Lattice Points in $n$-Dimensional Solids Like-Ellipsoid
Elementary Methods

Rafael Jakimczuk

División Matemática, Universidad Nacional de Luján
Buenos Aires, Argentina

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Abstract

In a former article the author studied, by use of elementary methods, the number of solutions $(x_1, \ldots, x_n)$, where the $x_i$ are positive integers, to the general inequality

$$r_1 x_1^{k_1} + r_2 x_2^{k_2} + \cdots + r_n x_n^{k_n} \leq x,$$

where the $r_i$ and $k_i$ are positive numbers. The main result is that it number of solutions is asymptotically equal to the number of lattice points in the corresponding $n$-dimensional solid like-ellipsoid, namely

$$r_1 x_1^{k_1} + r_2 x_2^{k_2} + \cdots + r_n x_n^{k_n} \leq x, \quad (x_i \geq 0)$$

In this note we study the case where all $k_i = k$, where $k$ is a positive integer, and $r_i = 1$ ($i = 1, \ldots, n$). In particular, we study the volume of the solid like-ellipsoid

$$x_1^k + x_2^k + \cdots + x_n^k \leq 1, \quad (x_i \geq 0)$$

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1 Introduction and Main Results

Let us consider the inequality, where $s \geq 1$ and $k \geq 2$ are arbitrary but fixed positive integers.

$$x_1^s + x_2^s + \cdots + x_k^s \leq x$$  \hspace{1cm} (1)

The number of solutions $(x_1, \ldots, x_k)$, where the $x_i$ ($i = 1, \ldots, k$) are positive integers will be denoted $S_{k,s}(x)$. The following theorem is well-known (see [1])

**Theorem 1.1** We have

$$S_{k,s}(x) \sim D_{k,s} x^{\frac{k}{s}}$$  \hspace{1cm} (2)

where $D_{k,s}$ is the volume of the domain

$$x_1^s + x_2^s + \cdots + x_k^s \leq 1, \quad (x_i \geq 0)$$  \hspace{1cm} (3)

and the following formula holds

$$D_{k,s} = \prod_{i=2}^{k} I_{i,s}$$  \hspace{1cm} (4)

where

$$I_{i,s} = \int_0^1 (1 - t^s)^{\frac{i-1}{s}} \, dt$$  \hspace{1cm} (5)

Proof. See [1]. The theorem is proved.

In this note we are interested in the constant $D_{k,s}$ and some results are obtained in short and elementary proofs. The case $s = 1$ (volume of the tetrahedron) is trivial, we have

$$D_{k,1} = \prod_{i=2}^{k} I_{i,1} = \frac{1}{k!}$$  \hspace{1cm} (6)

since

$$I_{i,1} = \int_0^1 (1 - t)^{i-1} \, dt = \frac{1}{i}$$  \hspace{1cm} (7)

**Theorem 1.2** Let $s \geq 2$ a fixed positive integer. The following limits hold.

$$\lim_{k \to \infty} I_{k,s} = 0$$  \hspace{1cm} (8)

$$\lim_{k \to \infty} D_{k,s} = 0$$  \hspace{1cm} (9)

Let $k \geq 2$ a fixed positive integer. The following limits hold.

$$\lim_{s \to \infty} I_{k,s} = 1$$  \hspace{1cm} (10)
Proof. Note that the function \( f(t) = (1 - t^s)^{\frac{k-1}{s}} \) is strictly decreasing on the interval \([0, 1]\), \( f(0) = 1 \) and \( f(1) = 0 \). Limit (9) is an immediate consequence of limit (8). Therefore we shall prove limit (8). Note that the sequence \( I_{s,k} \) is decreasing and therefore it has a limit, we shall prove that this limit is zero. If \( 0 < a < 1 \) is a fixed number then

\[
I_{k,s} = \int_0^1 (1 - t^s)^{\frac{k-1}{s}} \, dt = \int_0^a (1 - t^s)^{\frac{k-1}{s}} + \int_a^1 (1 - t^s)^{\frac{k-1}{s}} \leq a + (1 - a^s)^{\frac{k-1}{s}} \leq 2a \quad (k \geq k_a)
\]

Therefore limit (8) holds, since \( a \) can be arbitrarily small. Now, we shall prove (10). We have

\[
I_{k,s} = \int_0^1 (1 - t^s)^{\frac{k-1}{s}} \, dt = \int_0^1 (1 - t)^{\frac{k-1}{s}} \left( 1 + t + \cdots + t^{s-1} \right)^{\frac{k-1}{s}} \, dt
\]

