Abstract

A general theorem on the distribution of certain composite numbers is proved. This theorem include as particular cases another theorems on composite numbers proved by the author in another papers. The method used here is more direct, simple and short. This general theorem is applicable to the distribution of composite numbers with some restriction.

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1 Introduction and Main Result

Let us consider the prime factorization of a positive integer $n = p_1^{t_1} \cdots p_r^{t_r}$ where the $p_i \ (i = 1, \ldots, r)$ are the different primes in the prime factorization and the $t_i \ (i = 1, \ldots, r)$ are the multiplicities or exponents. Let $h \geq 2$ an arbitrary but fixed positive integer. A number such that $t_i \geq h \ (i = 1, \ldots, r)$ is called $h$-full number. The number of $h$-full numbers not exceeding $x$ will be denoted $A_h(x)$. We have the following well-known theorem, we use it as a lemma.

Lemma 1.1 The following asymptotic formula holds.

$$A_h(x) \sim c_h x^{\frac{1}{h}}$$

where $c_h$ is a positive constant depending of $h$. 
Proof. An elementary proof is given in [3]. More precise asymptotic formulae are given in [2]. The lemma is proved.

Let us consider a sequence of \((h + 1)\)-full numbers \(C_{n,h+1} (h \geq 1)\).

**Lemma 1.2** The following series converges.

\[
AC_{n,h+1} = \sum_{C_{n,h+1}} \frac{1}{(C_{n,h+1})^h}
\]

where \(AC_{n,h+1}\) denotes the sum of the series and the notation \(\sum_{C_{n,h+1}}\) mean that the sum run on all the numbers of the sequence \(C_{n,h+1}\).

Proof. Let \(s_n\) be the sequence of all \((h + 1)\)-full numbers. Substituting \(s_n\) in equation (1) we obtain that \(s_n \sim d_{h+1} n^{h+1}\), where \(d_{h+1}\) is a positive constant. Now, the series \(\sum_{n=1}^{\infty} \frac{1}{n^{h+1}}\) converges. The lemma is proved.

Let \(k\) be an arbitrary but fixed positive integer. Let us consider the numbers such that their prime factorization is of the form \(p_1p_2\cdots p_k\), where the \(p_i\) \((i = 1, 2, \ldots, k)\) are the distinct primes in the prime factorization. We denote a number with this prime factorization \(P_k\). Let \(\pi_k(x)\) be the number of these numbers not exceeding \(x\). If \(k = 1\) we obtain the prime numbers, and consequently \(\pi_1(x) = \pi(x)\) is the prime counting function. We have the following well-known theorem, we use it as a lemma.

**Lemma 1.3** The following asymptotic formulae hold.

\[
\pi_k(x) = \frac{x(\log \log x)^{k-1}}{(k-1)! \log x} + o \left( \frac{x(\log \log x)^{k-1}}{\log x} \right) \quad (k \geq 1)
\]

If \(k = 1\) we have the inequality

\[
\pi_1(x) = \pi(x) < D_1 \frac{x}{\log x} \quad (x \geq 2)
\]

where \(D_1\) is a positive constant.

If \(k \geq 2\) we have the inequality

\[
\pi_k(x) < D_k \frac{x(\log \log x)^{k-1}}{\log x} \quad (x \geq 3)
\]

where \(D_k\) is a positive constant.

Proof. See, for example, [1]. The lemma is proved.

Let \(m\) be an arbitrary but fixed positive integer. Let \(\pi_{k,m}(x)\) be the number of \(P_k\) numbers not exceeding \(x\) relatively prime to \(m\), that is \((P_k, m) = 1\). We have the following lemma.
Lemma 1.4 The following asymptotic formula holds.

\[ \pi_{k,m}(x) = \frac{x(\log \log x)^{k-1}}{(k-1)! \log x} + o\left(\frac{x(\log \log x)^{k-1}}{\log x}\right) \quad (k \geq 1) \]  

Proof. Suppose that the prime factorization of \( m \) is \( p_i^{t_i} \) where the \( p_i \) \( (i = 1, \ldots, r) \) are the different primes in the prime factorization and the \( t_i \) \( (i = 1, \ldots, r) \) are the multiplicities or exponents. If \( k = 1 \) the lemma is trivial since we have to eliminate the primes \( p_i \) \( (i = 1, \ldots, r) \) in the sequence of prime numbers. Suppose \( k \geq 2 \), from the inequality \( p_i P_{k-1}^{t_i} \leq x \) \( (p_i, P_{k-1}) = 1 \) we obtain that the number of \( P_k \) not exceeding \( x \) multiple of \( p_i \) does not exceed (see equation (3)) \( \pi_{k-1} \left( \frac{x}{p_i} \right) = o_{p_i} (\pi_k(x)) \), since \( o(1) \) depends of \( p_i \). Therefore the number of \( P_k \) not exceeding \( x \) such that \( (P_k, m) > 1 \) does not exceed \( \sum_{i=1}^{r} o_{p_i}(\pi_k(x)) = o(\pi_k(x)) \) and consequently this number is also \( o(\pi_k(x)) \). Now, equation (6) is an immediate consequence equation (3). The lemma is proved.

