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Solving the Linear Homogeneous One-Dimensional Wave Equation Using the Adomian Decomposition Method

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Abstract

The use of the standard Adomian decomposition method for obtaining the approximate solution of the linear homogeneous one-dimensional wave equation is investigated. The results are compared with the exact solutions obtained using d'Alembert's formula. The results obtained show that the method has a high degree of efficiency, validity and accuracy as it leads to the exact solution.

Mathematics Subject Classification: 35J05, 35L05, $60-08$

Keywords: Wave equation, Adomian decomposition method, Initial value problem, Homogeneous partial differential equation, D'Alembert's formula

1 Introduction

The wave equation is a non-trivial partial differential equation that seems to be everywhere in several places at the same time. Wave propagation represents one of the most common physical phenomena experienced in everyday life. Waves occur most frequently through sight and hearing, as well as through telecommunication, radar, medical imaging, etc. Industrial applications range from aero acoustics to music (acoustic waves), from oil prospecting to non-destructive testing (elastic waves), from optics to stealth technology (electromagnetic waves) and stabilization of ships and offshore platforms [1].

Waves possess several interesting characteristics. Typically, a wave is a disturbance that propagates in a medium. However, two important kinds of waves that have been studied extensively are *light waves* (special relativity) and *probability waves* (quantum mechanics), neither of which propagates in a conventional medium. Waves undergo interference and diffraction (e.g., bending around corners). The engine for these phenomena is superposition. Furthermore, waves tend to be spread out in space or in the medium in which they propagate. Waves satisfy the wave equation. Being a partial differential equation, the wave equation is a somewhat non-intuitive mathematical abstraction and encapsulates many properties of waves in a very convenient form. In this paper, we limit ourselves to linear waves in one-dimensional space. This is a simple prototype example of all other kinds of wave propagation models. There is a wide variety of numerical schemes for approximating the solution of the linear one-dimensional wave equation. The uniqueness of the solution of a wave equation is obtained by imposing additional conditions, viz. initial and/or boundary conditions.

Recently, the Adomian decomposition method (ADM) [2] has proved to be an effective mathematical tool for obtaining approximate and analytical solutions to many types of ordinary and partial differential equations and yields results that are exact or very close to exact in many situations. The ADM is a well-known systematic method for practical solution of linear or non-linear and deterministic or stochastic operator equations, including ordinary differential equations (ODEs), partial differential equations (PDEs), integral equations, integro-differential equations, etc. (Duan et al. [3]). The efficiency of the ADM for solution of these different kinds of equations is well known. The ADM is a great method in that it provides the solution as an infinite series in which each term can be easily determined. The method accurately computes a rapidly convergent series solution.

Other advantages of the ADM have been pointed out in the literature. It has been argued that the ADM maintains a high degree of accuracy of the numerical solutions while at the same time reducing the amount of computational work compared to traditional approaches (Bulut et al. [4]). Other advantages include its ability to solve non-linear problems without linearization, the wide applicability to several types of problems and scientific fields, and the development of a reliable, analytic solution. Additionally, this method does not linearize the problem, nor does it use assumptions of weak non-linearity [5]. For this reason the ADM can handle fairly general non-linearities and generates solutions that 'may be more realisitic than those achieved by simplifying the model...to achieve conditions required for other techniques.'

These advantages provide justification for the ADM's wide-ranging applicability in such fields as engineering, chemistry, biology and physics [6, 7, 8, 9]. It has recently been pointed out that the method has proved to be highly ap-

plicable to such diverse areas as non-linear optics, particle transport, mass and/or heat transfer and chaos theory [10]. Over against these advantages, the method also suffers from certain disadvantages. One major concern lies in the region and rate of convergence of the series produced by the ADM. While the series can be rapidly convergent in very small regions, it has a very slow convergence rate in the wider regions [11]. Also, because of truncation the series solution is inaccurate in that region and this greatly restricts the application area of the method.

The rest of this paper is organized as follows: Section 2 derives the model to be solved and outlines the proposed methods of solution. In Section 3 some results are presented and discussed based on selected examples and the paper closes with conclusions and possible extensions in Section 4.

