Abstract

In this paper, we introduce and investigate the concept of MS-ideals of MS-algebras. We reveal the connections between MS-ideals and several kinds of ideals as prime ideals, kernel ideals, e-ideals, dominator ideals and closure ideals. We show that many of these classes are proper subclasses of the class of MS-ideals. We round off with many properties of the homomorphic images, inverse homomorphic images of MS-ideals and congruence relations.

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1 Introduction

The study of MS-algebras was initiated by the seminal paper of T.S. Blyth and J.C. Varlet, [5], where they introduced the class of MS-algebras as a common generalization of both the class of de Morgan algebras and the class of Stone algebras. The study of ideals of MS-algebras has been proved to be a very satisfactory tool in investigating the structure of MS-algebras (see [5]). Recently, in [3], Badawy and Rao introduced the notions of dominator ideals and closure ideals. They characterized closure ideals in terms of principal
dominator ideals and studied some properties of closure ideals with respect to homomorphisms. In [10], Luo and Zheng defined e-ideals, tail ideals of MS-algebras and characterized those MS-algebras whose e-ideals are kernel ideals.

The goal of this paper is to introduce the class of MS-ideals of MS-algebras. We prove that this class is a generalization of many known classes of ideals. Also, we derive some results on the homomorphic, inverse homomorphic images of MS-ideals and congruence relations.

2 Preliminaries

In this section, we present the main definitions and results which are needed throughout this paper. We refer the reader to [5], [7], [8] and [9] for more details.

A de Morgan algebra is an algebra \((L; \lor, \land, ^-, 0, 1)\) of type \((2,2,1,0,0)\) where \((L; \lor, \land, 0, 1)\) is a bounded distributive lattice and the unary operation of involution \(^-\) satisfies:

\[
\overline{x} = x, \overline{x \lor y} = \overline{x} \land \overline{y}, \overline{x \land y} = \overline{x} \lor \overline{y}.
\]

An MS-algebra is an algebra \((L; \lor, \land, ^\circ, 0, 1)\) of type \((2,2,1,0,0)\) where \((L; \lor, \land, 0, 1)\) is a bounded distributive lattice and the unary operation \(^\circ\) satisfies:

\[
x \leq x^{\circ\circ}, (x \land y)^\circ = x^\circ \lor y^\circ, 1^\circ = 0.
\]

The following theorem assembles the basic properties of MS-algebras.

**Theorem 2.1.** [5],[8]. For any two elements \(a, b\) of an MS-algebra \(L\), we have

1. \(0^\circ = 1\),
2. \(a \leq b \Rightarrow b^\circ \leq a^\circ\),
3. \(a^{\circ\circ\circ} = a^\circ\),
4. \((a \lor b)^\circ = a^\circ \land b^\circ\),
5. \((a \lor b)^{\circ\circ} = a^{\circ\circ} \lor b^{\circ\circ}\),
6. \((a \land b)^{\circ\circ} = a^{\circ\circ} \land b^{\circ\circ}\).

**Lemma 2.2.** Let \(L\) be an MS-algebra. Then

1. \(L^\lor = \{x \in L : x \leq x^\circ\}\) is an order ideal of \(L\). Moreover, \(L^\lor\) is an ideal whenever \(L \in K_2\), [5].
2. \(D(L) = \{x \in L : x^\circ = 0\}\) is a filter (filter of dense elements) of \(L\), [2].
The ideal \((a) = \{x \in L : x \leq a\}\) is called the principal ideal of \(L\) generated by \(a\).
For any ideal of an MS-algebra, let \(I_\infty = \{x \in L : x \leq a^{\infty}\) for some \(a \in I\) and \(I^\circ = \{x \in L : x \geq i^\circ\) for some \(i \in I\}\). It is known that \(I_\infty\) is an ideal of \(L\) and \(I^\circ\) is a filter of \(L\) (see [8]).

