Portfolio Optimization Problem
with Cooperation Behavior

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Abstract

This paper investigates the cooperation behavior between two fund managers with CARA (constant absolute risk aversion coefficient) utility. Each fund manager’s goal is not only chooses a investment strategy to maximize his/her terminal log return, but also maximize the cooperator’s joint log return. This optimization problem is formulated as a non-zero-sum stochastic differential game. By using stochastic control method, we derive the explicit expressions of the equilibrium investment for each investor. The results show that cooperation behavior will make the investor becoming more cautious, and adjust the position of the risky asset to hedge the investment risk. Finally, some numerical examples are conducted to illustrate our results.

Keywords: Non-zero-sum game; Cooperation behavior; Hamilton-Jacobi-Bellman (HJB) equation; Equilibrium investment strategy; CARA utility

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1 Introduction

Portfolio optimization problem is a classic problem in finance mathematics. Its origin can be traced to the works of Markowitz [1] and [2]. In the setting of a single period model, he measured the risk by using the variance, which provides a fundamental formulation of the portfolio construction. Based on this approach, Merton [3] and [4] studied the portfolio and consumption optimization problem in a continuous time model. Apply the stochastic control method, he derived the explicit form of the optimal investment and consumption strategy in a complete market. Since then, many papers have addressed the portfolio optimization problem under more general and more realistic model, such that Cox and Huang [5], Lee [6] and Pliska [7], for an incomplete market. Browne [8], Zhu et al. [9] and Zhou et al.[10] considered the optimal investment and reinsurance problem for an insurer.

However, in all of these literatures, no strategic interaction between investors is taken into account. In this paper, we analyze the dynamic investment strategies of fund managers in the presence of strategic interactions arising from each fund manager(investor) desire to protect their joint return. The two fund managers cooperatively choose their investment strategies to maximize their utilities based on their terminal joint weighted log-return. We formulate their optimal investment problem within a non-zero-sum differential game. More studies on the non-zero-sum differential investment games may be found in Basak and Makarov [11], Bensousan et al. [12], Espinosa and Touzi [13], Ma et al. [14] and Deng et al. [15].

The contributions of this paper are summarized as follows. We derive the explicit form of the equilibrium investment strategies and obtain the corresponding value functions by using the dynamic programming approach. From the results, we find that the cooperation behavior will make the fund manager’s investment strategy more cautious. That is , each fund manager will adjust the position of the risky asset to hedge the risk of the joint return.

Compared with the study by Ma et al. [14] and Deng et al. [15], our paper is different from Ma et al. [14] in at least two respects. First, we consider the optimization problem with the goal of maximizing the terminal log-return while Ma et al. [14] with maximizing the terminal wealth. Second, each fund manager can invest in the different risk asset while the same asset in Ma et al. [14]. This paper is also different from Deng et al. [15], where the competition behavior(relative concerns) is studied in a non-zero-sum game. Yet our paper explores the corporation behavior between two fund managers.

The remainder of the paper is organized as follows. Section 2 formulate the non-zero-sum game between two cooperative fund managers. In Section 3, we derive the expression for the equilibrium investment policies and corresponding value functions for each fund manager by using the stochastic control
method. In Section 4 we provide the numerical simulations analysis. Finally, we conclude the paper in Section 5.

2 Model formulation

2.1 Financial Market

We consider a continuous-time financial market consisting of two assets: a bond and two stocks. Suppose that the price processes of the risk-free bond \( B(t) \) and stock asset \( S_k(t) \), \( k = 1, 2 \), satisfy the following equations:

\[
\begin{align*}
    dB(t) &= rB(t) \, dt, \\
    dS_k(t) &= S_k(t) \left[ \mu_k dt + \sigma_k dW_k(t) \right],
\end{align*}
\]

where \( r > 0 \) is the constant interest rate.

\[
\begin{align*}
    dS_k(t) &= S_k(t) \left[ \mu_k dt + \sigma_k dW_k(t) \right],
\end{align*}
\]

where \( \mu_k > r \) and \( \sigma_k > 0 \) are the return rate and volatility processes of the \( k \)-th risky stock asset. \( W_1(t) \) and \( W_2(t) \) is two standard Brownian motions, and \( E[dW_1(t)dW_2(t)] = \rho dt \), \( S_k(0) = s_k > 0 \).

