Mathematical Approach to Thermoelastic Characteristics for Functionally Graded Rotating Circular Disks

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Abstract
Rotating circular disks were taken into account in order to evaluate their thermoelastic behaviors such as temperature distribution, strain, and thermal stress. A second-order ordinary differential equation was derived based on the two-dimensional thermal elastic theory for the thermoelastic characteristics of circular disks, and a finite element technique applied to obtain numerical solutions. In comparison with the composite material circular disk, a circular disk with functionally graded bond layer exhibit improved durability to failure mechanisms by showing the followings; (i) exposing the lower temperature distribution profiles at the functionally graded layer, (ii) under the loading of smaller radial pressure over the entire domain. The thermoelastic characteristics of a functionally graded circular disk are sensitive to the variations of angular speed and the ratio between the top coat and the substrate, of the disk.

Keywords: Circular disk, Composite material, Functionally graded layer, Finite element method, Thermoelastic characteristics

1. Introduction

The increasing use of composite materials (CMs) in many engineering applications is due to their high specific strength and stiffness, as well as high temperature resistance. Especially fiber-rein forced and resin-matrix composite materials possessing higher strength and stiffness are widely applied in applications sensitive to weight such as aircraft, aerospace, automotive, and
The reinforcement is required to be stronger and stiffer to form a sort of backbone, whereas the matrix protects the brittle or breakable nature of reinforcement in a set place. CMs are made of two or more distinct constituent materials, including different physical or chemical properties such that their constituents remain still distinguishable at the macroscopic or microscopic scale within the structure. The thermomechanical properties: elastic modulus, thermal conductivity, and coefficient of thermal expansion are dependent deeply on the deposition process for manufacturing a CM. Failure phenomena thus usually are appeared at the interface between the top and bond coats according to the mismatch of mechanical and thermal properties as the CM system cools from a high operating temperature to a room temperature. Functionally graded (FG) layer conception is employed to improve the risk of delamination appearing at the interface of the CM system for protecting hot components like combustors and turbine parts [2-4].

Functionally graded materials (FGMs) are made of two or more distinct material phases whose volume fractions continuously vary with space variables, presenting continuously varying mechanical and thermal properties in the FGM system. Owing to the continuously varying material properties FGMs display improved bonding strength, wear and corrosion resistance, and reduced residual and thermal stresses [5-7]. For the exploration of the outstanding advantages of FGMs over conventional composites and monolithic materials many potential applications as structural elements, such as FGM beams, plates, shells, and cylinders have been investigated extensively, and researchers move their interests toward the development of potential new structural and functional applications. Yongdong et. al.[8], and Zhong and Yu [9] developed traditional theories to analyzed the behaviors of functionally gradient material (FGM) beams, while the buckling behavior of functionally graded material (FGM) rectangular plates were studied by Chen et. al.[10], and Feldman and Aboudi [11]. Kim et. al. [12] presented an analytical formulation for vibration and buckling analysis of clamped conical shell made of FGMs under various uniform pressures. In the FGM application of cylinder, Obata and Noda [13] derived the exact solution for the problem of uniformly heating a cylinder whose elastic moduli and thermal expansion coefficient vary linearly with radius, and Liew et. al [14] analyzed the thermomechanical behavior of hollow circular cylinders based on a novel limiting process that employs the solutions of homogeneous hollow circular cylinders, with no recourse to the basic theory or the equations of non-homogeneous thermoelasticity.

Recently, the study for CMs and FGMs is extended to various engineering application problems. Wave propagation in inhomogeneous materials undergoing a loading was presented by Idesman [15]. Idesman used linear elements with limped mass matrix and explicit central difference method to obtain numerical solutions. Lee et al. [16] proposed a topology optimization method for the sequential design of material layout and fiber orientation in functionally graded fiber-reinforced composite structures. Li et al. [17] presented thermoelastic damping in free vibrating FGM micro beams with rectangular cross sections. Heat
conduction equation is solved using a layer-wise homogenization. Rayleigh wave behavior in functionally graded magneto-electro-elastic material was analyzed by Ezzin et al. [18]. They presented a dynamic solution for a wave propagation using stiffness matrix methods. Applying the context of Lagrangian and isotropic visco-hyperelasticity Pascon [19] announced a numerical for the analysis of viscoelastic FGMs under finite strains. However, even though many authors have been tried to analyze various behavior of FGM structures, failure mechanisms, and fatigue behavior, as referenced above, the generalization is still remained as a one of the biggest challenge in FGM structural elements.