2 Methods and Materials

If $u = \phi(x - vt)$ describes a wave, the question that arises is, 'What type of equation does it satisfy?' We can write $u = \phi(\alpha) = \phi(x - vt)$, i.e., $\alpha = x - vt$. We note that $\frac{\partial u}{\partial x} = \frac{d\phi}{d\alpha} \cdot \frac{\partial \alpha}{\partial x} = \frac{d\phi}{d\alpha}$ (since $\frac{\partial \alpha}{\partial x} = 1$). So

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2\phi}{d\alpha^2} \tag{1}$$

Similarly, $\frac{\partial u}{\partial t} = \frac{d\phi}{d\alpha} \cdot \frac{\partial \alpha}{\partial t} = -v\frac{d\phi}{d\alpha}$ (since $\frac{\partial \alpha}{\partial t} = -v$). Thus, $\frac{\partial^2 u}{\partial t^2} = -v\frac{\partial}{\partial t}\left(\frac{d\phi}{d\alpha}\right) = -v\frac{d^2\phi}{d\alpha^2}\frac{\partial \alpha}{\partial t} = v^2\frac{d^2\phi}{d\alpha^2}$ and so

$$\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = \frac{d^2\phi}{d\alpha^2} \tag{2}$$

From (1) and (2), we have that

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \tag{3}$$

or, in compact form, $u_{xx} = v^{-2}u_{tt}$, which is the linear second-order homogeneous wave equation that describes the propagation of waves with respect to space and time. Here, $v > 0$ is a constant representing the wave velocity which is determined by the physical properties of the material through which the wave propagates. The initial conditions associated with this wave equation are $u(0,t) = f(t),\ u_x(0,t) = g(t)$. (3) admits solutions of the form $u(x,t) = \phi(x - vt) + \psi(x + vt)$ where ϕ and ψ are arbitrary functions. No matter the shape of the function ϕ, the wave $u = \phi(x - vt)$ satisfies the wave equation, since $\frac{\partial \phi}{\partial x} = \phi',\ \frac{\partial^2 \phi}{\partial x^2} = \phi'',\ \frac{\partial \phi}{\partial t} = -v\phi',\ \frac{\partial^2 \phi}{\partial t^2} = (-v)^2\phi'' = v^2\phi''$. Thus,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = \phi'' - \frac{1}{v^2}(v^2\phi'') = 0$$

Similarly, any function of the form $u = \phi(x + vt)$ also satisfies the wave equation, i.e., $\frac{\partial \phi}{\partial x} = \phi^{'}$, $\frac{\partial^2 \phi}{\partial x^2} = \phi^{''}$, $\frac{\partial \phi}{\partial t} = v\phi^{'}$, $\frac{\partial^2 \phi}{\partial t^2} = v^2\phi^{''}$, giving, again,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = \phi^{''} - \frac{1}{v^2}(v^2\phi^{''}) = 0$$

It follows that any function of the form $u = \phi(x-vt)+\psi(x+vt)$ is a solution of the wave equation. We can write (3) as $u_{tt} = v^2 u_{xx}$. In the standard form, this is

$$L_t(u(x,t)) = L_x\left(v^2 u(x,t)\right) \tag{4}$$

where the differential operators are defined, respectively, as $L_t = \frac{\partial^2}{\partial t^2}$ and $L_x = \frac{\partial^2}{\partial x^2}$. Suppose the inverse operator L_t^{-1} exists. Then L_t^{-1} is the two-fold integral operator from 0 to t defined by

$$L_t^{-1} = \int_0^t \int_0^t (\cdot) ds ds$$

Applying this inverse operator on both sides of (4) gives

$$L_t^{-1}\left(L_t(u(x,t))\right) = L_t^{-1}\left[L_x\left(v^2 u(x,t)\right)\right]$$

Since v^2 is a constant, it can be factored out and the above equation can be written as

$$L_t^{-1}\left(L_t(u(x,t))\right) = v^2 L_t^{-1}\left[L_x\left(u(x,t)\right)\right] \tag{5}$$