In the financial market, we assume that two fund managers can participate in the financial market continuously. Suppose that for the fund manager \( k, k = 1, 2 \), the proportions of money invested in the risky stock at time \( t \) is denoted by \( \pi_k(t) \). Then, we derive the fund manager \( k \)'s dynamic wealth process as follows:

\[
\frac{dX_k(t)}{X_k(t)} = [r + \pi_k(t)\mu_k] dt + \sigma_k dW_k(t).
\]

with \( X_k(t) = x_k(0) \). Let The class of admissible strategies is denoted by \( \Pi_k \) and is given by

\[
\Pi_k := \{ \pi(t)_{t \in [0,T]} \in \mathcal{F}_t \text{ and } \int_t^T \| \pi \|^2 \, ds < \infty \text{ a.s.} \}.
\]

2.2 Formulation of a Non-zero-sum game

With joint interest, fund manager \( k \)'s goal is maximizes the expected utility on the sum between their log-return process. That is

\[
\sup_{\pi_k} \mathbb{E} \left[ U_k \left( \ln \frac{X_k^{\pi_k}(T)}{X_k(t)} + \lambda_k \ln \frac{X_j^{\pi_j}(T)}{X_j(t)} \right) \mid \mathcal{F}_t \right], \text{ for } j \neq k \in 1, 2,
\]

where \( U_k \) is a strictly increasing and strictly concave smooth utility function for fund manager \( k \) (i.e., \( U_k' > 0 \) and \( U_k'' < 0 \)). The parameter \( \lambda_k \in [0, 1] \), \( k = 1, 2 \), describes fund manager \( k \)'s degree of cooperative to its partner fund manager
$j (j \neq k \in \{1, 2\})$. A greater $\lambda_k$ means that fund manager $k$ cares more about their joint return.

Because $x_k(t) \in \mathcal{F}_t$, $k = 1, 2$, so the optimization problem (4) equivalent to the following problem

$$\sup_{\pi_k} E \left[ U_k \left( \ln X_k^\pi_k(T) + \lambda_k \ln X_j^\pi_j(T) \right) \bigg| \mathcal{F}_t \right], \quad \text{for } j \neq k \in \{1, 2\},$$

We assume that fund manager $k$ has the following CARA type utility:

$$U_k(x) = -\frac{1}{\gamma_k} e^{-\gamma_k x}, \quad \text{for } k = 1, 2,$$

where $\gamma_k > 0$ is the risk aversion parameter. Then, we can rewrite this CARA utility of log-wealth as following CRRA type utility form:

$$U_k \left( \ln X_k^\pi_k(T) + \lambda_k \ln X_j^\pi_j(T) \right) = -\frac{1}{\gamma_k} \left\{ \frac{X_k^\pi_k(T)}{[X_j^\pi_j(T)]^{1/\lambda_k}} \right\}^{-\gamma_k}.$$ (7)

**Problem 2.1** The classical non-zero-sum stochastic differential game problem is to find a Nash equilibrium $(\pi_1^*, \pi_2^*) \in \Pi_1 \times \Pi_2$ such that

$$J_1^{(\pi_1^*, \pi_2^*)}(t, x_1, x_2) \geq J_1^{(\pi_1, \pi_2)}(t, x_1, x_2),$$

and

$$J_2^{(\pi_1^*, \pi_2^*)}(t, x_1, x_2) \geq J_2^{(\pi_1^*, \pi_2)}(t, x_1, x_2).$$

If (8) and (9) hold, then we respectively define the value functions of investors 1 and 2 as follow:

$$J_1(t, x_1, x_2) = J_1^{(\pi_1^*, \pi_2^*)}(t, x_1, x_2) = \sup_{\pi_1 \in \Pi_1} J_1^{(\pi_1, \pi_2)}(t, x_1, x_2)$$

and

$$J_2(t, x_1, x_2) = J_2^{(\pi_1^*, \pi_2^*)}(t, x_1, x_2) = \sup_{\pi_2 \in \Pi_2} J_2^{(\pi_1^*, \pi_2)}(t, x_1, x_2).$$

We refer the admissible strategies $\pi_1^*$ and $\pi_2^*$ as the equilibrium optimal investment strategies.

To obtain a Nash equilibrium for the above problem, we first define $Z_k(t) = \ln X_k^\pi_k(t) + \lambda_k \ln X_j^\pi_j(t)$ be the sum process of fund manager $k$ against his cooperator $j$. According the wealth process (3) and the Itô’s lemma, we can derive

$$d \ln X_k(t) = \left[ r + (\mu_k - r)\pi_k(t) - \frac{1}{2} \pi_k^2(t)\sigma_k^2 \right] dt + \pi_k(t)\sigma_k dW_k(t).$$

Then, we have

$$dZ_k^\pi(t) = \left[ (1 + \lambda_k)r + (\mu_k - r)\pi_k(t) + \lambda_k(\mu_j - r)\pi_j(t) - \frac{1}{2} (\pi_k^2(t)\sigma_k^2 + \lambda_k\pi_j^2(t)\sigma_j^2) \right] dt + \pi_k(t)\sigma_k dW_k(t) + \lambda_k\pi_j(t)\sigma_j dW_j(t).$$