In the present paper, two types of circular disk were taken into account in order to evaluate their thermoelastic behaviors such as temperature distribution, displacement, and thermal stresses. The first CM circular disk is composed of primary layers: an Al2O3 top coat, an Al substrate, a mixture bond coat of Al2O3 (weight 50 %) and Al(weight 50 %), and the second type is replaced the bond coat with a FG layer in the first CM circular disk (see Fig. 1). A second-order ordinary differential equation was derived based on the two-dimensional thermal elastic theory and applied the governing equation for the thermoelastic characteristics of circular disks. Due to the complexity of the governing equation to attain exact solutions, a finite element method is developed to search numerical approximations.

2. Mathematical modeling

A rotating FGM circular disk with a concentric circular hole is considered (see Fig. 1). The origin of the polar coordinate system $r - \theta$ is assumed to be located at the center of the disk and hole. The CM circular disk is composed of three layers: the substrate Al, bond coat Al2O3 (weight 50 %) / Al (weight 50%), and top coat Al2O3. For the CM circular disk with FG bond coat, the distribution of each material are assumed to vary continuously along the radial direction only. The radii of the hole and outer surface of the disk are designated by $a$ and $b$, and the angular velocity $\omega$ can be determined from the relation $\omega = 2\pi N / 60$, $N$ is the revolutions per minute (rpm).

![Figure 1](image.png)

**Figure 1.** Schematic diagram of CM circular disk models for thermoelastic characteristics: (a) typical CM circular disk and (b) FG layer circular.
2.1. Temperature distribution profiles

Due to the assumption that the disk is under the loading of symmetric temperature to the radial direction, the differential equation expressing the temperature distribution profile in the polar coordinate is

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) = 0. \quad (1)$$

The general solution of Eq (1) is of the form

$$T(r) = c_{i1} r + c_{i2}, \quad (2)$$

where $c_{i1}$ and $c_{i2}$ are integral constants for $i^{th}$ layer and will be determined on account of the boundary conditions. However, only two boundary conditions, inner and outer surface temperature, are known. To determine a unique temperature distribution profile for each layer heat flux at each layer point and the continuity requirements at each layer interfaces are taken into consideration, the equations for $i^{th}$ layer are

$$q_i = \frac{k_i}{L_i} (T_{i-1} - T_i), \quad q_i = q_{i+1},$$

where $q_i$ is the heat flux into $i^{th}$ layer, $k_i$ is the thermal conductivity, and $L_i$ is the length of $i^{th}$ layer. Then, the integral constants $c_j$ (j=1, 2) for the temperature distribution profiles at each $i^{th}$ layer can be obtained uniquely by solving the following linear system;

$$c_{i1} r + c_{i2} = T_{i-1}$$

$$c_{i1} r + c_{i2} - T_i = 0$$

$$q_i + \frac{k_i}{L_i} T_i = \frac{k_i}{L_i} T_{i-1}, \quad q_i - q_{i+1} = 0$$

$$\vdots$$

$$q_{i-2} + \frac{k_{i-1}}{L_{i-1}} T_{i-1} - \frac{k_{i-1}}{L_{i-1}} T_{i-2} = 0, \quad q_{n-2} - q_n = 0$$

$$q_n - \frac{k_n}{L_n} T_{n-1} = -\frac{k_n}{L_n} T_{out}, \quad i = 1, 2, \ldots, n-1$$

The $(i-1)^{th}$ layer linear system determines $T_{i-1}$ value and the known $T_{i-1}$ is used to for the $i^{th}$ layer linear system. Since $T_{i-1}$ is known and $T_{out}$ is given initial value, the number of $2(n-i) + 3$ equations will uniquely decide $2(n-i) + 3$’s unknown coefficients. The $n$ represents the number of layer.
2.2. Mathematical formulation

The elastic modulus $E$, thermal expansion coefficient $\alpha$, and density $\rho$ are assumed to vary exponentially with the variable $r$ as

$$E = E_0 e^{\beta r}$$  \hspace{1cm} (3a)
$$\alpha = \alpha_0 e^{\gamma r}$$  \hspace{1cm} (3b)
$$\rho = \rho_0 e^{\mu r}.$$  \hspace{1cm} (3c)

