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Extension of the HPM for Quasilinear and Nonlinear Equations

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Abstract

In this paper, we apply He's [2] Homotopy Perturbation Method (HPM) for solving partial differential equations. Previous results deal largely with linear examples and numerical justifications of convergence, whereas we exactly solve both quasilinear and nonlinear equations, including those of intrinsic interest such as the Eikonal Equation. We use an equivalent, simplified version of He's equations for increased clarity.

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1. Introduction

The Homotopy Perturbation Method (HPM), first developed by Ji-Huan He [2], is a combination of perturbation and homotopy techniques. Unlike perturbation methods which depend on a small parameter ϵ, the HPM can be applied to problems without a small parameter. The HPM provides a series solution to a given problem, where the convergence of the series is discussed in [1]. In the event that the magnitudes of the series coefficients are contractive and the iterative scheme we use satisfies the partial differential equations and boundary conditions at each step, we must have a series converging to the correct solution.

Our motivation is to provide increased clarity in the technique with an elementary reformulation of He's technique into one with as few pieces as possible. We then demonstrate the HPM in a range of different scenarios to illustrate exactly how it might be used and allow the reader to obtain an intuitive understanding of its benefits and limitations.

2. The Homotopy Perturbation Method

To illustrate the HPM, we consider

$$L(u) + N(u) = 0 \tag{1}$$

L and N are differential operators. Define a convex homotopy $\mathcal{H}(u, p)$ by

$$\mathcal{H}(u, p) = (1 - p)[L(u) - L(u_0)] + p[L(u) + N(u)],\ p \in [0,1] \tag{2}$$

As embedding parameter p takes values from zero to unity, $\mathcal{H}(u, p) = 0$ changes from $L(u) = 0$ to the original problem $L(u) + N(u) = 0$. Assume that the solution of $\mathcal{H}(u, p) = 0$ can be defined as

$$u = \sum_{n=0}^{\infty} p^n u_n = u_0 + pu_1 + p^2 u_2 + \cdots \tag{3}$$

As $p \to 1$,

$$u = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \cdots$$

would be a solution of $\mathcal{H}(u, 1) = 0$, the approximate solution of (1). The convergence of series (3) is given in [1].

Equation (2) can be rewritten as

$$\mathcal{H}(u, p) = L(u) + pN(u),\ p \in [0,1] \tag{4}$$

if one chooses u_0 such that $L(u_0) = 0$.

3. Quasilinear Equations

3.1. Consider

$$(y + u)u_x + yu_y = x - y,\ \ u(x, 1) = 1 + x \tag{5}$$

Let $L(u) = yu_x + yu_y$, and $N(u) = uu_x + (y - x)$, so $L(u) + N(u) = 0$.

Construct the convex homotopy $\mathcal{H}(u,p) = L(u) + pN(u) = 0$:

$$yu_x + yu_y + p(uu_x + y - x) = 0 \tag{6}$$

Substituting $\sum_{n=0}^{\infty} p^n u_n$ into (6) and equating the like powers of p yields:

$$y(u_{0x} + u_{1x}p + u_{2x}p^2 + \cdots) + y(u_{0y} + u_{1y}p + u_{2y}p^2 + \cdots) +$$
$$p[(u_0 + u_1 p + u_2 p^2 + \cdots)(u_{0x} + u_{1x}p + u_{2x}p^2 + \cdots) + y - x] = 0$$

p^0: $yu_{0x} + yu_{0y} = 0$

p^1: $yu_{1x} + yu_{1y} + u_0 u_{0x} + y - x = 0$

p^2: $yu_{2x} + yu_{2y} + u_0 u_{1x} + u_1 u_{0x} = 0$

p^3: $yu_{3x} + yu_{3y} + u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x} = 0$

p^4: $yu_{4x} + yu_{4y} + u_0 u_{3x} + u_1 u_{2x} + u_2 u_{1x} + u_3 u_{0x} = 0$

$$u(x,1) = u_0(x,1) + pu_1(x,1) + p^2 u_2(x,1) + \cdots = x + 1 \tag{7}$$

Equation (7) yields the side conditions

$$u_0(x,1) = x + 1, \text{ and } u_n(x,1) = 0 \text{ for } n = 1, 2, 3, \ldots \tag{8}$$