(13)
with \(Z_k(0) = z_k = \ln x_k - \lambda_k \ln x_j, \ j \neq k \in 1, 2\). Furthermore, we can define the value function \(J_k(t, z_k), k = 1, 2\), as

\[
J_k(t, z_k) = \sup_{\pi_k} E \left[ U_k \left( \ln X_k^\pi_k(T) + \lambda_k \ln X_j^\pi_j(T) \right) \bigg| Z_k(t) = z_k \right].
\]

(14)

3 Nash equilibrium investment strategy

In this section, we will solve the non-zero-sum games using the dynamic programming method, the explicit form of the equilibrium investment strategies are obtained.

**Theorem 3.1.** For \(\forall t \in [0, T]\), the equilibrium investment strategies for investor \(k, k = 1, 2\), is given by

\[
\pi_k^*(t) = \frac{\tilde{\pi}_k^* - M_k \tilde{\pi}_j^*}{1 - M_1 M_2},
\]

(15)

where \(M_k = \frac{\rho \lambda_k \gamma_k \sigma_j}{(1 + \gamma_k) \sigma_k}\) and \(\tilde{\pi}_k^* = \frac{\mu_k - r}{(1 + \gamma_k) \sigma_k}\). The corresponding value function is describe by

\[
J_k(t, z_k) = -\frac{1}{\gamma_k} e^{-\gamma_k z_k + m_k(t)},
\]

(16)

where

\[
m_k(t) = \int_t^T \left[ \frac{1}{2} (\tilde{\pi}_k^*(s))^2 \gamma_k^2 - \gamma_k \left[ (1 + \lambda_k) r + (\mu_k - r) \pi_k^*(s) + \lambda_k (\mu_j - r) \pi_j^*(s) \right. \right.
\]

\[
- \left. \frac{1}{2} ((\tilde{\pi}_k^*(s))^2 \sigma_k^2 + \lambda_k (\pi_j^*(s))^2 \sigma_j^2) \right] ds.
\]

(17)

**Proof.** For the sake of convenience, we can decompose the Brown motion \(W_j(t)\) into two parts: \(W_j(t) = \rho W_k(t) + \sqrt{1 - \rho^2} \tilde{W}(t)\). Then we can rewrite the equation (18) as

\[
dZ_k^{\pi_k}(t) = \left[ (1 + \lambda_k) r + (\mu_k - r) \pi_k(t) + \lambda_k (\mu_j - r) \pi_j(t) - \frac{1}{2} (\pi_k^2(t) \sigma_k^2 + \lambda_k \pi_j^2(t) \right]
\]

\[
x \sigma_j^2 \right] dt + [\pi_k(t) \sigma_k + \lambda_k \pi_j(t) \sigma_j \rho] dW_k(t) + \lambda_k \pi_j(t) \sigma_j \sqrt{1 - \rho^2} d\tilde{W}(t).
\]

(18)

Applying standard dynamic programming techniques, we can derive the following HJB equation:

\[
\frac{\partial J_k}{\partial t} + \left[ (1 + \lambda_k) r + (\mu_k - r) \pi_k(t) + \lambda_k (\mu_j - r) \pi_j(t) - \frac{1}{2} (\pi_k^2(t) \sigma_k^2 + \lambda_k \pi_j^2(t) \right]
\]

\[
x \sigma_j^2 \right] \frac{\partial J_k}{\partial z_k} + \frac{1}{2} \left[ \pi_k^2(t) \sigma_k^2 + \lambda_k^2 \pi_j^2(t) \sigma_j^2 + 2 \lambda_k \rho \sigma_k \sigma_j \pi_k(t) \pi_j(t) \right] \frac{\partial^2 J_k}{\partial z_k^2} = 0
\]

(19)
We conjecture that the equilibrium value function \( J_k(t, z_k) \) takes the following form:

\[
J_k(t, z_k) = -\frac{1}{\gamma_k} e^{-\gamma_k z_k + m_k(t)}, \quad \text{for } t \in [0, T],
\]

with the boundary condition \( J_k(T, z_k) = U_k(z_k) \). Put all these derivatives of \( J_k(t, z_k) \) in the HJB equation (19), we have

\[
m_k'(t) + \left[ (1 + \lambda_k) r + (\mu_k - r) \pi_k(t) + \lambda_k (\mu_j - r) \pi_j(t) - \frac{1}{2} (\pi_k^2(t) \sigma_k^2 + \lambda_k \pi_j^2(t) \sigma_j^2) \right] (-\gamma_k) + \frac{1}{2} \left[ \pi_k^2(t) \sigma_k^2 + \lambda_k \pi_j^2(t) \sigma_j^2 + 2 \lambda_k \rho \sigma_k \sigma_j \pi_k(t) \pi_j(t) \right] \gamma_k^2 = 0.
\]