**Theorem 2.3.** [5]. Let \(I\) be an ideal of an MS-algebra \(L\). Then, \(I\) is a kernel ideal of \(L\) if and only if

(i) \(I = I_\infty\),

(ii) \(a \land j \in I\) implies \(a \in I\), for all \(a \in L\) and \(j \in I^\circ\).

**Lemma 2.4.** [8]. If \(I\) is a kernel ideal of an MS-algebra \(L\), then \(I = I_\infty\) and \(I \cap I^\circ = \emptyset\).

**Definition 2.5.** [10]. Let \(I\) be an ideal of an MS-algebra \(L\). Then, \(I\) is called an \(e\)-ideal of \(L\) if \(I = I_\infty\) and \(I \cap I^\circ = \emptyset\).

**Definition 2.6.** [3]. An ideal \(I\) of an MS-algebra \(L\) is called a dominator ideal if \(I = I_\infty\).

**Definition 2.7.** [3]. An ideal \(I\) of an MS-algebra \(L\) is called a closure ideal if \(\overline{\sigma}(I) = I\), where \(\sigma(I) = \{(i)_\infty : i \in I\}\) and \(\overline{\sigma}(I) = \{x \in L : (x)_\infty \in I\}\).

**Definition 2.8.** A congruence on an MS-algebra is a lattice congruence \(\theta\) such that \(x \equiv y(\theta)\) implies \(x^\circ \equiv y^\circ(\theta)\)

## 3 MS-ideals of MS-algebras

**Definition 3.1.** An ideal \(I\) of an MS-algebra \(L\) is called an MS-ideal if \(x^{\infty} \in I\) for every \(x \in I\).

**Lemma 3.2.** Let \(L\) be an MS-algebra from \(K_2\). Then \(L^\uparrow\) is an MS-ideal.

**Proof.** For any \(x \in L^\uparrow\), \(x \leq x^\circ\). Therefore, \(x^{\infty} \leq x^\circ = x^{\infty}\). Hence, \(x^{\infty} \in L^\uparrow\) and \(L^\uparrow\) is an MS-ideal. \(\square\)

For any ideal \(I\) of an MS-algebra, let \(\ell(I) = \{x \in L : x^\circ \not\in I\}\).

**Lemma 3.3.** Let \(L\) be an MS-algebra. Then \(\ell(P)\) is an MS-ideal for any prime ideal \(P\) of \(L\).

**Proof.** \(0^\circ = 1 \not\in P\), so \(0 \not\in P\). Let \(x, y \in \ell(P)\), then \(x^\circ, y^\circ \not\in P\). As \(P\) is a prime ideal, we have \(x^\circ \land y^\circ = (x \lor y)^\circ \not\in P\). Then, \(x \lor y \in \ell(P)\). Let \(x \in \ell(P)\) and \(y \in L\) such that \(y \leq x\). Then, \(y^\circ \geq x^\circ\). Since \(x^\circ \not\in P\), then \(y^\circ \not\in P\) and \(y \in P\). Hence, \(\ell(P)\) is an ideal of \(L\). Finally, for any \(x \in \ell(P)\) we have \(x^{\infty} = (x^{\infty})^\circ = x^\circ \not\in P\). Hence, \(x^{\infty} \in \ell(P)\) and \(\ell(P)\) is an MS-ideal. \(\square\)
Definition 3.4. For any ideal $I$ of an $MS$-algebra $L$. The perpendicular of $I$ is defined to be the set $I^\perp = \{ x \in L : x^\circ \land a = 0, \text{ for every } a \in I \}$.

Lemma 3.5. Let $L$ be an $MS$-algebra. Then $I^\perp$ is an $MS$-ideal for any ideal $I$ of $L$.