From (5), it follows, by application of the initial conditions, that

$$u(x,t) = f(t) + xg(t) + v^2 L_t^{-1}\left[L_x\left(u(x,t)\right)\right] \tag{6}$$

where the unknown function u is decomposed into a sum of components defined by the decomposition series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

with u_0 identified as $u(x,0)$ and the components $u_n(x,t)$ obtained from the recursive formula

$$\sum_{n=0}^{\infty} u_n(x,t) = f(t) + xg(t) + v^2 L_t^{-1}\left[L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right] \tag{7}$$

or

$$\begin{aligned} u_0(x,t) &= f(t) + xg(t) \\ u_{n+1}(x,t) &= v^2 L_t^{-1}\left[L_x(u_n(x,t))\right],\ n \geq 0 \end{aligned}$$

i.e.,

$$\begin{aligned} u_0 &= f(t) + xg(t) \\ u_1 &= v^2 L_t^{-1}\left[L_x(u_0)\right] \\ u_2 &= v^2 L_t^{-1}\left[L_x(u_1)\right] \\ &\vdots \\ u_{n+1} &= v^2 L_t^{-1}\left[L_x(u_n)\right] \end{aligned} \tag{8}$$

If the PDE contains a non-linear term, this term is represented as Nu and decomposed into a series

$$Nu = \sum_{n=0}^{\infty} A_n$$

where the A_n, depending on $u_0,\ u_1,\ u_2, \ldots,\ u_n$, are called Adomian polynomials obtained, for the non-linearity $Nu = \beta(u)$, by the formula

$$A_n = \frac{1}{n!} L_\lambda \left[\beta\left(\sum_{i=0}^{\infty} u_i \lambda^i\right)\right]_{\lambda=0},\ n = 0,\ 1,\ 2, \ldots$$

where λ is a grouping parameter of convenience and $L_\lambda = \frac{\partial^n}{\partial \lambda^n}$. In this way, the components $u_0,\ u_1,\ u_2, \ldots$ are identified and the series solution to the wave equation is completely determined. The exact solution may be determined using the approximation

$$u(x,t) = \lim_{n\to\infty} \Phi_n$$

where $\Phi_n = \sum_{k=0}^{n-1} u_k$. It is important to note that the recursive relationship (7) is constructed on the basis that the zeroth component $u_0(x,t)$ is defined by all the terms that arise from the initial conditions and from integrating the source term. However, since we are dealing with a *homogeneous* equation, the source term is zero. The remaining components $u_n(x,t),\ n \geq 1$ are completely determined such that each term is computed by using the immediately preceding term. Accordingly, considering the first few terms only, the recursive relation (8) gives $u_0(x,t),\ u_1(x,t),\ u_2(x,t), \ldots$.

Remark 2.1. Adomian and Rach [12] *and* Wazwaz [13] *investigated the phenomenon of self-cancelling 'noise' terms where some terms in the series vanish*

in the limit. These 'noise' terms do not appear for homogeneous equations but only for specific types of non-homogeneous equations. Furthermore, it has formally been shown that if some terms in u_0 are cancelled by identical terms in u_1 with opposite signs, even though u_1 includes additional terms, the remaining non-cancelled terms in u_0 constitute the exact solution of the given equation [14, 15]. *However, it is necessary and essential to verify that the remaining non-canceled terms satisfy the given equation.*

Since in the examples given in the next section of this paper we will be comparing the approximate solution from the ADM with the exact solution obtained by d'Alembert's formula, the following theorem is pertinent.

Theorem 2.2. *(d'Alembert's Formula) For the Cauchy initial value problem*

$$u_{tt} = v^2 u_{xx},\ u(x,0) = f(x),\ u_t(x,0) = g(x),\ x \in \mathbb{R},\ t > 0,\ v > 0 \tag{9}$$

where $f,\ g \in C^2(-\infty,\infty)$ are given by the initial conditions, there exists a unique $C^2(\mathbb{R}\times\mathbb{R})$-solution given by d'Alembert's formula

$$u(x,t) = \phi(x-vt)+\psi(x+vt) = \frac{1}{2}[f(x-vt)+f(x+vt)]+\frac{1}{2v}\int_{x-vt}^{x+vt} g(s)ds, \tag{10}$$

$\phi,\ \psi$ being arbitrary C^2-functions.