The first-order condition for minimizing the quantity in (21) obtains an equilibrium investment strategy \( \pi_k, k = 1, 2 \), as follows

\[
(1 + \gamma_1) \sigma_1^2 \pi_1^*(t) + \lambda_1 \rho \sigma_1 \sigma_2 \gamma_1 \pi_2^*(t) = \mu_1 - r,
\]

\[
(1 + \gamma_2) \sigma_2^2 \pi_2^*(t) + \lambda_2 \rho \sigma_1 \sigma_2 \pi_1^*(t) = \mu_2 - r.
\]

In the case that without cooperation behavior (\( \lambda_k = 0 \)), we obtain \( \pi_k^*(t) = \tilde{\pi}_k = \frac{\mu_k - r}{(1 + \gamma_k) \sigma_k^2} \). In the case \( \lambda_k \in (0, 1] \), the equilibrium investment strategy \( \pi_k^*(t) \) is obtained in (15).

**Lemma 3.1** If \( \lambda_1 = 0 \) and \( \lambda_2 \neq 0 \), then the equilibrium can rewrite as

\[
\pi_1^*(t) = \tilde{\pi}_1^*, \quad \pi_2^*(t) = \tilde{\pi}_2^* - M_2 \tilde{\pi}_1^*.
\]

The equilibrium strategy \( \pi_2^* \) indicates that the fund manager 2 will adjust the shares of the risky asset to hedge the cooperator’s risk.

**Remark 3.1** Compared the equilibrium policies in Ma et al. (2015), we can find two interesting results. First, our equilibrium investment strategy is independent of the time \( t \). Because each fund manager’s goal is maximizing the log return while maximizing the terminal wealth in Deng et al. (2018). Secondly, the competition makes the fund managers investing more in risky asset, while the cooperation will reduce the demand of the risk asset for \( \rho > 0 \).

Next, we will provide a verification theorem as follows, which can guarantee that a solution of the HJB equation is indeed the value function.

**Theorem 3.2** For \( k = 1, 2 \), let \( \Phi_k(t, z_k) \in C^{1,2} \) be a classical solution to the HJB equation (19), and the investment strategy \( \pi_k^* \) be described in Theorem 3.1. Then \( \pi_k^* \) is an optimal investment strategy for problem (14), and the concave function \( J_k(t, z_k) \) given by (16) is the corresponding value function.

**Proof.** This proof is similar to the theorem in Bensousan et al. [12]; we omit it here.
4 Numerical simulation

In this section, we will construct numerical example to investigate the effect of cooperation behavior on fund manager’s investment decision. Next, we adopt reasonable values for the parameters, as shown in the following Table 1.

Table 1. Model parameter values.

<table>
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<th>Parameters</th>
<th>Value</th>
<th>Parameters</th>
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<tr>
<td>$\mu_1$</td>
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<td>$\sigma_2$</td>
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<tr>
<td>$\gamma_1$</td>
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<tr>
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<td>$\lambda_2$</td>
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<tr>
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<td>$r$</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Fig. 1. Effects of $\lambda_1$ and $\gamma_1$ on the equilibrium strategy $\pi^*_1$.

Fig. 2. Effects of $\lambda_2$ and $\gamma_2$ on the equilibrium strategy $\pi^*_2$.

Figures 1-2 show the effect of cooperation behavior on fund manager’s investment decision. From the Figures 1-2, we can find that each fund manager will reduce the investment in risk asset. Moreover, the fund manager $k, k = 1, 2$, will invest more in the risk asset as the cooperation parameter $\lambda_k$ increase when $\rho > 0$. However, if $\rho < 0$, each fund manager will hold less shares of risk asset as the parameter $\lambda_k$ increase. In order to protect the joint return, each fund manager is optimal to hedge investment risks by adjusting the positions of risky assets.
5 Conclusion

In this paper, we discuss a non-zero-sum games between two cooperative fund managers. Each manager can invest in two different risk assets, and they all care about their joint return. The goal of each fund manager is maximizing the utility of the terminal joint weighting log-return. Applying the dynamic programming method, we obtain the Nash equilibrium investment strategy for the non-zero-sum game, and corresponding value functions. From the equilibrium results, we find that the cooperation behavior can reduce the joint investment risk, because each fund manager will adjust the position of the risk asset to hedge the risk.

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