Note that

\[
1 \leq \left( 1 + t + \cdots + t^{s-1} \right)^{\frac{k-1}{s}} \leq s^{\frac{k-1}{s}} \to 1
\]

Therefore

\[
1 - 2\epsilon \leq \frac{1}{s^{\frac{k-1}{s}} + 1} = \int_0^1 (1 - t)^{\frac{k-1}{s}} \, dt \leq I_{k,s} \leq (1 + \epsilon) \int_0^1 (1 - t)^{\frac{k-1}{s}} \, dt = (1 + \epsilon) \frac{1}{s^{\frac{k-1}{s}} + 1} \leq 1 + 2\epsilon
\]

Equation (10) is proved, since \( \epsilon > 0 \) can be arbitrarily small. The theorem is proved.

In the following theorem we prove more precise results. Before, we need a lemma.

**Lemma 1.3** Let \( n \geq 2 \) and \( s \geq 2 \) arbitrary but fixed positive integers. Let us consider the function

\[
f(x) = \left( 1 - \frac{n^s}{x} \right)^{\frac{k-1}{s}} \quad (x > n^s)
\]

Then the following inequality holds

\[
0 < f(x) = \left( 1 - \frac{n^s}{x} \right)^{\frac{k-1}{s}} < e^{-\frac{n^2}{x}} \quad (x > n^s)
\]
Proof. We need the following well-known power series
\[
\log(1 - y) = - \left( \sum_{j=1}^{\infty} \frac{y^j}{j} \right) \quad (|y| < 1)
\]
\[
\frac{1}{1 - y} = \sum_{j=0}^{\infty} y^j \quad (|y| < 1)
\]

We have
\[
f'(x) = \frac{1}{s} \left( 1 - \frac{n^s}{x} \right)^{\frac{x-1}{s}} \left( \log \left( 1 - \frac{n^s}{x} \right) - \frac{n^s}{x^2} \frac{1}{1 - \frac{n^s}{x}} + \frac{n^s}{x^2} \frac{1}{1 - \frac{n^s}{x}} \right)
\]
\[
= \sum_{h=2}^{\infty} (n^s)^{h-1} \left( \frac{h-1}{h} n^s - 1 \right) \frac{1}{x^h} > 0 \quad (x > n^s)
\]
since \(\frac{h-1}{h} n^s - 1 > 0\). Besides, we have
\[
\lim_{x \to n^s} f(x) = 0, \quad \lim_{x \to \infty} f(x) = e^{-\frac{n^s}{s}}
\]
The lemma is proved.

**Theorem 1.4** Let \(s \geq 2\) an arbitrary but fixed positive integer. The magnitude order of \(I_{k,s}\) is \(\frac{1}{k^2}\). That is, there exist two positive constants \(C_{1,s} < 1\) and \(C_{2,s} > 1\) depending of \(s\) such that
\[
C_{1,s} \leq \frac{I_{k,s}}{k^2} \leq C_{2,s} \quad (k \geq 2)
\]
The following inequality holds
\[
\frac{(C_{1,s})^k}{(k!)^{\frac{1}{s}}} \leq D_{k,s} \leq \frac{(C_{2,s})^k}{(k!)^{\frac{1}{s}}} \quad (k \geq 2)
\]
The following inequality holds
\[
\frac{C_{3,s}}{(k!)^{\frac{1}{s} + \epsilon}} \leq D_{k,s} \leq \frac{C_{4,s}}{(k!)^{\frac{1}{s} - \epsilon}} \quad (k \geq 2)
\]
where \(C_{3,s}\) and \(C_{4,s}\) depend of \(\epsilon\) and \(\epsilon > 0\) can be arbitrarily small.
\[
\log D_{k,s} \sim -\frac{1}{s} k \log k
\]
\[
D_{k,s} = e^{(-\frac{1}{s} + o(1))k \log k}
\]
Proof. Inequality (12) is an immediate consequence of inequality (11) and equation (4). Inequality (13) is an immediate consequence of inequality (12). Equation (14) is also an immediate consequence of inequality (12), since

\[ \log(k!) \sim k \log k \]

Equation (15) is an immediate consequence of equation (14). Therefore we shall prove inequality (11).

We recall that the function

\[ f(t) = (1 - t^s)^{\frac{k-1}{s}} \]

is strictly decreasing on the interval \([0, 1] \), \( f(0) = 1 \) and \( f(1) = 0 \). The domain below is curve we denote \( D \).