Proof. Clearly $0 \in I^\perp$. Let $x, y \in I^\perp$, then $(x \lor y)^\circ \land a = (x^\circ \land a) \lor (y^\circ \land a) = 0$. We conclude that $x \lor y \in I^\perp$. Let $x \in I^\perp$ and $y \leq x$, then $y^\circ \land a \leq x^\circ \land a = 0$. Then, $y^\circ \land a = 0$. Hence, $y \in I^\perp$. Moreover, $(x^\circ)^\circ \land a = x^\circ \land a = 0$. So, $x^\circ \in I^\perp$. Hence, $I^\perp$ is an $MS$-ideal of $L$.

Lemma 3.6. For any ideal $I$ of an $MS$-algebra $L$, $I^\circ$ is an $MS$-ideal of $L$.

Proof. Let $x \in I^\circ$, then $x \leq a^\circ$ for some $a \in I$. This implies that $x^\circ \leq a^\circ$ for some $a \in I$. Therefore, $x^\circ \in I^\circ$. Hence, $I^\circ$ is an $MS$-ideal of $L$.

The following corollaries follow directly from the previous lemma and the definitions of kernel ideal and $e$-ideal of an $MS$-algebra.

Corollary 3.7. Any kernel ideal of an $MS$-algebra is an $MS$-ideal.

Corollary 3.8. Any $e$-ideal of an $MS$-algebra is an $MS$-ideal.

For any filter $F$ of an $MS$-algebra $L$, let $\delta(F) = \{ x \in L : x^\circ \in F \}$. We call an ideal $I$ of $L$ a $\delta$-ideal if $I = \delta(F)$ for some filter $F$ of $L$, see [1].

Lemma 3.9. Any $\delta$-ideal $I$ of an $MS$-algebra $L$ is an $MS$-ideal.

Proof. Let $I = \delta(F)$ for some filter $F$ of $L$ and let $x \in I$. We have $x^\circ \in F$ and so $x^\circ \in \delta(F) = I$. Hence, $I$ is an $MS$-ideal.

Lemma 3.10. Let $L$ be an $MS$-algebra, then any principal ideal of the form $(x^\circ)$ is an $MS$-ideal of $L$ for every $x \in L$.

Proof. Let $y \in (x^\circ)$. Then, $y \leq x^\circ$. This is equivalent to $y^\circ \leq x^\circ$ and so $y^\circ \in (x^\circ)$.

Lemma 3.11. A proper $MS$-ideal of an $MS$-algebra contains no dense element.

Proof. Let $I$ be a proper $MS$-ideal of an $MS$-algebra $L$. Suppose $x \in I$ is a dense element. Then $x^\circ = 1 \in I$. Hence $I = L$, which is a contradiction.

Let $L$ be an $MS$-algebra. For any non empty subset $S$ and any element $a$ of $L$, define $S_a = \{ x \in L : x \land a \in S \}$. It is easy to see that $S_a$ is not an ideal in general but it is an ideal if $S$ is an ideal. In the following we give some properties of these sets.

Lemma 3.12. Let $I$ be an ideal of an $MS$-algebra $L$. Then
(1) For any element $a \in L$, $I \subseteq I_a$.

(2) $I_1 = I$.

(3) For any element $a \in L$, $I_a = L$ if and only if $a \in I$.

(4) Let $a, b \in L$. If $a \leq b$, then $I_b \subseteq I_a$.

(5) For any element $a \in L$, $I_a^{\infty} \subseteq I_a$.

**Proof.** (1) Let $x \in I$. Then $x \wedge a \in I$, $\forall a \in L$. Therefore, $x \in I_a$. Hence, $I \subseteq I_a$.

(2) It is obvious.

(3) If $I_a = L$, then $x \wedge a \in I, \forall x \in L$. Taking $x = 1$, we get $a \in I$. Conversely, assume $a \in I$. Then, $x \wedge a \in I, \forall x \in L$. Accordingly, $x \in I_a, \forall x \in L$. So, $L \subseteq I_a$ and hence $I_a = L$.