In the CM circular disk with functionally graded bond layer, the inner area of the circular disk ($r = r_2$) consists of 100% of mixture bond coat of $Al_2O_3$ and $Al$, whereas the outer area of the disk ($r = r_3$) has 100% of $Al2O3$ top coat material. Thus, the constants in Eq. (3) can be determined as

$$E_0 = E_{mix} e^{-\beta_0}$$  \hspace{1cm} (4a)
$$\alpha_0 = \alpha_{mix} e^{-\gamma_0}$$  \hspace{1cm} (4b)
$$\rho_0 = \rho_{mix} e^{-\mu_0}$$  \hspace{1cm} (4c)

$$\beta = \frac{1}{r_2 - r_3} \ln \left( \frac{E_{mix}}{E_T} \right)$$  \hspace{1cm} (5a)
$$\gamma = \frac{1}{r_2 - r_3} \ln \left( \frac{\alpha_{mix}}{\alpha_T} \right)$$  \hspace{1cm} (5b)
$$\mu = \frac{1}{r_2 - r_3} \ln \left( \frac{\rho_{mix}}{\rho_T} \right)$$  \hspace{1cm} (5c)

The subscripts $T$ and $mix$ on a variable represent the properties of the constituent materials in the top coat and the bond coat, respectively.

Since the FGM disk experiences a change in temperature, the eigenstrain is related with the thermal expansion. Under the assumption that the FGM disk is isotropic, the total strain can be expressed as

$$\varepsilon_r = e_r + \alpha(r)T(r)$$
$$\varepsilon_\theta = e_\theta + \alpha(r)T(r),$$

where $T(r)$ is the change in temperature at any point $r$, and $\varepsilon_r$ and $\varepsilon_\theta$ are the total radial and circumferential strain components, and $e_r$ and $e_\theta$ are the elastic radial and circumferential strain components, respectively. Shear strain component is ignored due to the symmetric deformation of the disk. The relations between strains and stresses are, according to Hooke’s law,

$$\varepsilon_r = \frac{1}{E} (\sigma_r - \nu \sigma_\theta) + \alpha(r)T(r)$$
$$\varepsilon_\theta = \frac{1}{E} (\sigma_\theta - \nu \sigma_r) + \alpha(r)T(r),$$

where $\sigma_r$ and $\sigma_\theta$ are the radial and circumferential stress components, $E$ and $\nu$ are Young’s modulus and Poisson’s ratio, respectively. By considering the
inertia force due to the rotation of the disk as the only body force, the two-
dimensional equilibrium in polar coordinates can be written as:
\[
\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta r}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + \rho \omega^2 r = 0
\]
\[
(7)
\]
\[
\frac{\partial \sigma_\theta}{\partial \theta} + \frac{1}{r} \frac{\partial \tau_{r \theta}}{\partial r} + 2 \tau_{\theta r} = 0.
\]
\[
(8)
\]
But, the symmetry yields that \(\frac{\partial \tau_{\theta r}}{\partial r} = 0\) and \(\sigma_r\) and \(\sigma_\theta\) are independent of \(\theta\). Thus, the equation (8) is identically satisfied and the equation (7) is reduced to
\[
\frac{d}{dr} (r \sigma_r) - \sigma_\theta + \rho \omega^2 r^2 = 0.
\]
\[
(9)
\]
For the axisymmetric problem the strain-displacement relations are given by
\[
\varepsilon_r = \frac{du_r}{dr}
\]
\[
\varepsilon_\theta = \frac{u_r}{r},
\]
which provides the following two strain component relation
\[
\varepsilon_r = \frac{d}{dr} (r \varepsilon_\theta).
\]
\[
(10)
\]
By the substitution of \(F = r \sigma_r\), equations (6), and (9) can be rewritten by
\[
\sigma_\theta = \frac{dF}{dr} + \rho \omega^2 r^2
\]
\[
\varepsilon_r = \frac{1}{E} \left( \frac{F}{r} - \nu \frac{dF}{dr} \right) - \frac{1}{E} \nu \rho \omega^2 r^2 + \alpha(r) T(r)
\]
\[
\varepsilon_\theta = \frac{1}{E} \left( \frac{dF}{dr} - \frac{F}{r} - \frac{1}{E} \rho \omega^2 r^2 + \alpha(r) T(r). \right.
\]
\[
(11)
\]
The combination of equations (10) and (11) gives the following second order
governing equation
\[
\frac{d^2 F}{dr^2} + \left( \frac{1}{r} - \beta \right) \frac{dF}{dr} + \frac{1}{E} \left( \beta \nu - \frac{1}{r} \right) F = \rho \omega^2 r (\beta \nu - \mu \nu - 3) - E \alpha \left( \gamma T + \frac{dT}{dr} \right).
\]
\[
(12)
\]
For the present rotating disk problem, the following boundary conditions are applied
(i) \(\sigma_r(a) = 0 \Rightarrow F(a) = 0,\)
(ii) \(\sigma_r(b) = 0 \Rightarrow F(b) = 0.\)
\[
(13)
\]
2.3. Finite Element Formulation
A finite element method is adopted to obtain numerical approaches since the
governing equation is too involved to solve analytically. According to the
variational approach, equation (12) is multiplied by a trial function \(w\) and
integrated over the domain. Then, the governing equation is expressed with
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\[ \int_a^b w \frac{d^2 F}{dr^2} \, dr + \int_a^b \frac{1}{r} \beta \frac{dF}{dr} \, dr + \int_a^b \frac{1}{r} (\beta \nu - 1) F \, dr = \int_a^b w f(r) \, dr, \]