Since the rectangle of basis \( \frac{1}{k^s} \) and height \( f \left( \frac{1}{k^s} \right) \) is included in the domain \( D \) we have

\[
I_{k,s} = \int_0^1 (1 - t^s)^{\frac{k-1}{s}} \geq \frac{1}{k^s} \left( 1 - \left( \frac{1}{k^s} \right)^s \right)^{\frac{k-1}{s}} = \frac{1}{k^s} \left( 1 - \frac{1}{k} \right)^{\frac{k-1}{s}} \geq \frac{1}{2} e^{-\frac{1}{2}} \frac{1}{k^s}
\]

Note that the domain \( D \) is included in the rectangle of basis \( \frac{1}{k^s} \) and height 1, the rectangles of basis \( \frac{1}{k^s} \) and height \( f \left( j \frac{1}{k^s} \right) \) (\( j = 2, 3, \ldots, \lfloor \frac{1}{k^s} \rfloor \)) and the areas of the little domains like-triangles above of the rectangles do not exceed the area of the rectangle of basis \( \frac{1}{k^s} \) and height 1. Therefore we have (Lemma 1.3)

\[
I_{k,s} = \int_0^1 (1 - t^s)^{\frac{k-1}{s}} \leq \frac{2}{k^s} + \frac{1}{k^s} \sum_{j=2}^{\lfloor k^s \rfloor} \left( 1 - \left( \frac{j}{k^s} \right)^s \right)^{\frac{k-1}{s}}
\]

\[
= \frac{2}{k^s} + \frac{1}{k^s} \sum_{j=2}^{\lfloor k^s \rfloor} \left( 1 - \left( \frac{j}{k} \right)^s \right)^{\frac{k-1}{s}} \leq \frac{2}{k^s} + \frac{1}{k^s} \sum_{j=2}^{\lfloor k^s \rfloor} e^{-j^s}
\]

\[
\leq \frac{2}{k^s} + \frac{1}{k^s} \sum_{j=2}^{\lfloor k^s \rfloor} e^{-\frac{j}{k}} \leq \frac{2}{k^s} + \frac{1}{k^s} \int_1^\infty e^{-\frac{x}{k}} \, dx = \frac{1}{k^s} (2 + se^{-\frac{s}{k}})
\]

The theorem is proved.

**Corollary 1.5** The following formula holds (\( s \geq 1 \)).

\[
\frac{D_{k,s+1}}{D_{k,s}} = e^{(\frac{1}{s(s+1)} + o(1))k \log k}
\]

(16)

\[
\lim_{k \to \infty} \frac{D_{k,s+1}}{D_{k,s}} = \infty
\]

(17)
Proof. Equation (16) is an immediate consequence of equation (15) and limit (17) is an immediate consequence of (16). The corollary is proved.

**Theorem 1.6** Let $s$ be an arbitrary but fixed positive integer. We have (compare with (2))

$$S_{k,s}(x) \leq D_{k,s}x^\frac{k}{s} \quad (x \geq 0)$$

Proof. We use mathematical induction. Let us consider the case $k = 2$. If $x \geq 1$ then we have

$$S_{2,s}(x) = \sum_{i=1}^{\left\lfloor \frac{x}{s} \right\rfloor} (x - i^s) \frac{1}{s} \leq \int_0^{\frac{x}{s}} (x - t^s) \frac{1}{s} \, dt = x^\frac{2}{s} \int_0^1 (1 - t^s) \frac{1}{s} \, dt = D_{2,s}x^\frac{2}{s}$$

If $0 \leq x \leq 1$ then $S_{2,s}(x) = 0$ and consequently the inequality holds. The case $k = 2$ is proved.

Suppose that the theorem is true for $k - 1 \geq 2$ then we shall prove that the theorem is also true for $k$. If $x \geq 1$ then (see (4) and (5))

$$S_{k,s}(x) = \sum_{i=1}^{\left\lfloor \frac{x}{s} \right\rfloor} S_{k-1,s}(x - i^s) \leq \sum_{i=1}^{\left\lfloor \frac{x}{s} \right\rfloor} D_{k-1,s}(x - i^s)^\frac{k-1}{s}$$

$$\leq D_{k-1,s} \int_0^{\frac{x}{s}} (x - t^s) \frac{k-1}{s} \, dt = D_{k-1,s}x^\frac{k}{s} \int_0^1 (1 - t^s) \frac{k-1}{s} \, dt = D_{k,s}x^\frac{k}{s}$$

If $0 \leq x \leq 1$ then $S_{k,s}(x) = 0$ and consequently the inequality holds. The theorem is proved.

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**References**


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