Proof. Assume that there is a solution $u(x,t)$ of the general linear homogeneous wave equation (9). We start by writing the general solution of the wave equation as

$$u(x,t) = \phi(x-vt) + \psi(x+vt). \tag{11}$$

Now we choose ϕ and ψ in such a way that the initial conditions are satisfied. Substituting $\phi(x-vt)+\psi(x+vt)$ for u in the initial conditions results in

$$u(x,0) = \phi(x) + \psi(x) = f(x) \tag{12}$$

$$u_t(x,0) = -v\phi^{'}(x) + v\psi^{'}(x) = g(x) \tag{13}$$

By differentiating (12), we obtain

$$\phi^{'}(x) + \psi^{'}(x) = f^{'}(x). \tag{14}$$

Now (13) and (14) form an algebraic system of two equations in two unknowns $\phi^{'}(x)$ and $\psi^{'}(x)$ since $f,\ g$ and v are all known from the given PDE and the initial conditions. Multiplying (14) by v gives

$$v\phi^{'}(x) + v\psi^{'}(x) = vf^{'}(x). \tag{15}$$

Adding (13) and (15) eliminates $\phi^{'}(x)$ so that we have

$$2v\psi^{'}(x) = vf^{'}(x) + g(x). \tag{16}$$

Now subtracting (13) from (15) eliminates $\psi^{'}(x)$ to give

$$2v\phi^{'}(x) = vf^{'}(x) - g(x). \tag{17}$$

Dividing (16) and (17) by $2v$ gives, respectively,

$$\begin{aligned} \psi^{'}(x) &= \frac{1}{2}f^{'}(x) + \frac{1}{2v}g(x) \\ \phi^{'}(x) &= \frac{1}{2}f^{'}(x) - \frac{1}{2v}g(x) \end{aligned}$$

which, upon integration, become

$$\begin{aligned} \psi(x) &= \frac{1}{2}f(x) + \frac{1}{2v}\int_0^x g(s)ds \\ \phi(x) &= \frac{1}{2}f(x) - \frac{1}{2v}\int_0^x g(s)ds \end{aligned}$$

Therefore,

$$\begin{aligned} u(x,t) &= \phi(x-vt) + \psi(x+vt) \\ &= \frac{1}{2}f(x-vt) - \frac{1}{2v}\int_0^{x-vt} g(s)ds + \frac{1}{2}f(x+vt) + \frac{1}{2v}\int_0^{x+vt} g(s)ds \\ &= \frac{1}{2}[f(x-vt) + f(x+vt)] + \frac{1}{2v}\int_{x-vt}^{0} g(s)ds + \frac{1}{2v}\int_0^{x+vt} g(s)ds \\ &= \frac{1}{2}[f(x-vt) + f(x+vt)] + \frac{1}{2v}\int_{x-vt}^{x+vt} g(s)ds \end{aligned}$$

Thus, each C^2-solution of the Cauchy IVP is given by d'Alembert's formula. On the other hand, the function $u(x,t)$ defined by the RHS of (10) is a solution of the given IVP. □

3 Examples

In this section we give three numerical examples of linear homogeneous wave equations and show their solution by the ADM. All the computations associated with these examples were performed using a Samsung Series 3 PC with an Intel Celeron CPU 847 at 1.10GHz and 6.0GB internal memory. The figures were constructed using MATLAB R2016a.

EXAMPLE 1. Consider the wave equation $u_{tt} - v^2 u_{xx} = 0$ with initial conditions $u(x,0) = f(x) = \sin x$, $u_t(x,0) = g(x) = x^2$. Applying d'Alembert's formula (Theorem 2.2) to the given equation we have the solution

$$u(x,t) = \frac{1}{2}\left[\sin(x - vt) + \sin(x + vt)\right] + \frac{1}{2v}\int_{x-vt}^{x+vt} s^2 ds$$

By the trigonometric identities $\sin(A - B) = \sin A \cos B - \cos A \sin B$ and $\sin(A + B) = \sin A \cos B + \cos A \sin B$, we have

$$\begin{aligned}\sin(x - vt) + \sin(x + vt) &= \sin x \cos(vt) - \cos x \sin(vt) + \sin x \cos(vt) + \cos x \sin(vt)\\ &= 2 \sin x \cos(vt)\end{aligned}$$

i.e.,

$$\begin{aligned}u(x,t) &= \frac{1}{2}\left[2 \sin x \cos(vt)\right] + \frac{1}{2v}\left[\frac{s^3}{3}\right]_{x-vt}^{x+vt}\\ &= \sin x \cos(vt) + \frac{1}{6v}\left[(x + vt)^3 - (x - vt)^3\right]\\ &= \sin x \cos(vt) + x^2 t + \frac{1}{3} v^2 t^3\end{aligned}$$