Solving the above equations together with (8) gives

$$u_0 = x - y + 2,$$
$$u_1 = -2\ln y,$$
$$u_2 = \frac{2}{2!}(\ln y)^2,$$
$$u_3 = -\frac{2}{3!}(\ln y)^3,$$
$$u_4 = \frac{2}{4!}(\ln y)^4,$$
$$\ldots$$
$$u_n = \frac{2(-1)^n}{n!}(\ln y)^n.$$

Therefore,

$$u = \lim_{p\to 1} \sum_{n=0}^{\infty} p^n u_n =$$
$$x - y + 2\left[1 - \ln y + \frac{1}{2!}(\ln y)^2 - \frac{1}{3!}(\ln y)^3 + \frac{1}{4!}(\ln y)^4 - \cdots\right],$$
$$u(x,y) = x - y + 2\left(e^{-\ln y}\right) = x - y + \frac{2}{y}.$$

One can directly verify this solves (5), or one can consider [1] and note that having solved the iterative scheme and obtained u_n whose magnitudes decrease contractively on any bounded domain – it must have solved (5). This second criterion is useful in the event that a closed-form expression for the series cannot be found.

Using the HPM to solve a differential equation, one may choose different differential operators for L and N. In general, the choice of L and N is important with regards to the calculations of the sequence of approximations $u_0, u_1, u_2, \dots$ and its convergence. Let us use the HPM to generate a sequence of approximations for (5), choosing

$$L(u) = y(1 + u_y) - x, \quad \text{and} \quad N(u) = (y + u)u_x, \text{ so we consider}$$
$$L(u) + pN(u) = y(1 + u_y) - x + p(y + u)u_x = 0, \tag{9}$$

Substituting $\sum_{n=0}^{\infty} p^n u_n$ into (9) and equating the same powers of p on both sides of the equation:

$$y\left(1 + \sum_{n=0}^{\infty} p^n u_{ny}\right) - x + p\left(y + \sum_{n=0}^{\infty} p^n u_n\right)\left(\sum_{n=0}^{\infty} p^n u_{nx}\right) = 0$$

This yields

p^0: $\quad y(1 + u_{0y}) - x = 0, \; u_{0y} = -\dfrac{1}{y}(y - x)$

p^1: $\quad yu_{1y} + (y + u_0)u_{0x} = 0, \; u_{1y} = -\dfrac{1}{y}(y + u_0)u_{0x}$

p^2: $\quad yu_{2y} + u_0u_{1x} + u_1u_{0x} = 0, \; u_{2y} = -\dfrac{1}{y}(u_0u_{1x} + u_1u_{0x})$

$\dots$

p^n: $\quad u_{ny} = -\dfrac{1}{y}\sum_{j=0}^{n-1} u_j u_{(n-j-1)x}, n \geq 2.$

Solving u_{ny} for u_n with $u_0(x, 1) = x + 1$ and $u_n(x, 1) = 0$ yields $u_0 = x\ln y - y + (x + 2)$, $\quad u_1 = -\frac{1}{3}x(\ln y)^3 + (x + 1) + (\ln y)^2 + (x + 2)\ln y$.

It seems u is of the form $u = xf(\ln y) + g(\ln y)$, and making the substitution into the PDE (5) yields $(y + yf + g)f + (xf' + g) = x - y$, or $x(f^2 + f' - 1) + gf + g' + yf + y = 0$. This must be true $\forall x$, so we have $f^2 + f' - 1 = 0$,
$gf + g' + yf + y = 0$, $f(0) = 1$, and $g(0) = 1$, giving $f = 1$ and $g(z) = -e^z + 2e^{-z}$, so $g(\ln y) = -y + \frac{2}{y}$. Hence, $= x - y + \frac{2}{y}$.