where

\[ f(r) = \rho \omega^2 r (\beta r - \mu r - \nu - 3) - E \alpha (\gamma T + \frac{dT}{dr}). \]

Integration by part yields

\[ \int_a^b \frac{dF}{dr} \frac{dF}{dr} \, dr - \int_a^b \frac{1}{r} \beta \frac{dF}{dr} \, dr + \int_a^b \frac{1}{r} (\beta \nu - 1) F \, dr \]

\[ = - \int_a^b w f(r) \, dr + w(b) \frac{dF}{dr} + (b) - w(a) \frac{dF}{dr} (a) \quad (14) \]

The boundary conditions (14) are used for the present rotating disk problem.

Now, the radial domain of the disk \( \Omega = (a, b) \) is divided into \( N \) number of subdomains \( \Omega^e = (r_e, r_{e+1}) \), where \( e = 1, 2, \ldots N \). The validation of the equation (14) throughout the domain \( \Omega \) approves the validation over the subdomains \( \Omega^e \). The variational form, thus, is

\[ B(w, F) = l(w), \quad (15) \]

where

\[ B(w, F) = \int_{r_e}^{r_{e+1}} \frac{dF}{dr} \, dr - \int_{r_e}^{r_{e+1}} \frac{1}{r} \beta \frac{dF}{dr} \, dr + \int_{r_e}^{r_{e+1}} \frac{1}{r} (\beta \nu - 1) F \, dr \]

\[ l(w) = - \int_{r_e}^{r_{e+1}} w f(r) \, dr + w(r_{e+1}) \frac{dF}{dr} (r_{e+1}) - w(r_e) \frac{dF}{dr} (r_e). \]

To apply the Ritz method for the present problem, the solution is assumed in the form of

\[ F = \sum_{j=1}^{2} \phi_j^e \]

where

\[ \phi_1^e = \frac{r_{e+1} - r}{r_{e+1} - r_e}, \quad \phi_2^e = \frac{r - r_e}{r_{e+1} - r_e}, \]

Then equation (15) can be written as

\[ \sum_{j=1}^{2} K_{ij}^e F_j^e = l_i^e, \quad i = 1, 2; e = 1, 2, \ldots, N, \]

where

\[ K_{ij}^e = B(\phi_j^e, \phi_i^e) \]

\[ l_i^e = l(\phi_i^e). \]

Equation (16) is a system of algebraic equations and the continuity condition \( F_2^e = F_{1+1}^e \) will be applied to form a global algebraic system.
3. Numerical results and discussion

The mechanical and thermal properties shown in Table 1 are applied for evaluating the temperature distribution profile and thermoelastic characteristics. The temperature distributions are displayed in Fig. 2 according to the process in the section 2. The FG circular disk displays lower temperature loading at the top and bond layers in comparison with the CM circular disk. Generally, failure mechanisms are occurred at the interfaces due to the mismatch of mechanical and thermal properties. The FG circular disk weakens the sharp temperature change at the interfaces, which reduce the risk of delamination. The temperature distributions demonstrate that the rate of temperature decrease is getting slower as metal concentration.

Table 1. Mechanical and thermal properties used for analyzing thermoelastic characteristics of rotating CM circular disks.

<table>
<thead>
<tr>
<th>Material/Property</th>
<th>Elastic module (MPa)</th>
<th>Thermal expansion coefficient (10^{-6}/\text{oC})</th>
<th>Thermal conductivity (W/m(^{-}\text{oC}))</th>
<th>Density (g/cm(^{3}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Substrate (Al)</td>
<td>71</td>
<td>23.1</td>
<td>237</td>
<td>2.70</td>
</tr>
<tr>
<td>Bond coating (Al/(\text{Al}_2\text{O}_3))</td>
<td>164.3</td>
<td>13.6</td>
<td>84.3</td>
<td>1.61</td>
</tr>
<tr>
<td>Top ((\text{Al}_2\text{O}_3))</td>
<td>380</td>
<td>8.0</td>
<td>30</td>
<td>0.96</td>
</tr>
</tbody>
</table>

![Figure 2](image-url)  
Figure 2. Temperature distribution profiles under the symmetric loading of 150–20 °C.