(4) Let $x \in I_b$, then $x \wedge b \in I$. Since $x \wedge a \leq x \wedge b$, then $x \wedge a \in I$. Hence, $x \in I_a$ and $I_b \subseteq I_a$.

(5) The proof is a direct consequence of (4) with $a \leq a^{\infty}$. \hfill\(\Box\)

**Proposition 3.13.** Let $L$ be an MS-algebra. If $I$ is an MS-ideal of $L$, then $I_a$ is an MS-ideal of $L$ for any $a \in L$.

**Proof.** Let $x \in I_a$. As $x \wedge a \in I$, then $(x \wedge a)^{\infty} = x^{\infty} \wedge a^{\infty} \in I$. Since $x^{\infty} \wedge a \leq x^{\infty} \wedge a^{\infty}$, then $x^{\infty} \wedge a \in I$. Consequently, $x^{\infty} \in I_a$. Hence, $I_a$ is an MS-ideal of $L$. \hfill\(\Box\)

**Proposition 3.14.** Let $I$ be an MS-ideal of an MS-algebra $L$. Then, $I_a = I$ for any $a \in D(L)$.

**Proof.** Let $a \in D(L)$ and $x \in I_a$. Then, $x \wedge a \in I$. So, $(x \wedge a)^{\infty} = x^{\infty} \wedge a^{\infty} = x^{\infty} \wedge 1 \in I$. Then, $x^{\infty} \in I_1 = I$ and so $x \in I$. Hence, $I_a \subseteq I$ and thus $I_a = I$. \hfill\(\Box\)

**Proposition 3.15.** Let $I$ be an ideal of an MS-algebra $L$. If $I$ is maximal then $I_a$ is maximal. Moreover, $I = I_a$ for every $a \notin I$.

**Proof.** Suppose $I_a \subseteq J$ for some ideal $J$ of $L$. Then $I \subseteq I_a \subseteq J$. Consequently, $I = J$ or $J = L$. Therefore, $I_a = J$ or $J_a = L$ and so $I_a$ is maximal. If $a \notin I$, then $I_a \neq L$ (by lemma 3.12 ). Besides, we have $I \subseteq I_a$. Hence, $I = I_a$. \hfill\(\Box\)

**Proposition 3.16.** Let $I$ be an ideal of an MS-algebra $L$. If $I$ is prime then $I_a$ is prime.

**Proof.** Suppose $x, y \in L$ such that $x \wedge y \in I_a$. We have $(x \wedge y) \wedge a = (x \wedge a) \wedge (x \wedge a) \in I$. So, $x \wedge a \in I$ or $y \wedge a \in I$. Then, $x \in I_a$ or $y \in I_a$. Hence, $I_a$ is prime. \hfill\(\Box\)
The following examples have two goals. First, they show that the converses of many results are not true. Second, they clarify some of these results.

**Example 3.17.** Consider the following MS-algebra:

```
1 = 0°
\quad y \quad z = c°
\quad a = b° = d° \quad x \quad b = a°
\quad c = x° = z° \quad d
\quad 0 = 1° = y°
```

We have
1. \( I = \{0, b, c, d, x, z\} \) is an MS-ideal while it is not a \( \delta \)-ideal. So, the converse of Lemma 3.9 is not true.
2. \( I = \langle d \rangle \) is not an MS-ideal. So, not every principal ideal is an MS-ideal (see Lemma 3.10).
3. \( I = \{0, b, c, d, x, z\} \) is an MS-ideal as \( I = \langle c\rangle \) (see Lemma 3.10).
4. \( I = \langle d \rangle \) is not an MS-ideal although \( I_a = \{0, b, d\} \) is an MS-ideal. So, the converse of Proposition 3.13 does not hold.
5. Consider \( I = \{0, c, d, x\} \). Then \( I_a = \{0, b, c, d, x, z\} \) is maximal while \( I \) is neither maximal nor prime. Hence, the converses of Proposition 3.15 and Proposition 3.16 are not true.