This is the exact solution against which the solution obtained by the ADM will be compared. The recursive relationship of the ADM is given by

$$\begin{aligned}u_0 &= f(x) + tg(x)\\ u_{n+1} &= v^2 L_t^{-1}\left[L_x\left(\sum_{n=0}^{\infty} u_n\right)\right] = v^2 L_t^{-1}\left[\left(\sum_{n=0}^{\infty} u_n\right)_{xx}\right]\end{aligned}$$

i.e.,

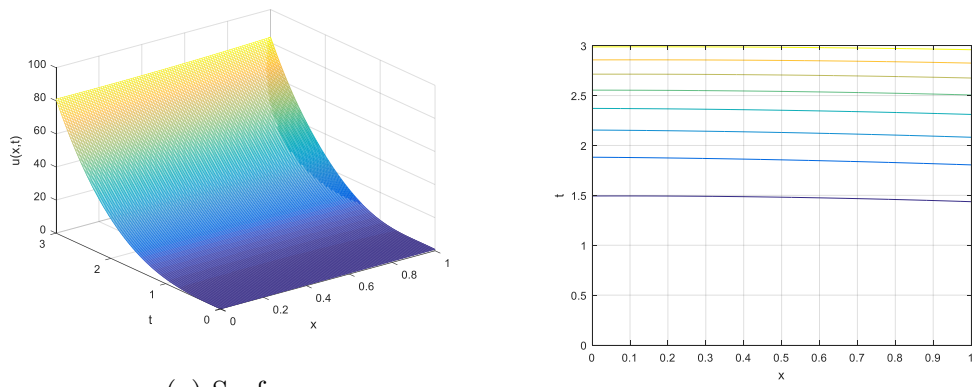
$$\begin{aligned}u_0 &= \sin x + tx^2\\ u_1 &= v^2 L_t^{-1}[L_x(u_0)] = v^2 L_t^{-1}\left[L_x\left(\sin x + tx^2\right)\right] = -\frac{v^2 t^2}{2}\sin x + \frac{v^2 t^3}{3}\\ u_2 &= v^2 L_t^{-1}[L_x(u_1)] = v^2 L_t^{-1}\left[L_x\left(-\frac{v^2 t^2}{2}\sin x + \frac{v^2 t^3}{3}\right)\right] = \frac{v^4 t^4}{24}\sin x\\ u_3 &= v^2 L_t^{-1}[L_x(u_2)] = v^2 L_t^{-1}\left[L_x\left(\frac{v^4 t^4}{24}\sin x\right)\right] = -\frac{v^6 t^6}{720}\sin x\end{aligned}$$

and so on. Adding the approximants $u_0,\ u_1,\ u_2,\ u_3, \ldots$ gives the approximate

solution as

$$\begin{aligned} u &= u_0 + u_1 + u_2 + u_3 + \cdots \\ &= \sin x + tx^2 - \frac{v^2t^2}{2}\sin x + \frac{v^2t^3}{3} + \frac{v^4t^4}{24}\sin x - \frac{v^6t^6}{720}\sin x + \cdots \\ &= \sin x\left(1 - \frac{(vt)^2}{2!} + \frac{(vt)^4}{4!} - \frac{(vt)^6}{6!} + \cdots\right) + x^2t + \frac{v^2t^3}{3} \\ &= \sin x\cos(vt) + x^2t + \frac{1}{3}v^2t^3 \end{aligned}$$

This is, in fact, the exact solution obtained earlier by d'Alembert's formula. The results for $v = 3,\ 0 < x < 1$ and $0 < t < 3$ are given in Figure 1.



