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Sensitivity of Risk Measures Defined on L^1-Spaces

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Abstract

In this paper, we introduce the notion of the Strong Sesitivity for a risk measure, being defined on a L^1-space. We prove Strong Sensitivity of the Expected Shortfall, and consequently of any spectral risk measure. Moreover, we deduce the Strong Sensitivity of the pointwise limit of spectral risk measures.

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1. Introduction

Expected Shortfall ES_a is confirmed to be the risk measure that replaces Value-at-Risk according to the Basel Committe (BCBS (2013)) for regulatory capital purposes in the trading book. In this paper, we show that ES_a is the building block for risk measures with interesting accuracy properties. We consider a non-atomic probability space $(\Omega, \mathcal{F}, \mu)$. A risk measure $\rho : E \to \mathbb{R}$, where E is a subspace of $L^0(\Omega, \mathcal{F}, \mu)$, is called *law-invariant* if $Z =^d Z'$ implies $\rho(Z) = \rho(Z')$, $Z, Z' \in E$. Also, ρ is called *strictly sensitive* if $Z \geq Z', \mu$- a.e., $\rho(Z) = \rho(Z') \Rightarrow Z = Z', \mu$-a.e. [3]. From [6], it is well -known known that law-invariant coherent risk measures on L^∞ admit the representation

$$\rho(X) = \sup_{\mu \in \mathcal{M}} \int_0^1 ES_a(X) d\mu(a), (1)$$

where $\mathcal{M}$ is a set of probability measures on $[0, 1]$. This representation is extended for L^p spaces $p \in [1, \infty)$ in [7] for the closure of $\mathcal{M}$ in the weak topology (of the probability measures). We selected the space L^1, because it is the **maximal** L^p -space which admits at least one *locally-convex topology* and includes heavy -tailed random variables. In this brief paper, we state a definition of sensitivity close to the one in [3], which is valid for the most of the risk measures which rely on Expected Shortfall.

2. Sensitivity on Classes of Risk Measures

By the notation L^1, we mean $L^1(\Omega, \mathcal{F}, \mu)$, where $(\Omega, \mathcal{F}, \mu)$ is a non-atomic probability space. A partially ordered vector space E is a *vector lattice* if for any $x, y \in E$, the supremum and the infimum of $\{x, y\}$ with respect to the partial ordering defined by E_+ exist in E. In this case $\sup\{x, y\}$ and $\inf\{x, y\}$ are denoted by $x \vee y$, $x \wedge y$ respectively. If so, $|x| = \sup\{x, -x\}$ is the *absolute value* of x and if E is also a normed space such that $\| \, |x| \, \| = \|x\|$ for any $x \in E$, then E is called *normed lattice*. If a normed lattice is a Banach space, then it is called *Banach lattice*. For more details on these spaces, see [1, Ch.7,8].

Definition 2.1. *A monetary risk measure ρ on L^1 is* ***strongly sensitive****, if and only if $X = Y, \mu$-a.e. $\Rightarrow \rho(X) = \rho(Y)$.*

Theorem 2.2. *For any $a \in (0, 1]$, the Expected Shortfall ES_a is strongly sensitive.*

Proof. By the dual representation theorem of Expected Shortfall [4, Th.4.1], we have

$$ES_a(X) = \max_{\pi \in [0, \frac{1}{a}\mathbf{1}]} \pi(-X),$$

such that π is a Radon -Nikodym derivative of some probability measure $Q_\pi << \mu$,

$$ES_a(|X|) \leq \frac{1}{a}\|X\|_1,$$

which implies that ES_a is strongly sensitive. □

Theorem 2.3. *For any $a \in (0, 1], b > 1$, such that $\frac{1}{b} < \frac{1}{a}$, Adjusted Expected Shortfall is strongly sensitive.*

Proof. By [5, Lem.6] regarding the dual representation of Adjusted Expected Shortfall, we get that

$$AES_{a,b}(|X|) \leq \frac{1}{a}\|X\|_1,$$

hence the specific risk measure is strongly sensitive on L^1. □

Theorem 2.4. *Any spectral risk measure of the form*

$$M_m(X) = \int_0^1 aES_a(X)dm(a),$$

defined on L^1 is strongly sensitive.

Proof. By [2, Th.2.5], any spectral risk measure M_m is a continuous, coherent risk measure on L^1, since $\int_0^1 adm(a) = 1$ and ES_a is a continuous, coherent risk measure on L^1. The coherence of M_m is implied by [2]. The continuity of M_m is implied by relation (6) in [2]. More specifically,

$$|M_m(|X|)| \leq \int_0^1 ES_a(|X|)adm(a) \leq \frac{1}{a}\|X\|_1 \cdot \int_0^1 adm(a) \leq$$

$$\leq \frac{1}{a} \cdot \|X\|_1.$$

□

Theorem 2.5. *The pointwise limit of spectral risk measures on L^1 under the same measure of risk spectrum m is a strongly sensitive coherent risk measure, if there exists some $b > 0$ such that $\frac{1}{a_n} \leq \frac{1}{b}$, for any $n \in \mathbb{N}$.*