**Example 3.18.** In the MS-algebra:

```
1 = 0°
\quad d
\quad a = a° = c°
\quad c
\quad 0 = 1° = d°
```

Consider the ideal \( I = \{0, a, c\} \). We see that \( I° = \{1, a, d\} \). Then, \( I \cap I° = \{a\} \neq \phi \). It follows that \( I \) is not an \( e \)-ideal. Consequently, \( I \) is neither a kernel ideal nor a tail ideal. Hence, the converses of Corollary 3.7 and Corollary 3.8 are not valid.

**Lemma 3.19.** Let \( I \) be an MS-ideal of an MS-algebra \( L \). Then \( I_a = I_{a°°} \), for all \( a \in L \).
Proof. By lemma 3.12, we have $I_{a^{\omega}} \subseteq I_a$. Let $x \in I_a$. Then, $x \wedge a \in I$. This gives $x^{\omega} \wedge a^{\omega} \in I$. Since $x \wedge a^{\omega} \leq x^{\omega} \wedge a^{\omega}$, then $x \wedge a^{\omega} \in I$. This means that $x \in I_{a^{\omega}}$. Hence, $I_a = I_{a^{\omega}}$. \hfill \Box

**Lemma 3.20.** Let $I$ and $J$ be two ideals of an $MS$-algebra $L$. Then, for every $a \in L$, we have

(1) $(I \cap I)_a = I_a \cap J_a$,

(2) $I \subseteq J$ implies $I_a \subseteq J_a$.

**Proof.** (1) $x \in (I \cap I)_a \iff x \wedge a \in I$ and $x \wedge b \in I \iff x \in I_a \cap J_a$.

(2) Let $x \in I_a$. Then $x \wedge a \in I \subseteq J$. So, $x \in J_a$. Hence, $I_a \subseteq J_a$. \hfill \Box

**Theorem 3.21.** Let $I$ be an ideal of an $MS$-algebra $L$. Let $I_L = \{I_a : a \in L\}$ ordered by set inclusion. Then

(1) $I_L$ is a bounded distributive lattice,

(2) If $I$ is an $MS$-ideal of $L$, then $I_L$ is a de Morgan algebra with $\bar{I}_a = I_{a^{\omega}}$.

**Proof.** (1) The greatest and least elements of $I_L$ are $I_0$ and $I_1$, respectively. We show that $I_a \cap I_b = I_{a \wedge b}$ and $I_a \cup I_b = I_{a \vee b}$ for any two elements $a, b \in L$. For the first, we know that $I_{a \vee b} \subseteq I_a$ and $I_{a \wedge b} \subseteq I_b$ (as $a, b \leq a \vee b$) and so $I_{a \vee b} \subseteq I_a \cap I_b$. Let $x \in I_a \cap I_b$, then $x \wedge a, x \wedge b \in I$. Then, $(x \wedge a) \vee (x \wedge b) = x \wedge (a \vee b) \in I$. Therefore, $x \in I_{a \vee b}$. Hence, $I_a \cap I_b = I_{a \vee b}$. For the second equality, we have $I_a, I_b \subseteq I_{a \vee b}$. Let $I_a, I_b \subseteq I_{a \wedge b}$. Then $x \wedge a \wedge b \in I$. We conclude that $x \wedge a \in I_b \subseteq I_z$ and $x \wedge b \in I_a \subseteq I_z$. Then, $(x \wedge a \wedge z) \vee (x \wedge b \wedge z) = (x \wedge z) \wedge (a \vee b) \in I$. Accordingly, $x \wedge z \in I_{a \vee b} = I_a \cap I_b \subseteq I_z$. So, $x \wedge z \wedge z \in I$. Therefore, $x \in I_z$. Hence, $I_{a \wedge b} \subseteq I_z$ and $I_a \vee I_b = I_{a \wedge b}$. Now, we prove distributivity. For any $a, b, c \in L$, we have

$(I_a \vee I_b) \wedge I_c = I_{a \wedge b} \wedge I_c = I_{(a \wedge b) \vee c} = I_{(a \wedge c) \vee (b \vee c)} = I_{a \wedge c} \vee I_{b \vee c} = (I_a \wedge I_c) \vee (I_b \wedge I_c)$.