3.2. Given the problem

$$(y^2 - u^2)u_x - xyu_y - xu = 0, u(x, x) = x, \tag{10}$$

we apply the HPM to the problem, recalling from previous experience we generally have many options for choosing $L(u)$ and $N(u)$. Typically these problems are easier if $L(u)$ is simple, so we choose $L(u) = y^2u_x$ and $N(u) = -u^2u_x - xyu_y - xu$, and set up the homotopy $L(u) + pN(u) = 0$, assuming the HPM yields a solution for $p \in [0,1]$. Substitute

$$u = \sum_{n=0}^{\infty} p^n u_n$$

into (10), and equate like powers of p.

For $n = 0$ we have $y^2 u_{0x} = 0$, and $u_0(x,x) = x$ gives $u_0(x,y) = y$, and for $n > 0$ we have

$$y^2 \sum_{n=1}^{\infty} u_{nx} p^n - p\left(\left(\sum_{n=0}^{\infty} p^n u_n\right)^2 \left(\sum_{n=0}^{\infty} p^n u_{nx}\right) - \sum_{n=0}^{\infty} \left(xyu_{ny} + xu_n\right) p^n\right) = 0$$

$$\sum_{n=1}^{\infty} \left[y^2 u_{nx} - xyu_{(n-1)y} - xu_{n-1} - \sum_{j=0}^{n-1} \sum_{l=0}^{j} u_l u_{j-l} u_{(n-j-1)x}\right] p^n = 0 \qquad (11)$$

so the expression inside the bracket must be zero. Thus,

$$u_{nx} = \frac{1}{y^2}\left[xyu_{(n-1)y} + xu_{n-1} + \sum_{j=0}^{n-1} \sum_{l=0}^{j} u_l u_{j-l} u_{(n-j-1)x}\right], p > 1,$$

$$\text{and } u_{nx}(x,x) = 0. \qquad (12)$$

Equation (12), together with $u_{0x} = 0$, gives the following PDE's

$u_{0x} = 0,$
$u_{1x} = 2xy^{-1},$
$u_{2x} = 0,$
$u_{3x} = 2x^5 y^{-5} - 4x^3 y^{-3} + 2xy^{-1},$
where $u_0(x,x) = x$ and $u_n(x,x) = 0, n \geq 1$.

Hence,
$u_0 = y,$
$u_1 = x^2 y^{-1} - y,$
$u_2 = 0,$
$u_3 = \frac{1}{3} x^6 y^{-5} - x^4 y^{-3} + x^2 y^{-1} - \frac{1}{3} y.$

It seems that $u = xg(z)$, where $z = \frac{y}{x}$, $u_x = g(z) - zg'(z)$, and $u_y = g'(z)$.

Substituting u_x, u_y into equation (10) yields the ODE

$$zg'(g^2 - z^2 - 1) = g(g^2 - z^2 + 1), g(1) = 1. \qquad (13)$$

Equation (13) gives the implicit solution $4g^2 - 12zg + 4z^2 + 4 = 0$, where $z = \frac{y}{x}$, and $g = \frac{u}{x}$, so $u^2 + y^2 + x^2 = 3yu$. This is easily verified to be the solution to the PDE in (10).

3.3. Consider the equations

$$uu_x + u_y = 1 \text{ and } u(2a^2, 2a) = 0 \ \forall a \in \mathbb{R}. \tag{14}$$

Let $L(u) = u_y - 1$ and $N(u) = uu_x$. Construct the homotopy $L(u) + pN(u) = 0$, and define

$$u = \sum_{n=0}^{\infty} p^n u_n \text{, for } p \in [0,1]. \tag{15}$$

Substituting (15) into (14) and equating the like powers of p, taking into account that $u(2a^2, 2a) = 0$, we obtain

$$u_{0y} = 1, u_{ny} = -\sum_{j=0}^{n-1} u_j u_{(n-j-1)x} \text{, and } u_n(2a^2, 2a) = 0, n = 0, 1, 2 \ldots \tag{16}$$