Proof. If

$$\rho(X) = \lim_n \int_0^1 ES_{a_n}(X)a_n dm(a_n),$$

then since $ES_{a_n}(|X|) \leq \frac{1}{a_n}\|X\|_1$. If there exists some $b > 0$ such that $\frac{1}{a_n} \leq \frac{1}{b}$, for any $n \in \mathbb{N}$, while $\int_0^1 a_n dm(a_n) = 1$ for any $n \in \mathbb{N}$ as well,

$$ES_{a_n}(X) \leq \frac{1}{b}\|X\|_1,$$

for any $n \in \mathbb{N}$. The last inequality implies

$$\rho(X) \leq \frac{1}{b}\|X\|_1,$$

which completes the proof. □

Theorem 2.6. *Any Kusuoka Representable risk measure on L^1 is strongly sensitive.*

Proof. By [7, Pr.1], any such risk measure ρ is a coherent risk measure on L^1, which admits the representation

$$\rho(Z) = \sup_{\mu \in \overline{\mathcal{M}}} \int_0^1 ES_a(Z)d\mu(a),$$

where $Z \in L^1$ and $\overline{\mathcal{M}}$ denotes the closure under weak topology, for a set of probability measures $\mathcal{M}$ defined on $[0,1]$, which implies that the set $\overline{\mathcal{M}}$ is norm-bounded. More specifically,

$$|\rho(|Z|)| \leq \int_0^1 ES_a(|Z|)d\mu(a) \leq \frac{c}{a}\|Z\|_1,$$

where $\int_0^1 d\mu(a) \leq c$, due to the fact that the closure of $\mathcal{M}$ is norm -bounded.

□

Theorem 2.7. *The pointwise limit of a sequence of Kusuoka Representable risk measures on L^1 is a strongly sensitive coherent risk measure, if $c_n = \sup_{\mu \in \overline{\mathcal{M}_n}} \int_0^1 d\mu(a)$ is upper bounded, where $\overline{\mathcal{M}_n}$ denotes the closure under weak topology of a set of probability measures $\mathcal{M}_n$, defined on $[0,1]$, for any Kusuoka Representable risk measure $\rho_n, n \in \mathbb{N}$.*

Proof. By [7, Pr.1], any such risk measure ρ_n of the sequence $(\rho_n)_{n\in\mathbb{N}}$, where any of its terms is a coherent risk measure on L^1, which admits the representation

$$\rho_n(Z) = \sup_{\mu \in \overline{\mathcal{M}_n}} \int_0^1 ES_a(Z) d\mu(a).$$

This definition denotes that $Z \in L^1$ and $\overline{\mathcal{M}_n}$ denotes the closure under weak topology of a set of probability measures $\mathcal{M}_n$, defined on $[0,1]$, for any $n \in \mathbb{N}$. More specifically,

$$|\rho_n(Z)| \leq \int_0^1 ES_a(|Z|) d\mu(a) \leq \frac{c_n}{a} \|Z\|_1,$$

where $c_n = \sup_{\mu \in \overline{\mathcal{M}_n}} \int_0^1 d\mu(a)$, for any $n \in \mathbb{N}$. Since $\rho_n(Z) \to \rho(Z)$, this implies that the risk measure ρ is a coherent risk measure, by the following Proposition of the present paper. □

We may quote on the coherence of pointwise limit of a sequence of coherent risk measures, being defined on L^1 at this point.

Proposition 2.8. *The pointwise limit ρ of a sequence of coherent risk measures $(\rho_n)_{n\in\mathbb{N}} : L^1 \to \mathbb{R}$, is a coherent risk measure on L^1.*

Proof. It suffices to prove that ρ satisfies the four properties of coherence. $\rho_n(Z + a\mathbf{1}) = \rho_n(Z) - a, Z \in L^1, a \in \mathbb{R}$ denotes the Translation Invariance of $\rho_n, n \in \mathbb{N}$. From the uniqueness of the limit of the sequence of real numbers $(\rho_n(Z + a\mathbf{1}))_{n\in\mathbb{N}}$, which is $\rho(Z + a\mathbf{1})$ is equal to $\rho(Z) - a$. By the same way we deduce the Positive Homogeneity of ρ. About Subadditivity of ρ, we notice that for any $X, Y \in L^1$ and $n \in \mathbb{N}$, the inequality

$$\rho_n(X + Y) - \rho_n(X) - \rho_n(Y) \leq 0,$$

holds in the set on real numbers. This implies that for the limit of this sequence

$$\rho(X + Y) - \rho(X) - \rho(Y),$$

the same inequality is true. Finally, if $X \geq Y, \mathbb{P} - a.e.$ for any $n \in \mathbb{N}$, the inequality

$$\rho_n(X) - \rho_n(Y) \leq 0,$$

holds in the set of real numbers. This implies that for the limit $\rho(X) - \rho(Y)$, the same inequality holds. □

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