Hence, $I_L$ is a bounded distributive lattice.

(2) We have $\bar{I}_a = I_{a^{\omega}} = I_a$, $\forall a \in L$. Also, $\bar{I}_a \lor \bar{I}_b = I_{a \wedge b} = I_{a^{\omega} \vee b^{\omega}} = I_a^{\omega} \cap I_b^{\omega} = \bar{I}_a \cap \bar{I}_b$. Finally, $\bar{I}_0 = \bar{I}_1 = I$. Hence, $I_L$ is a de Morgan algebra. \hfill \Box

**Theorem 3.22.** Any dominator ideal of an $MS$-algebra $L$ is an $MS$-ideal.

**Proof.** Assume that $I$ is a dominator ideal of $L$. Then, $I = I_{\omega}$. It follows, by Lemma 3.6, that $I$ is an $MS$-ideal. \hfill \Box

**Corollary 3.23.** Any closure ideal of an $MS$-algebra $L$ is an $MS$-ideal.

**Proof.** This is an immediate consequence of the previous result and Theorem 3.7 of [3]. \hfill \Box
4 MS-ideals and Homomorphisms

In this section, we present some results on the homomorphic images and inverse homomorphic images of MS-ideals. We show that any isomorphism between MS-algebras \( L_1 \) and \( L_2 \) induces an isomorphism between \( I_{L_1} \) and \( I_{L_2} \) for any ideal \( I \) of \( L_1 \).

**Theorem 4.1.** Let \( f : L_1 \to L_2 \) be a homomorphism of MS-algebras. Then,

1. \( \text{Ker}(f) \) is an MS-ideal of \( L_1 \),
2. The image of an MS-ideal of \( L_1 \) is an MS-ideal of \( L_2 \),
3. The inverse image of an MS-ideal of \( L_2 \) is an MS-ideal of \( L_1 \).

**Proof.** (1) Let \( x \in \text{Ker}(f) \), then \( f(x) = 0 \). This implies that \( (f(x))^\circ = f(x^\circ) = 0 \). Hence, \( x^\circ \in \text{Ker}(f) \).

(2) Let \( I \) be an MS-ideal of \( L_1 \). It is known that \( f(I) \) is an ideal of \( L_2 \). Let \( y \in f(I) \). Then there exists \( x \in I \) such that \( f(x) = y \). Consequently, \( (f(x))^\circ = y^\circ \). Equivalently, \( f(x^\circ) = y^\circ \). Since \( x^\circ \in I \), then \( y^\circ \in f(I) \).

(3) Let \( J \) be an MS-ideal of \( L_2 \). Then \( f^{-1}(J) \) is an ideal of \( L_1 \). Assume that \( x \in f^{-1}(J) \), then \( f(x) \in J \). As \( J \) is an MS-ideal, we conclude that \( f(x^\circ) \in J \).

Hence, \( x^\circ \in f^{-1}(J) \) and \( f^{-1}(J) \) is an MS-ideal of \( L_1 \). \( \square \)

**Theorem 4.2.** Let \( f : L_1 \to L_2 \) be a homomorphism of MS-algebras. Let \( I \) be an ideal of \( L_1 \) and \( J \) an ideal of \( L_2 \). Then, for every \( a \in L_1 \):

1. \( f(I_a) \subseteq (f(I))_{f(a)} \),
2. \( f^{-1}(J_{f(a)}) = (f^{-1}(J))_a \),
3. If \( f \) is an isomorphism, then \( f(I_a) = (f(I))_{f(a)} \).