Fig. 3 exhibits the stress distribution profiles. As shown in Fig. 3(a), only tensile radial stress is developed over the entire domain for the both types of circular disk. In the radial stress distribution the CM circular disk is under the loading of larger pressure, implying the improved durability of the FG circular disk. The top coat of FG circular disk experiences of compressive circumferential stress, while the substrate was of the tensile circumferential stress (see Fig. 3(b)).
The magnitude of circumferential stress in CM circular disk is smaller at the substrate and the top coat except near the edge. The strain distribution profiles are displayed in Fig. 4. The largest radial strain happens at the interface between the FG layer and the substrate of the FG circular disk and the interface between the bond coat and the top coat of the CM circular disk suffers from radial strain with the lowest radial strain ignoring near the center area (see Fig 4(a)). Fig. 4(b) presents the circumferential strain. The FG circular disk is afflicted with larger circumferential strain at the substrate, whereas lower circumferential strain is appeared at the FG layer and the top coat except near the edge.

**Figure 3.** Stress distribution profiles of circular disks: (a) radial, (b) circumferential.

The effects of rotating speed for the FG circular disk are investigated through Figs. 5-6. The representative rotating speeds are 200, 600, and 1000 rpm. As shown in Fig. 5(a), the magnitude of the radial stress is getting large with the increase of rotating speed, implying the increase of centrifugal force as larger rotating speed applies on the circular disk. Similar thermoelastic behavior are appeared in the circumferential stress distribution profiles to the rotating speed (see Fig. 5(b)). The magnitude of the circumferential stress is getting larger in accordance to the increase of rotating speed. The characteristics of FG circular disk to the rotating speed signifies that the rotating speed is needed to be controlled to reduce the risk of failure mechanisms. Fig. 6 represents the strain distributions to the rotating speed. At the top and bond coats the magnitude of the radial strain increases with the increase of rotating speed and larger strain is developed at the substrate as rotating speed increases except near the center (see Fig. 6(a)). But, as
shown in Fig. 6(b), different phases are appeared in the circumferential strain at the top coat and the center of the circular disk. With the increase of the rotating speed the magnitude of the circumferential strain decreases at the top coat, whereas the center undergoes larger circumferential strain. Moreover, the largest influence to rotating speed occurs at the center, which signifies that the inner area of the circular disk is more sensitive to rotating speed.

![Graph showing stress distribution](image1)

**Figure 5.** Effects of rotating speed on the components of stress for FG layer circular disk: (a) radial, (b) circumferential.

![Graph showing strain distribution](image2)

**Figure 6.** Effects of rotating speed on the components of strain for FG layer circular disk: (a) radial, (b) circumferential.

The influences of the radial thickness are investigated using values of the ratio $T/S$, representing the ratio between the top coat and the substrate. The $T$ and $S$ imply the top coat and substrate, respectively. The representative values of the ratio $T/S$ are 33/87, 39/81, and 45/75. Fig. 7 presents the stress distributions. As the value of $T/S$ decreases the circular disk is under the larger pressure loading over the entire domain (see Fig. 7(a)). In the circumferential stress, as shown in Fig. 7(b), similar phenomena are appeared with the variation of the values of the ratio $T/S$. The magnitude of the circumferential stress increases with the decrease of the value of $T/S$. However, different phases are developed in the strain distribution profiles. With the increase of the value of $T/S$, the FG layer suffers larger radial strain distribution (see Fig. 8(a)), and FG layer and the top coat larger circumferential strain distribution (see Fig. 8(b)).
4. Conclusions

Circular disks have been taken into account to analyze the thermoelastic characteristics, and the effects to rotating speed and the radial thickness. Circular disks with FG layer reduce the failure mechanisms by decreasing the pressure loading due to the high temperature. By the radial stress distribution profiles intensity of critical shocks was declined improving the durability and stability of FG layer circular disks. Based on the results of the thermoelastic characteristics to the rotating speed the substrate of FG layer circular disk is more sensitive to rotating speed. The magnitude of the stress increases with the variation of the value of $T/S$, which implies that the structure formation of CM circular disk played a vital role to prong life time. Consequently, the results obtained demonstrate that the thermal durability and stability of the CM circular disk can be improved by introducing an FG layer considering the rotating speed and the structure formation of CM circular disk.
Acknowledgments. This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2018-0182).

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Received: April 29, 2019; Published: May 30, 2019