Solving equation (16) for u_{ny} yields

$$u_0 = y - \sqrt{2}\sqrt{x},$$
$$u_1 = -y + \frac{1}{\sqrt{2}}\sqrt{x} + \frac{1}{2\sqrt{2}}\frac{y^2}{\sqrt{x}},$$
$$u_2 = y - \frac{3}{2\sqrt{2}}\sqrt{x} - \frac{3}{4\sqrt{2}}\frac{y^2}{\sqrt{x}} + \frac{1}{16\sqrt{2}}\frac{y^4}{\sqrt{x^3}},$$
$$u_3 = -\frac{7}{4}y - \frac{26}{15\sqrt{2}}\sqrt{x} + \frac{9}{8\sqrt{2}}\frac{y^2}{\sqrt{x}} + \frac{1}{24}\frac{y^3}{\sqrt{x^2}} - \frac{7}{32\sqrt{2}}\frac{y^4}{\sqrt{x^3}} + \frac{1}{80}\frac{y^5}{\sqrt{x^4}} + \frac{1}{64\sqrt{2}}\frac{y^6}{\sqrt{x^5}}.$$

After observing the emerging pattern, we will prove that

$$u = \sum_{n=0}^{\infty} \alpha_n y^n x^{\frac{1-n}{2}} \text{, for some } \alpha_n \in \mathbb{R}. \tag{17}$$

Substituting (17) into (14), one can obtain $\alpha_0 = -1$, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{2^3}$, $\alpha_3 = 0$, $\alpha_4 = \frac{1}{2^7}$, $\alpha_5 = 0$, $\alpha_6 = \frac{1}{2^{10}}$, $\alpha_7 = 0$, $\alpha_8 = \frac{5}{2^{15}}$. It seems we have a pattern where for odd $n \geq 3, \alpha_n = 0$, and for even $n \geq 4$ we have

$$\alpha_n = \frac{n-3}{2n}\alpha_1\alpha_{n-2} = \frac{n-3}{4n}\alpha_{n-2},$$

and we would have

$$\alpha_{2n} = \frac{(2n-3)!}{(n-2)!\,(n)!\,4^{2n-1}}$$

for $n \geq 2$.

Equation (17) can be written as

$$u = -\sqrt{x} + \frac{1}{2}y + \frac{1}{8}\frac{y^2}{\sqrt{x}} + \sqrt{x}\sum_{n=2}^{\infty} \frac{(2n-3)!}{(n-2)!\,(n)!\,4^{2n-1}}\left(\frac{y^2}{x}\right)^n,$$

thus

$$u = -\sqrt{x} + \frac{1}{2}y + \frac{1}{8}\frac{y^2}{\sqrt{x}} + \sqrt{x}\left[1 - \frac{\sqrt{x - \frac{1}{4}y^2}}{\sqrt{x}} - \frac{1}{8}\frac{y^2}{x}\right],$$

which gives

$$u = \tfrac{1}{2}y - \sqrt{x - \tfrac{1}{4}y^2}. \quad (18)$$

One can easily verify that (18) is the solution to (14).

3.4. Consider the quasilinear system

$$u_t + uu_x = 0, u(x,0) = x \quad (19)$$

with $L(u) = u_t$ and $N(u) = uu_x$. Define

$$\mathcal{H}(u,p) = L(u) + pN(u) = 0. \quad (20)$$

Suppose u is of the form

$$u = \sum_{n=0}^{\infty} p^n u_n. \quad (21)$$

Substituting (21) into (20) and equating like powers of p, we find

$$u_{0t} = 0, u_{nt} + \sum_{j=0}^{n-1} u_{(n-j-1)}u_{jx} = 0,\ u_0(x,0) = x \text{ and } u_n(x,0) = 0, \forall n \geq 1. \quad (22)$$

Using equation (22), we obtain

$$u_0 = x, u_1 = -xt, u_2 = xt^2, \ldots$$

At this point, one has sufficient evidence to conjecture $u_n = x(-t)^n$. .

$u = x + x(-t) + x(-t)^2 + x(-t)^3 + \cdots$

$u = x(1 - t + t^2 - t^3 + \cdots)$

$u = \dfrac{x}{1-(-t)}$

$u = \dfrac{x}{1+t}, |t| < 1.$

4. Nonlinear Equations

4.1. Consider the nonlinear problem

$$u_t + u_x^2 = 0, u(x,0) = x \quad (23)$$

Define

$$L(u) = u_t,\ N(u) = u_x^2, \text{ and homotopy } \mathcal{H}(u,p) = L(u) + pN(u) = 0. \quad (24)$$

Suppose

$$u = \sum_{n=0}^{\infty} p^n u_n. \quad (25)$$

Substituting (25) into (24) and equating the coefficients of p^n yields $u_{0t} = 0$, and

$$u_{nt} + \sum_{j=0}^{n-1} u_{(n-j-1)}u_{jx} = 0,\ u_0(x,0) = x \text{ and } u_n(x,0) = 0, \text{for } n \geq 1. \quad (26)$$

Using equation (26) and induction, one can easily show that $u_0 = x, u_1 = -t$, and $u_n = 0$ for $n \geq 2$. Hence, $u = x - t$.

