Explicit Solution of a Nonlinear Black-Scholes Partial Differential Equation: Tanh Method

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Abstract

We study a nonlinear Black-Scholes partial differential equation developed by Cetin et al (2004) whose nonlinearity arise in option price theory as a result of transaction costs. This is a transaction cost model for modeling illiquid markets. The hyperbolic tangent (Tanh) method is a technique used to solve nonlinear evolution and wave equations. We transform the transformed nonlinear partial differential equation to an ordinary differential equation by assuming a traveling wave solution. We then solve the Ordinary differential equation by assuming a power series in tanh form to obtain an analytic solution of the nonlinear equation.

Keywords: Nonlinear Black-Scholes equation, analytic solution, tanh method, transaction costs, illiquid markets

1 Introduction

The current financial markets began with Black-Scholes partial differential equation obtained under several model assumptions. When some of the assumptions are relaxed, the linear Black-Scholes equation is replaced by a nonlinear one. Nonlinear Black-Scholes equations arise from considering some parameters such as putting nontrivial transaction costs into option pricing models.
to have a more accurate option price. For equity derivatives on emerging mar-
kets transaction costs are large and important therefore pricing and hedging
must take into account incurred costs. When transaction costs are incurred
pricing and hedging is nonlinear hence nonlinear transaction cost models. The
value of a portfolio of options is generally not the same as the sum of values
of individual components.

Some numerical methods have been used to solve such nonlinear Black-Scholes
equations such as the finite difference schemes, binomial trees and Monte Carlo
Simulation. Analytical techniques such as direct integration method has also
been used to obtain exact solutions. In this paper, we solve a nonlinear Black-
Scholes partial differential equation whose nonlinearity arise in option price
theory as a result of transaction costs. The study of partial differential equa-
tions has shown that solitons are essential physical phenomena connected with
nonlinear equations. Solitary wave solution represent a localized wave traveling
with unchanged shape. The hyperbolic tangent method (Tanh) is a technique
used to compute traveling wave solutions. An analytic solution to the nonlinear
Black-Scholes partial differential equation via the hyperbolic tangent method
is currently unknown. This will be done by converting the transformed partial
differential equation to ordinary differential equation by assuming a traveling
wave solution. We then find the solution of the ODE in tanh form and with
the substitution using original variables we obtain the explicit solution of the
equation.

This paper is outlined as follows: Section 2 describes the linear Black-Scholes
equation and Section 3 gives the modified form of the nonlinear equation.
Section 4 presents the solution of the equation via tanh method and Section 5
concludes the paper.

2 Linear Black-Scholes Equation

Option pricing theory has made a great step since the inception of the Black-
Scholes pricing model by Fischer Black and Myron Scholes in 1973. The solu-
tion of the linear Black-Scholes equation

\[ V_t + \frac{1}{2} \sigma^2 S^2 V_{SS} + rS V_S - rV = 0 \]  

(2.1)

where \( S = S(t) > 0 \) and \( t \in (0, T) \) provides an option pricing formula assuming
that:

i. Price of the underlying asset follows a Geometric Brownian motion; \( W_t \)
ii. The drift $\mu$, the volatility $\sigma$ and riskless interest rate $r$ are constant for $0 \leq t \leq T$ and no dividends are paid in that period of time.

iii. The market is frictionless, thus no transaction costs, the interest rates for borrowing and lending money are equal. All parties have immediate access to any information and all securities and credits are available at any time and any size.

iv. There are no arbitrage opportunities.

Under the above assumptions the market is complete i.e. any derivative and any asset can be replicated or hedged with a portfolio of other assets in the market.

The Black Scholes formula for the prices at time zero of a European call option $C(S, t)$ and a European put option $P(S, t)$ on a non dividend paying stock are derived by solving the PDE in (2.1). A European call option is a contract where at a prescribed time in future, known as expiry date $T$, the holder of the option may purchase the underlying asset $S(t)$ for a prescribed amount known as the strike price $K$. On the other hand, a European Put option is the right to sell the underlying asset $S(t)$ at the expiry date $T$ for the strike price $K$. We use boundary conditions to specify the values of the derivative at the boundaries where $S$ and $t$ lie. For a European call option, the boundary conditions are:

i) $C(0, t) = 0$ for $0 \leq t \leq T$,

ii) $C(S, t) \sim S - Ke^{-r(T-t)}$ as $S \to \infty$.

where $T - t$ is time to maturity.