(a) Surface
$u(x,t) = \sin x\cos(vt) + x^2t + \frac{1}{3}v^2t^3$

(b) Contour diagram

Figure 1: Approximate and exact solution and contour diagram for given IVP

EXAMPLE 2. Consider the following one-dimensional wave equation: $u_{tt} = 4u_{xx},\ u(0,t) = u(1,t) = 0, u(x,0) = f(x) = 0,\ u_t(x,0) = g(x) = 2\pi\sin(\pi x)$. Here $v = 2$ and by d'Alembert's formula we have the exact solution

$$\begin{aligned} u(x,t) &= \frac{1}{4}\int_{x-2t}^{x+2t} 2\pi\sin(\pi s)ds = -\frac{1}{2}[\cos(\pi s)]_{x-2t}^{x+2t} \\ &= -\frac{1}{2}[\cos(\pi(x+2t)) - \cos(\pi(x-2t))] \\ &= -\frac{1}{2}[\cos(\pi x)\cos(2\pi t) - \sin(\pi x)\sin(2\pi t) - \cos(\pi x)\cos(2\pi t) - \sin(\pi x)\sin(2\pi t)] \\ &= \sin(\pi x)\sin(2\pi t) \end{aligned}$$

Now applying the concept of ADM to the given wave equation, we have

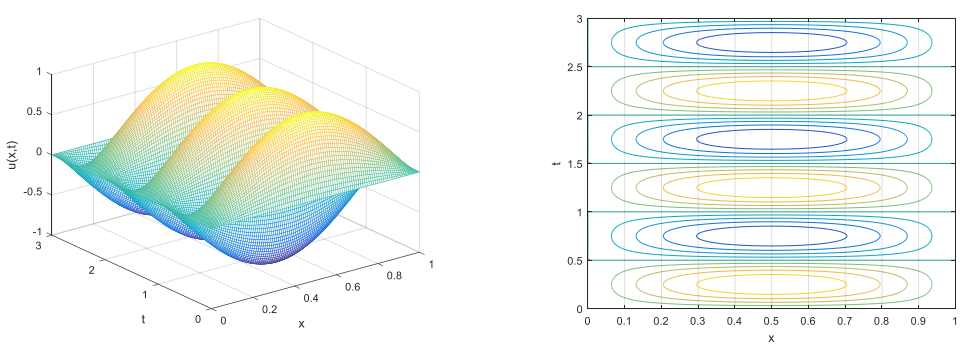
the recursive relationship

$$
\begin{aligned}
u_0 &= tg(x) = 2\pi t \sin(\pi x) \\
u_1 &= 4L_t^{-1}\left[L_x(u_0)\right] = 4L_t^{-1}\left[L_x(2\pi t \sin(\pi x))\right] = -\frac{8}{6}(\pi t)^3 \sin(\pi x) \\
u_2 &= 4L_t^{-1}\left[L_x(u_1)\right] = 4L_t^{-1}\left[L_x\left(-\frac{8}{6}(\pi t)^3 \sin(\pi x)\right)\right] = \frac{32}{120}(\pi t)^5 \sin(\pi x) \\
u_3 &= 4L_t^{-1}\left[L_x(u_2)\right] = 4L_t^{-1}\left[L_x\left(\frac{32}{120}(\pi t)^5 \sin(\pi x)\right)\right] = -\frac{128}{5040}(\pi t)^7 \sin(\pi x)
\end{aligned}
$$

and so on. Adding the approximants gives the solution

$$
\begin{aligned}
u &= u_0 + u_1 + u_2 + u_3 + \cdots \\
&= 2\pi t \sin(\pi x) - \frac{8}{6}(\pi t)^3 \sin(\pi x) + \frac{32}{120}(\pi t)^5 \sin(\pi x) - \frac{128}{5040}(\pi t)^7 \sin(\pi x) + \cdots \\
&= 2\pi t \sin(\pi x) - \frac{8}{6}(\pi t)^3 \sin(\pi x) + \frac{32}{120}(\pi t)^5 \sin(\pi x) - \frac{128}{5040}(\pi t)^7 \sin(\pi x) + \cdots \\
&= \sin(\pi x)\sin(2\pi t)
\end{aligned}
$$

which is the exact solution obtained from d'Alembert's formula. Figure 2 gives the results for $0 < x < 1$ and $0 < t < 3$.