4.2. Consider

$$u_x^2 + u_y^2 = 1, u(a, 2a) = 0, \quad (27)$$

the 2D Eikonal equation with a Dirichlet boundary condition. We re-write the PDE as $\left(u_x + u_y\right)^2 - 2u_x u_y = 1$, and choose $L(u) = \left(u_x + u_y\right)^2 - 1$, and $N(u) = -2u_x u_y$.

Suppose

$$L(u) + pN(u) = 0, \quad (28)$$

and define

$$u = \sum_{n=0}^{\infty} p^n u_n. \quad (29)$$

Substituting (29) into (28) with $u_n(a, 2a) = 0$ gives $u_0 = 2x - y$, $u_1 = -4(2x - y)$, $u_2 = 24(2x - y), \ldots$

$$u_{nx} + u_{ny} = 2u_{(n-1)y} + \sum_{j=1}^{n-1} \left(u_{jx} u_{(n-j-1)y} - \tfrac{1}{2}\left(u_{jx} + u_{jy}\right)\left(u_{(n-j)x} + u_{(n-j)y}\right)\right) \quad (30)$$

Noting that u_0, u_1, and u_2 are linear in x and y, and that all occurrences of u in equation (30) are in terms of one of its derivatives, it can be easily proven via induction that

$$u_n = a_n x + b_n y \ \ \forall n.$$

Since we now have

$$u = \sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} a_n x + \sum_{n=0}^{\infty} b_n y,$$

we find $u = f(x) + g(y)$ for some f and g. Substituting $u = f(x) + g(y)$ into the original PDE, equation (27) gives $f_x^2 = 1 - g_y^2 = c^2$ for some $c \in \mathbb{R}$. Then $f = \pm cx + d_1, g = \pm y\sqrt{1 - c^2} + d_2$, and putting it all together we have

$$u = \pm\left(cx + y\sqrt{1 - c^2}\right) + d.$$

Imposing $u(a, 2a) = 0$ gives $d = 0$ and $= -\frac{2}{\sqrt{5}}$. Hence, $u = \pm\frac{1}{\sqrt{5}}(y - 2x)$.

5. Choosing an Initial Guess Outside the Kernel

In all previous PDE's, the function u_0 was chosen in such a way that $L(u_0) = 0$. There are PDE's in which one could pick a different convenient choice for the function u_0 where $L(u_0) \neq 0$; in this case, the convex homotopy $\mathcal{H}(u, p)$ is defined as

$$\mathcal{H}(u, p) = (1 - p)[L(u) - L(u_0)] + p[L(u) + N(u)]. \quad (31)$$

5.1. Consider the quasilinear equation

$$u^2 + u_x + u_t = 0, u\left(-1 + \sqrt{2}, 0\right) = \sqrt{2}, \quad (32)$$

and let

$$L(u) = u^2, \tag{33}$$

$$N(u) = u_x + u_t, \tag{34}$$

and let $u_0 = x + t + 1$. Constructing homotopy (31) using (33) and (34) gives

$$\mathcal{H}(u,p) = u^2 - (x+t+1)^2 + (x+t+1)^2 p + (u_x + u_t)p = 0. \tag{35}$$

Substituting $u = \sum_{n=0}^{\infty} p^n u_n$ into (35) and equating the powers of p yields

p^0: $u_0^2 - (x+t+1)^2 = 0$

p^1: $2u_0u_1 + (x+t+1)^2 + u_{0x} + u_{0t} = 0$

p^n: $\left(u_{(n-1)x} + u_{(n-1)t} + \sum_{j=0}^{n} u_j u_{n-j}\right) p^n = 0, n = 2, 3, \ldots$