Let \( k \) and \( h \) be arbitrary but fixed positive integers. Let us consider the numbers of the form \( P_k^h C_{n,h+1} \), where \( (P_k, C_{n,h+1}) = 1 \). The number of these numbers not exceeding \( x \) will be denoted \( B_{k,h,C_{n,h+1}}(x) \), since the sequence of these numbers depends of \( k \), \( h \) and the sequence of \( (h+1) \)-full numbers \( C_{n,h+1} \).

In [4], [5] and [6], the author proved some theorems on the distribution of composite numbers. All these theorems are particular cases of the following general theorem.

Theorem 1.5 The following asymptotic formula holds (see Lemma 1.2)

\[ B_{k,h,C_{n,h+1}}(x) \sim \frac{A_{C_{n,h+1}}}{(k-1)!} \frac{h x^{\frac{1}{h}} (\log \log x)^{k-1}}{\log x} \]  

Proof. We have

\[ \log \left( \frac{x^{\frac{1}{h}}}{(C_{n,h+1})^{\frac{1}{h}}} \right) = \frac{1}{h} \log x \left( 1 - \frac{\log(C_{n,h+1})}{\log x} \right) \sim \frac{1}{h} \log x \]  

and

\[ \log \log \left( \frac{x^{\frac{1}{h}}}{(C_{n,h+1})^{\frac{1}{h}}} \right) \sim \log \log x \]  

From the inequality

\[ P_k^h C_{n,h+1} \leq x \quad (P_k, C_{n,h+1}) = 1 \]
we obtain \( P_k \leq \frac{x^{\frac{1}{h}}}{(C_{n,h+1})^{\frac{1}{h}}} \). Therefore (see Lemma 1.4) we have

\[
\pi_{k,C_{n,h+1}} \left( \frac{x^{\frac{1}{h}}}{(C_{n,h+1})^{\frac{1}{h}}} \right) = \frac{h x^{\frac{1}{h}} (\log \log x)^{k-1}}{(C_{n,h+1})^{\frac{1}{h}} (k-1)! \log x} + o \left( \frac{x^{\frac{1}{h}} (\log \log x)^{k-1}}{\log x} \right)
\] (11)

Let \( \epsilon > 0 \). There exists \( M \) depending of \( \epsilon \) such that (see Lemma 1.2)

\[
\sum_{C_{n,h+1} \geq M+1} \frac{1}{(C_{n,h+1})^{\frac{1}{h}}} < \epsilon
\] (12)

Consequently we have (see equation (11), (12) and Lemma 1.2)

\[
B_{k,h,C_{n,h+1}}(x) = \sum_{C_{n,h+1} \leq M} \pi_{k,C_{n,h+1}} \left( \frac{x^{\frac{1}{h}}}{(C_{n,h+1})^{\frac{1}{h}}} \right) + E_{k,h,C_{n,h+1}}(x)
\]

\[
= \sum_{C_{n,h+1} \leq M} \frac{h x^{\frac{1}{h}} (\log \log x)^{k-1}}{(C_{n,h+1})^{\frac{1}{h}} (k-1)! \log x} + o \left( \frac{x^{\frac{1}{h}} (\log \log x)^{k-1}}{\log x} \right) + E_{k,h,C_{n,h+1}}(x)
\]

\[
= \frac{A_{C_{n,h+1}} h x^{\frac{1}{h}} (\log \log x)^{k-1}}{(k-1)! \log x} + o \left( \frac{x^{\frac{1}{h}} (\log \log x)^{k-1}}{\log x} \right) + E_{k,h,C_{n,h+1}}(x)
\]

\[
- \sum_{C_{n,h+1} \geq M+1} \frac{h x^{\frac{1}{h}} (\log \log x)^{k-1}}{(C_{n,h+1})^{\frac{1}{h}} (k-1)! \log x}
\] (13)