(a) Surface $u(x,t) = \sin(\pi x)\sin(2\pi t)$ (b) Contour diagram

Figure 2: Approximate and exact solution and contour diagram or given IVP

EXAMPLE 3. Consider the Dirichlet problem $u_{tt} = u_{xx}$ for $0 < x < \pi$, $u(x,0) = \sin 3x$, $u_t(x,0) = 0$, $u(0,t) = u(\pi,t) = 0$. To find the solution,

we first use d'Alembert's formula which gives

$$\begin{aligned} u(x,t) &= \frac{1}{2}[f(x-t)+f(x+t)] \\ &= \frac{1}{2}[\sin 3(x-t)+\sin 3(x+t)] \\ &= \frac{1}{2}[\sin 3x\cos 3t-\cos 3x\sin 3t+\sin 3x\cos 3t+\cos 3x\sin 3t] \\ &= \sin 3x\cos 3t \end{aligned}$$

By the ADM, we have, recursively, the approximants

$$\begin{aligned} u_0 &= \sin 3x \\ u_1 &= L_t^{-1}\left[L_x(u_0)\right] = L_t^{-1}\left[L_x(\sin 3x)\right] = -\frac{9}{2}t^2\sin 3x \\ u_2 &= L_t^{-1}\left[L_x(u_1)\right] = L_t^{-1}\left[L_x\left(-\frac{9}{2}t^2\sin 3x\right)\right] = \frac{81}{24}t^4\sin 3x \\ u_3 &= L_t^{-1}\left[L_x(u_2)\right] = L_t^{-1}\left[L_x\left(\frac{81}{24}t^4\sin 3x\right)\right] = -\frac{729}{720}t^6\sin 3x \end{aligned}$$

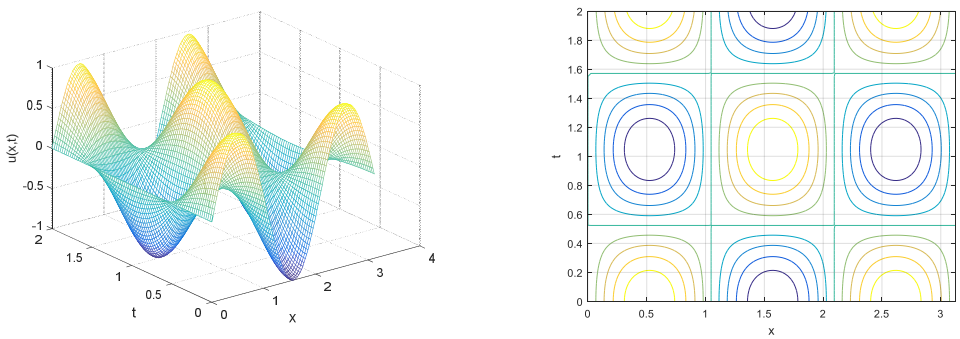
and so on. The sum of the approximants $u_0,\ u_1,\ u_2,\ldots$ is the desired approximate solution, i.e.,

$$\begin{aligned} u &= u_0+u_1+u_2+u_3+\cdots \\ &= \sin 3x-\frac{9}{2}t^2\sin 3x+\frac{81}{24}t^4\sin 3x-\frac{729}{720}t^6\sin 3x+\cdots \\ &= \sin 3x-\frac{(3t)^2}{2!}\sin 3x+\frac{(3t)^4}{4!}\sin 3x-\frac{(3t)^6}{6!}\sin 3x+\cdots \\ &= \sin 3x\left[1-\frac{(3t)^2}{2!}+\frac{(3t)^4}{4!}-\frac{(3t)^6}{6!}+\cdots\right] \\ &= \sin 3x\cos 3t \end{aligned}$$

which is simply the exact solution. The results for $0<x<\pi$ and $0<t<2$ are shown in Figure 3.

4 Conclusion

In this paper we proposed the application of the Adomian decomposition method for solving the linear homogeneous one-dimensional wave equation. This is a semi-analytical method that lends itself well to the solution of ordinary and partial differential equations with both linear and non-linear powers. Illustrative numerical examples were given and from the results it is concluded that the ADM leads to both exact and approximate numerical solutions for



(a) Surface $u(x,t) = \sin 3x \cos 3t$ (b) Contour diagram

Figure 3: Approximate and exact solution and contour diagram for given IVP

the wave equation. The work of this paper can be improved by (1) using one of the many modifications of the ADM, e.g., [15, 16, 17], instead of the standard ADM. (2) applying the ADM to the solution of Volterra integro-differential and integral equations that frequently occur in insurance mathematics; (3) applying the method to wave equations with a source term; (4) applying the ADM to partial differential equations with non-linear terms.

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