolation equations is likely to be a good choice for representing laboratory measurements.

Conclusion

Let four positive numbers define a rectangle. Many such rectangles can be interpolated by second-, third-, or fourth-power polynomial equations. That idea is an advance over the widespread impression that the bilinear equation is the singular instrument for interpolating a four-point rectangle. The new equations provide a variety of response surfaces for four-point rectangles defined by positive numbers. The alternative equations can also render curvature estimations. The bilinear equation cannot do that.

References

[1] G. L. Silver, Operational equations for data in rectangular array, *Computational Statistics and Data Analysis*, **28** (1998), 211-215. https://doi.org/10.1016/s0167-9473(98)00029-2

[2] G. L. Silver, Operational Method of Data Treatment, *J. Computational Chemistry*, **6**(3) (1985), 229-236. https://doi.org/10.1002/jcc.540060310

[3] G. L. Silver, Deriving Operational Equations, *Applied Mathematical Sciences*, **2**(9) (2008), 397-406.

Received: April 23, 2019; Published: May 25, 2019