Using $u_0 = x + t + 1$, we can find $u_1 = -(x+t+1)^{-1} - \frac{1}{2}(x+t+1)$. Note that u_1 is a function of $(x+t+1)$, and N is symmetric in x and t, so we will be able to show that u_2 is a function of $(x+t+1)$. Continuing this process inductively, all the u_n are functions of $(x+t+1)$, and the series $u_0 + \cdots + u_n + \cdots$ will retain this property, so we can consider u as simply some arbitrary function of $(x+t+1)$. Making the substitutions $z = x+t+1$ and $u = g(z)$ for some g, we obtain

$$g^2 + 2g_z = 0, \tag{36}$$

whose solution is

$$u = g = \frac{2}{x+t+c} \tag{37}$$

for some $c \in \mathbb{R}$. Applying our boundary condition, we conclude

$$u = \frac{2}{x+t+1}. \tag{38}$$

5.2. Consider the nonlinear PDE

$$6u_t + u_x - 3u_t^2 = t + 3, u(x,0) = 0. \tag{39}$$

Let $L(u) = 6u_t + u_x$, $N(u) = -3u_t^2 - (t+3)$, and $u_0 = t$. Then

$$\mathcal{H}(u,p) = 6u_t + u_x - 6 + 6p - 3pu_t^2 - pt - 3p = 0. \tag{40}$$

Substituting $u = \sum_{n=0}^{\infty} p^n u_n$ into (40) and equating the same powers of p on both sides of the equation, we find

p^0: $6u_{0t} + u_{0x} - 6 = 0$

p^1: $6u_{1t} + u_{1x} + 6 - 3u_{0t}^2 - t - 3 = 0$

p^n: $\left(6u_{nt} + u_{nx} - 3\sum_{j=0}^{n-1} u_{jt} u_{(n-j-1)t}\right) p^n = 0, n = 2, 3, \ldots$

Given $u_0 = t$, we can easily find $u_1 = \frac{1}{12}t^2$, $u_2 = \frac{1}{12}t^2$, $u_3 = t + \frac{1}{12}t^2$, etc.

Induction shows that if the resulting series $\sum_{n=0}^{\infty} u_n$ is convergent, there exists a solution to the PDE which is simply a function of t. Let $g(t)$ be such a function, and substitute this into equation (39) to find

$$6g' - 3g'^2 = t + 3. \tag{41}$$

Equation (41) gives

$$u = g = t\left(1 \pm \frac{2}{3}\sqrt{-\frac{t}{3}}\right) + C.$$

6. Concluding Remarks

Working with the Homotopy Perturbation Method found an equivalent characterization of the problem as it pertains to quasilinear partial differential equations. In particular, in the many cases where an assumption of $L(u_0) = 0$ suffices, a homotopy of the form

$$\mathcal{H}(u,p) = L(u) + pN(u)$$

represents a compact, easily manipulated alternative to the presentation by He [2]:

$$\mathcal{H}(u,p) = (1-p)[L(u) - L(u_0)] + p[L(u) + N(u)]$$

Additionally, we show in detail how this presentation lends itself to the solution of many quasilinear and nonlinear PDE's. We extend the notion of the HPM and consider its use not just toward finding a solution, but also toward finding a necessary functional form for a solution to a PDE. This functional form can then be used to finish a problem.

Lastly, we showed that our reformulations and extensions are applicable not just in special instances of these classes of problems, but also in important areas of application and theoretical interest like the Eikonal Equation.

References

[1] Z. Ayati and J. Biazar, On the convergence of homotopy perturbation method, *Journal of the Egyptian Mathematical Society*, **23**(2) (2015), 424-428. https://doi.org/10.1016/j.joems.2014.06.015

[2] J.-H. He, Homotopy perturbation method: a new nonlinear analytical technique, *Applied Mathematics and Computation*, **135**(1) (2003), 73-79. https://doi.org/10.1016/s0096-3003(01)00312-5

[3] W. Williams, *Partial Differential Equations*, Clarendon Press, Oxford University Press, Oxford, England, New York, 1980.

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