Note that inequality (10) implies

\[
2^k \leq P_k \leq \left( \frac{x}{C_{n,h+1}} \right)^{\frac{1}{h}}
\] (14)

and

\[
C_{n,h+1} \leq \frac{x}{2^{kh}}
\] (15)

Therefore, we have (see equations (14), (15), (3), (4) and (5))

\[
0 \leq E_{k,h,C_{n,h+1}}(x) \leq \sum_{M+1 \leq C_{n,h+1} \leq \frac{x}{2^{kh}}} \pi_k \left( \frac{x^{\frac{1}{h}}}{(C_{n,h+1})^{\frac{1}{h}}} \right)
\]

\[
= D_k \sum_{M+1 \leq C_{n,h+1} \leq \frac{x}{2^{kh}}} \frac{x^{\frac{1}{h}}}{(C_{n,h+1})^{\frac{1}{h}}} \left( \log \log \left( \frac{x^{\frac{1}{h}}}{(C_{n,h+1})^{\frac{1}{h}}} \right) \right)^{k-1}
\]
where \( \alpha = \frac{2h+1}{2h+2} \).

Note that the inequality \( C_{n,h+1} \leq x^\alpha \) implies the inequality \( \frac{1}{1-\frac{1}{\log C_{n,h+1}}} \leq \frac{1}{1-\alpha} \). Hence, we find that (see equation (16) and (12))

\[
D_k h \sum_{M+1 \leq C_{n,h+1} \leq x^\alpha} \frac{x^{\frac{1}{h} \log \log x}}{(C_{n,h+1})^{\frac{1}{k} \log x}} \leq \frac{D_k h}{1-\alpha} \epsilon \frac{(\log \log x)^{k-1}}{x}
\]

(17)

Note that the inequality \( x^\alpha < C_{n,h+1} \) implies the inequality \( \frac{x}{(C_{n,h+1})^{\frac{1}{h} \log x}} < x^{\frac{1-\alpha}{h}} \) and the inequality (14) implies the inequality \( \frac{1}{1-\frac{1}{\log x}} \leq \frac{1}{k \log 2} \). Consequently we find that (see (16))

\[
D_k \sum_{x^\alpha < C_{n,h+1} \leq \frac{x}{2\pi h}} \frac{x^{\frac{1}{h} \log \log x}}{(C_{n,h+1})^{\frac{1}{k} \log x}} \leq D_k \frac{1}{k \log 2} \frac{1-x}{x^{1-\frac{1}{k} \log 2}} \sum_{x^\alpha < C_{n,h+1} \leq \frac{x}{2\pi h}} 1
\]

(18)

where we have used Lemma 1.1. Substituting equations (17) and (18) into equation (16) and then substituting equation (16) into equation (13) we obtain

\[
\left| \frac{B_{k,h,C_{n,h+1}}(x)}{x^{\frac{1}{h} \log \log x} (k-1)!} - \frac{A_{C_{n,h+1}} h}{(k-1)!} \right| \leq \epsilon + \frac{h}{(k-1)!} \epsilon + \frac{D_k h}{1-\alpha} \epsilon + \epsilon \quad (x \geq x_\epsilon)
\]
That is, equation (7), since $\epsilon$ can be arbitrarily small. The theorem is proved.

Let us consider the prime factorization of a positive integer $n = p_1^{t_1} \cdots p_r^{t_r}$ where the $p_i$ ($i = 1, \ldots, r$) are the different primes in the prime factorization and the $t_i$ ($i = 1, \ldots, r$) are the multiplicities or exponents. As usual, let $\omega(n)$ be the number of distinct prime factors in the prime factorization of $n$ and let $\Omega(n)$ be the total number of prime factors in the prime factorization of $n$. That is, $\omega(n) = r$ and $\Omega(n) = t_1 + \cdots + t_r$. Note that $\omega(n) \leq \Omega(n)$ and from the trivial inequality $2^{\Omega(n)} \leq n$ we obtain $\Omega(n) \leq \frac{\log n}{\log 2}$.

As in Theorem 1.5 in the following two theorems we consider the numbers of the form $P_k^hC_{n,h+1}$, where $(P_k, C_{n,h+1}) = 1$.