The pay-off function is given by

$$C(S, T) = (S_T - K)^+ = \max\{S_T - K, 0\}$$

for $0 \leq S$ because it can only be exercised if $S_T > K$.

The call option’s value is given as

$$C(S, t) = S_0 N(d_1) - Ke^{-r(T-t)} N(d_2),$$

(2.2)

For a European put option the boundary conditions are

i) $P(t, 0) = Ke^{-r(T-t)}$ for $0 \leq t \leq T$,

ii) $P(S, t) \to 0$ as $S \to \infty$. 
The payoff function is given by
\[ P(S, T) = (K - S_T)^+ = \max\{K - S_T, 0\} \text{ for } 0 \leq S \]
because it can only be exercised if \( K > S_T \).

The put option’s price is given by
\[ P(S, t) = Ke^{-r(T-t)}N(-d_2) - S_0N(-d_1), \quad (2.3) \]
where
\[ d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \quad \text{and} \]
\[ d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} = d_1 - \sigma \sqrt{T-t} \quad (2.4) \]

\( N(.) \) is the cumulative distribution function of standard normal distribution.

3 Transaction Cost Model

We consider a market with one share denoted \( S_t \), and a risk free money market account with spot rate of interest \( r \geq 0 \) whose value at time \( t \) is \( B_t \equiv 1 \). Stock is illiquid (its price is affected by trading) while money market account is assumed to be liquid. The model we are going to focus on is the model due to Cetin et al [3] where a fundamental stock price \( S_0^t \) follows the dynamics
\[ dS_0^t = \mu S_0^t + \sigma S_0^t dW_t, \quad 0 \leq t \leq T. \]

An investor who wants to trade \( \alpha \) shares at time \( t \) has to pay the transaction price \( S_t \) which is given by
\[ \bar{S}_t(\alpha) = e^{\rho \alpha} S_0^t \]
where \( \rho \) is the liquidity parameter with \( 0 \leq \rho < 1 \). This models a bid-ask-spread whose size depends on \( \alpha \) (number of shares traded).

Consider a Markovian trading strategy (i.e. a strategy of the form \( \Phi_t = \phi(t, S_0^t) \)) for a smooth function \( \phi = u_S \) where \( \phi \) is the hedge ratio. Then we have \( \phi_S = u_{SS} \). If the stock and bond positions are \( \Phi_t \) and \( \eta_t \) respectively then the value of this strategy at time \( t \) is
\[ V_t = \Phi_t S_0^t + \eta_t. \]
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$\Phi_t$ is a semi-martingale with quadratic variation of the form

$$[\Phi]_t = \int_0^t (\phi_S(\tau, S^0_\tau) S^0_\tau)^2 d\tau$$

whose change is given by $d[\Phi]_t = (u_{SS}(t, S^0_t) S^0_t)^2$ since $\phi_S = u_{SS}$.

Applying Itô formula to $u(t, S^0_t)$ gives

$$du(t, S^0_t) = u_S(t, S^0_t) dS^0_t + \left(u_t(t, S^0_t) + \frac{1}{2} \sigma^2(S^0_t)^2 u_{SS}(t, S^0_t)\right) dt$$  \hspace{1cm} (3.1)

For a continuous semi-martingale $\Phi$ with quadratic variation $[\Phi]_t$, the wealth dynamics of a self-financing strategy becomes

$$dV_t = \Phi_t dS^0_t - \rho S^0_t d[\Phi]_t.$$  \hspace{1cm} (3.2)

Substituting $d[\Phi]_t$ into (3.2) yields the following dynamics

$$dV_t = \phi(t, S^0_t) dS^0_t - \rho S^0_t (\phi_S(t, S^0_t) S^0_t)^2 dt.$$  \hspace{1cm} (3.3)

Since $V_t = u(t, S^0_t)$, equating the deterministic components of (3.3) and (3.1) and taking $\phi_S = u_{SS}$ gives the nonlinear PDE

$$u_t + \frac{1}{2} \sigma^2 S^2 u_{SS}(1 + 2 \rho Su_{SS}) = 0, \quad u(S^0_T) = h(S^0_T)$$ \hspace{1cm} (3.4)

where $h(S^0_T)$ is the payoff of the value claim at maturity date $T$.