**Proof.** (1) Let \( y \in f(I_a) \), then there exists \( x \in I_a \) such that \( y = f(x) \). Since \( x \in I_a \), then \( x \land a \in I \). So, \( y \land f(a) = f(x \land a) \subseteq f(I) \). Consequently, \( y \in f(I)_{f(a)} \).

(2) Observe that \( x \in f^{-1}(J_{f(a)}) \Leftrightarrow f(x) \in J_{f(a)} \Leftrightarrow f(x \land a) \subseteq J \Leftrightarrow x \land a \in f^{-1}(J) \Leftrightarrow x \in (f^{-1}(J))_a \).

(3) We know, from part (1), that \( f(I_a) \subseteq (f(I))_{f(a)} \). Let \( y \in (f(I))_{f(a)} \). Then \( y \land f(a) \subseteq f(I) \). We conclude that \( f^{-1}(y \land f(a)) = f^{-1}(y) \land a \subseteq I \). Then, \( y \in f(I_a) \). Hence, \( f(I_a) = (f(I))_{f(a)} \). \( \square \)

**Theorem 4.3.** Let \( f : L_1 \to L_2 \) be an isomorphism of MS-algebras. Then:

1. \( I_{L_1} \) is isomorphic to \( f(I)_{L_2} \), for any ideal \( I \) of \( L_1 \),
2. \( J_{L_2} \) is isomorphic to \( f^{-1}(J)_{L_1} \), for any ideal \( J \) of \( L_2 \).
Proof. (1) Define $\varphi : I_{L_1} \to f(I)_{L_2}$ by $\varphi(I_a) = f(I_a)$, for any $a \in L_1$. Then, for any $a, b \in L_1$, we have
\[
\varphi(I_a \cap I_b) = \varphi(I_{a \lor b}) = f(I_{a \lor b}) = f(I)_{f(a \lor b)} \\
= f(I)_{f(a) \lor f(b)} = f(I)_{f(a) \cap f(I)_{f(b)}} \\
= f(I_a) \cap f(I_b) = \varphi(I_a) \cap \varphi(I_b).
\]

Also,
\[
\varphi(I_a \lor I_b) = \varphi(I_{a \land b}) = f(I_{a \land b}) = f(I)_{f(a \land b)} \\
= f(I)_{f(a) \land f(b)} = f(I)_{f(a) \lor f(I)_{f(b)}} \\
= f(I_a) \lor f(I_b) = \varphi(I_a) \lor \varphi(I_b).
\]

Now, let $I_a, I_b \in I_{L_1}$ such that $\varphi(I_a) = \varphi(I_b)$. Equivalently, $f(I_a) = f(I_b)$. Assume $x \in I_a$, then $f(x) \in f(I_a) = f(I_b)$. This gives $f(x) = f(y)$ for some $y \in I_b$. Since $f$ is one to one, then $x = y \in I_b$. Therefore, $I_a \subseteq I_b$. Analogously, we can show that $I_b \subseteq I_a$. Hence, $I_a = I_b$ and $\varphi$ is one to one. It remains to prove that $\varphi$ is onto. Let $f(I)_{b} \in f(I)_{L_2}, b \in L_2$. Since $f$ is onto, then there exists $a \in L_1$ with $f(a) = b$. Therefore, $f(I)_{b} = f(I)_{f(a)} = f(I_a)$. As $I_a \in I_{L_1}$, then $\varphi$ is onto. Hence, $\varphi$ is an isomorphism between $I_{L_1}$ and $f(I)_{L_2}$.