**Theorem 1.6** The following asymptotic formulae hold.

$$
\sum_{P_k^hC_{n,h+1} \leq x} \omega \left( P_k^h \right) \sim k A_{C_{n,h+1}} h x^\frac{1}{2} \left( \log \log x \right)^{k-1} \frac{\log x}{(k-1)!} \tag{19}
$$

and

$$
\sum_{P_k^hC_{n,h+1} \leq x} \omega \left( C_{n,h+1} \right) \sim \frac{B_{C_{n,h+1}} h x^\frac{1}{2} \left( \log \log x \right)^{k-1}}{(k-1)!} \frac{\log x}{(k-1)!} \tag{20}
$$

where

$$
B_{C_{n,h+1}} = \sum_{C_{n,h+1}} \frac{\omega \left( C_{n,h+1} \right)}{(C_{n,h+1})^{\frac{1}{2}}}
$$

Proof. Equation (19) is an immediate consequence of equation (7), since $\omega \left( P_k^h \right) = k$. The proof of equation (20) is the same as the proof of equation (7). The theorem is proved.

**Theorem 1.7** The following asymptotic formulae hold.

$$
\sum_{P_k^hC_{n,h+1} \leq x} \Omega \left( P_k^h \right) \sim h k A_{C_{n,h+1}} h x^\frac{1}{2} \left( \log \log x \right)^{k-1} \frac{\log x}{(k-1)!} \tag{21}
$$

and

$$
\sum_{P_k^hC_{n,h+1} \leq x} \Omega \left( C_{n,h+1} \right) \sim \frac{D_{C_{n,h+1}} h x^\frac{1}{2} \left( \log \log x \right)^{k-1}}{(k-1)!} \frac{\log x}{(k-1)!} \tag{22}
$$

where

$$
D_{C_{n,h+1}} = \sum_{C_{n,h+1}} \frac{\Omega \left( C_{n,h+1} \right)}{(C_{n,h+1})^{\frac{1}{2}}}
$$
Equation (21) is an immediate consequence of equation (7), since \( \Omega \left( P_h^k \right) = h k \).

The proof of equation (22) is the same as the proof of equation (7). The theorem is proved.

Let us consider the prime factorization of a positive integer \( n = p_1^{t_1} \cdots p_r^{t_r} \) where the \( p_i (i = 1, \ldots, r) \) are the different primes in the prime factorization and the \( t_i (i = 1, \ldots, r) \) are the multiplicities or exponents. We denote \( u(n) \) the kernel of \( n \), that is, the greatest squarefree that divides \( n \), that is, \( u(n) = p_1 \cdots p_r \).

We need the following lemma.

**Lemma 1.8** Let us consider a strictly increasing sequence of positive integers \( A_n \) such that

\[
\sum_{A_n \leq x} 1 = \frac{x \left( \log \log x \right)^{k-1}}{(k-1)! \log x} + o \left( \frac{x \left( \log \log x \right)^{k-1}}{\log x} \right)
\]

Then

\[
\sum_{A_n \leq x} A_n = \frac{x^2 \left( \log \log x \right)^{k-1}}{2(k-1)! \log x} + o \left( \frac{x^2 \left( \log \log x \right)^{k-1}}{\log x} \right)
\]

Proof. Use Abel’s summation. Besides note that (L’Hospital’s rule)

\[
\lim_{x \to \infty} \int_a^x \frac{t \left( \log \log t \right)^{k-1} \, dt}{t \log t} = 1
\]

The lemma is proved.

As in Theorem 1.5 in the following theorem we consider the numbers of the form \( P_h^k C_{n,h+1} \), where \( \left( P_h^k, C_{n,h+1} \right) = 1 \).

**Theorem 1.9** The following asymptotic formulae hold.

\[
\sum_{P_h^k C_{n,h+1} \leq x} u \left( P_h^k C_{n,h+1} \right) \sim \frac{E_{C_{n,h+1}} h x^{\frac{2}{k}} \left( \log \log x \right)^{k-1}}{2(k-1)! \log x}
\]  \( \quad (23) \)

where

\[
E_{C_{n,h+1}} = \sum_{C_{n,h+1}} \frac{u \left( C_{n,h+1} \right)}{\left( C_{n,h+1} \right)^{\frac{2}{k}}}
\]

and

\[
\sum_{P_h^k C_{n,h+1} \leq x} u \left( P_h^k \right) \sim \frac{F_{C_{n,h+1}} h x^{\frac{2}{k}} \left( \log \log x \right)^{k-1}}{2(k-1)! \log x}
\]  \( \quad (24) \)
where

\[ F_{C_{n, b+1}} = \sum_{C_{n, b+1}} \frac{1}{(C_{n, b+1})^{\frac{2}{2}}} \]

Proof. The proof is the same as the proof of Theorem 1.5 using Lemma 1.8. The theorem is proved.

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References


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