4 Explicit Solution to the Nonlinear Black-Scholes Equation

We differentiate (3.4) twice with respect to $S$ and let $u_{SS} = w$ to obtain

$$w_t + \frac{\sigma^2 S^2}{2} (1 + 4 \rho Sw) w_{SS} + 2 \rho \sigma^2 S^3 w_S^2 + 2 \sigma^2 S (1 + 6 \rho Sw) w_S + \sigma^2 (1 + 6 \rho Sw) w = 0$$ \hspace{1cm} (4.1)

Applying the transformations, $w = \frac{u}{\rho S}$ and $x = \ln S$ then (4.1) becomes

$$u_t + \frac{\sigma^2}{2} (1 + 4u) u_{xx} + 2 \sigma^2 u_x^2 + \frac{\sigma^2}{2} (1 + 4u) u_x = 0$$ \hspace{1cm} (4.2)

Letting $u = \frac{V - 1}{4}$ then $u_t = \frac{V_t}{4}$, $u_x = \frac{V_x}{4}$, $u_{xx} = \frac{V_{xx}}{4}$

Substituting the above in equation (4.2) gives

$$V_t + \frac{\sigma^2}{2} \{V V_{xx} + V_x^2 + V V_x\} = 0$$ \hspace{1cm} (4.3)
We assume a traveling wave solution of the form $\epsilon = k(x - \lambda t)$ where $k$ (positive) and $\lambda$ represents the wave number and velocity of the traveling wave respectively. The dependent variable $V(x, t)$ is replaced by $V(\epsilon)$ to transform the PDE in (4.3) to the ODE

$$-k\lambda \frac{dV}{d\epsilon} + \frac{\sigma^2}{2} \left\{ V k^2 \frac{d^2V}{d\epsilon^2} + \left( k \frac{dV}{d\epsilon} \right)^2 + V k \frac{dv}{d\epsilon} \right\} = 0 \quad (4.4)$$

We now find the exact solution of the ODE in Tanh form by introducing the independent variable $Y = \tanh \epsilon$ into the ODE then $V(\epsilon) = S(Y)$. $dY = \text{sech}^2 \epsilon = 1 - Y^2 \frac{d}{dY}$ hence equation (4.4) becomes

$$-k\lambda(1 - Y^2) \frac{dS(Y)}{dY} + \frac{\sigma^2}{2} S(Y) k^2 (1 - Y^2) \frac{d}{dY} \left\{ 1 - Y^2 \frac{dS(Y)}{dY} \right\}$$

$$+ \frac{\sigma^2 k^2}{2} \left[ (1 - Y^2) \frac{dS(Y)}{dY} \right]^2 + \frac{\sigma^2}{2} S(Y) k (1 - Y^2) \frac{dS(Y)}{dY} = 0 \quad (4.5)$$

The two possible highest powers of $Y$ which gives a positive integer of $N$ appears in the first summation and the second part of the third summation as $N + 1$ and $2N$ respectively. Equating $2N = N + 1$ gives a positive value of $N = 1$. To find a possible solution we assume a finite series of the form

$$S(Y) = \sum_{n=0}^{N} a_n Y^n \quad (4.6)$$

which incorporates solitary wave and shock wave profiles. Therefore $S(Y) = a_0 + a_1 Y$ and $S'(Y) = a_1$

Substituting above into equation (4.5) and simplifying we obtain

$$-\lambda a_1 + \frac{\sigma^2}{2} \left\{ k S(Y) \frac{d}{dY} \{(1 - Y^2)a_1\} + k(1 - Y^2)a_1^2 + S(Y)a_1 \right\} = 0 \quad (4.7)$$

Hence

$$S(Y) = \frac{\lambda - \frac{k\sigma^2}{2} (1 - Y^2)a_1}{\sigma^2(\frac{1}{2} - Yk)}$$

$$= \frac{2\lambda}{\sigma^2 - k(1 - Y^2)a_1}$$

$$= \frac{2\lambda}{\sigma^2 - k(1 - 2Yk)} \quad (4.8)$$

In its original variables we get

$$V(x, t) = \frac{2\lambda}{\sigma^2 - ka_1 \text{sech}^2 k(x - \lambda t)} \quad (1 - 2k \tanh k(x - \lambda t)) \quad (4.9)$$
which is the explicit solution of the nonlinear partial differential equation in Tahn form.

5 Conclusion

In this work we focussed on a transaction cost model for modeling illiquid markets. The Tanh method was applied successfully in solving the nonlinear Black-Scholes partial differential equation. Thus we conclude that the proposed method can be used to solve other nonlinear Black-Scholes models such as the Howison and Bakstein (2003) model (see [1]) which is an extension of Cetin et al model with $r > 0$.

References


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