(2) Define $\varphi : J_{L_2} \to f^{-1}(J)_{L_1}$ by $\varphi(J_{f(a)}) = f^{-1}(J_{f(a)})$, for any $a \in L_1$. Then, for any $a, b \in L_1$, we have
\[
\varphi(J_{f(a)} \cap J_{f(b)}) = f^{-1}(J_{f(a)} \cap J_{f(b)}) = f^{-1}(J_{f(a) \lor b}) = (f^{-1}(J))_{a \lor b} \\
= f^{-1}(J)_{a} \lor f^{-1}(J)_{b} = \varphi(J_{f(a)}) \lor \varphi(J_{f(b)}).
\]

Also,
\[
\varphi(J_{f(a)} \lor J_{f(b)}) = f^{-1}(J_{f(a)} \lor J_{f(b)}) = f^{-1}(J_{f(a) \land b}) = (f^{-1}(J))_{a \land b} \\
= (f^{-1}(J))_{a} \land f^{-1}(J)_{b} = \varphi(J_{f(a)}) \land \varphi(J_{f(b)}).
\]

Let $J_{f(a)}, J_{f(b)} \in J_{L_2}$ with $\varphi(J_{f(a)}) = \varphi(J_{f(b)})$. That is, $f^{-1}(J_{f(a)}) = f^{-1}(J_{f(b)})$. Let $f(c) \in J_{f(a)}$, then $c \in f^{-1}(J_{f(a)}) = f^{-1}(J_{f(b)})$. Then, $f(c) \in J_{f(b)}$. Whence, $J_{f(a)} \subseteq J_{f(b)}$. Similarly, we can show that $J_{f(b)} \subseteq J_{f(a)}$. Hence, $J_{f(a)} = J_{f(b)}$ and $\varphi$ is one to one. Now, let $(f^{-1}(J))_{a} \in f^{-1}(J)_{L_1}$. As $\varphi(J_{f(a)}) = f^{-1}(J_{f(a)}) = (f^{-1}(J))_{a}$, then $\varphi$ is onto. Hence, $J_{L_2}$ is isomorphic to $f^{-1}(J)_{L_1}$.

Let $\theta$ be a congruence on an $MS$-algebra $L$. The kernel of $\theta$ is defined by $\text{Ker}(\theta) = \{x \in L : x \equiv 0(\theta)\}$. It is known that $\text{Ker}(\theta)$ is an ideal of $L$.

**Lemma 4.4.** Let $\theta$ be a congruence on an $MS$-algebra $L$. Then $\text{Ker}(\theta)$ is an $MS$-ideal.
Proof. Let $x \in \text{Ker}(\theta)$. Then $x \equiv 0 (\theta)$. As $\theta$ is a congruence on $L$, then $x^{oo} \equiv 0(\theta)$. Hence, $x^{oo} \in \text{Ker}(\theta)$.

Theorem 4.5. Let $I$ be an MS-ideal of an MS-algebra $L$. Define $\theta(I)$ on $L$ by $x \equiv y \ (\theta(I))$ if and only if $I_x = I_y$. Then

1. $\theta(I)$ is a lattice congruence,
2. $\theta(I)$ is not a congruence on $L$ in general.

Proof. (1) It is clear that $\theta(I)$ is an equivalence relation. Let $a \equiv b \ \theta(I)$. For any $c \in L$, we have

$$x \in I_{a \land c} \iff x \land (a \land c) \in I \iff x \land c \in I_a = I_b \iff x \in I_{b \land c}.$$

Therefore $I_{a \land c} = I_{b \land c}$. Hence, $a \land c \equiv b \land c \ \theta(I)$. Also,

$$I_{a \lor c} = I_a \cap I_c = I_b \cap I_c = I_{b \lor c}.$$

Consequently, $a \lor c \equiv b \lor c \ \theta(I)$. Hence, $\theta(I)$ is a lattice congruence.

(2) We give a counter example. Consider example 3.17. Let $I = \{0, b, d\}$. Then $I_a = \{0, b, d\} = I_x$ and so $a \equiv x \ \theta(I)$. On the other hand, we have $I_a^o = I_b = L$ and $I_x^o = I_c = \{0, b, d\}$. This implies that $a^o \not\equiv x^o \ \theta(I)$. Hence, $\theta(I)$ is not a congruence on $L$.

References


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