Mergers and Acquisitions Terms under Stochastic Demand

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Abstract

This paper develops a dynamic model of corporate acquisition. In our model, we assume that the company’s price depends on the stochastic demand, and we determine the optimal terms, through Markovian perfect Nash equilibrium, based on the real option theory. Moreover, we analyze the impact of the model parameters on optimal solutions.

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1 Introduction

Mergers and Acquisitions (M&As) constitute a broad category of financial operations, aimed to redesign company structure. Nevertheless, these are complex and risky operations that only rarely achieve the stated objectives (see, e.g., Meleware and Harrold [11], where the authors examine the importance of the role played by corporate identity in the success of M&A activity). M&As spread from the end of the nineteenth century and initially involved both the US and the English markets. In the European market, the first
M&As transactions only appeared after the second post-war period; In Italy, it is difficult to find a significant presence of these operations, due to an excessively familiar corporate model and an exceedingly decisive state involvement in the social capital of most large companies.

In order to successfully achieve a growth strategy, a company can choose two possible approaches: growth by an internal line (i.e. expanding production) and growth by an external line (i.e. acquisition and reorganization). In the first case, it will take time before the investments yield a return while if the company acquires another company, the profit flows can be generated more quickly. Observing that M&As can make a company’s competitiveness degree grow, to the detriment of competitors, it is possible to incentivize the M&As with other companies for reasons of scale economy and market power (see, e.g., Lambrecht [9]). When a company wants to merge or acquire another company, we need to know:

1. the price that will be paid by the buyer;

2. under what terms the seller could accept the offer from the buyer.

As in Betton and Morán [3] (and later Lukas and Welling [10]), we apply the real-options theory to analyze a M&A evaluating problem. The idea to use the real-options approach was studied by Myers [14]: in his paper, the growth opportunities can be analyzed as a call options problem depending on future investments of the company. Consequently, it is possible to assume that the company’s value is determined by the value of assets placed and the value of growth options.

Of particular interest is the paper of Dapena and Fidalgo [6] where the authors analyze the M&As problems by focusing on the acquisition of a controlling / majority stake of the company and the control premium value, using the real options theory. In particular, their study reaches beyond the acquisition process, to the possibility of making an investor become a minority/majority shareholder of a company. This approach has brought to use the real options theory to analyze the managerial flexibility. Morellec and Zhdanov in [12] present a dynamic model of M&A, analyzing the stock market valuations of merging companies. Moreover, the authors determine the terms and the optimal timing in order for the mergers to occur. A few years later, the same authors, in [13], develop a dynamic model of acquisitions with three different companies; the terms and the optimal timing for the acquisitions are jointly determined. Hackbarth and Morellec [8] analyze the behaviour of stock returns in mergers and acquisitions. In real options framework, the timing and terms of acquisitions are endogenous and are determined from value-maximizing decisions. In [3], bargaining between the parties is described as a two person non-cooperative game. The seller offers a price and the counterparty decides
(possibly not immediately) if to accept or reject the offer. This model has been extended by [10], but with the difference that

1. the total gain of the acquisition is known by both parties; the percentages of each agent’s gain are known,

2. there are transaction costs and they are paid by both parties,

3. also the buyer can fix the offer.

The aim of our paper is to determine the optimal terms of a M&A; in our model, the price of the company depends on the variations in demand. This assumption is consistent with the fact that if the market reveals an increase (or decrease) in demand, the company’s price undergoes an increase (or decrease). In particular, in our paper, we assume that the cumulative demand, observed in the market, is a Markovian process. To our knowledge, there are many papers that concern the study of cumulative demand as a Markovian process. For instance, in Bather [1], the cumulative demand is Brownian motion in order to take advantage of independent increments property of the process. An immediate extension has been studied in Constantinides and Richard [5], where the authors model the cumulative demand as an arithmetic Brownian motion. More recently, we can cite the article of Bensoussan at al. [2], where the cumulative demand is defined as the sum of a diffusive process and a compound Poisson process.

In our paper, the cumulative demand is represented by a mean-reverting processes in order to exhibit its autoregressive properties (see, e.g., Bouasker ar al. [4]). In particular, if demand grows, the price will grow. Moreover, we also consider the arithmetic Brownian motion as a particular case. Through the Markovian perfect Nash equilibrium, we determine the optimal terms of M&A: In the first step, we determine the optimal premium placed by one or both companies and, in the second step, the optimal price at which the offer will be accepted.

The paper is structured as follows: Section 2 is devoted to the acquisition problem as a two players game, where one of the two players (the seller or the buyer) makes a bid on the premium; consequently, the counterparty fixes the price. If the price is believed fair, the acquisition can be completed. Under the particular assumptions, we can determine the optimal terms to complete the acquisition. Moreover, we establish that the total surplus is shared out among the parties. In Section 3, we analyze the behavior of the optimal terms and shares of the surplus with respect to model parameters.
2 Mergers and Acquisitions Framework

Consider a model with two agents:

1. A seller $S$ owning a company whose price is $V_t$;
2. A buyer $B$ who values the same company with a higher price; that is $\theta V_t$ with $\theta > 1$.

We determine an equilibrium strategy for the both agents, using the Markovian perfect Nash equilibrium theory. We assume that the part placing the bid firstly proposes a premium $\psi > 0$, then the counterpart proposes the price $V^*$ at which the offer will be accepted, without the possibility of counteroffers. With the sale of the company, the seller cashes $\psi V_t$ with $\psi > 0$ from the buyer and transfers the company which has value $V_t$ to the buyer; while the buyer gets the company whose value is $\theta V_t$ and pays $\psi V_t$. The transaction costs, which are constants, are equal to $A$, and, moreover, are allocated to both. It is easy to check that the both parts don’t experience loss if and only if $\theta - 1 > \frac{2A}{V_t}$.

Since the decision to accept the offer can be delayed, the valued $V^*$ corresponds to the value $V_{t^*}$ where $t^*$ is the first time such that $V_t > V^*$. Consequently, the problem can be interpreted as a perpetual option exercise problem; the offer will be accepted only if the option will be exercised. If the seller is the offering party, the premium $\psi = \psi_S$ is offered as a solution to the maximization problem

$$\max_{\psi} \mathbb{E}\left[((\psi_S - 1)V^*(\psi_S) - A)e^{-r t^*}\right],$$

where $V^*(\psi_S)$ is the optimal price chosen by the reacting party (i.e. the buyer). In turn, the buyer determines the optimal price to acquire the company as the solution of the following maximization problem:

$$\max_{\tau} \mathbb{E}\left[((\theta - \psi_S)V_\tau - A)e^{-r \tau}\right].$$

Otherwise, if the buyer proposes the premium $\psi = \psi_B$ as a solution to the maximization problem

$$\max_{\psi} \mathbb{E}\left[((\theta - \psi_B)V^*(\psi_B) - A)e^{-r t^*}\right],$$

the seller determines the optimal price as the solution of the following maximization problem:

$$\max_{\tau} \mathbb{E}\left[((\psi_B - 1)V_\tau - A)e^{-r \tau}\right].$$

For simplicity, we can reduce to solve the optimization problems

$$F(V, D) := \max_{\tau} \mathbb{E}\left[(g(\psi)V_\tau - A)e^{-r \tau}\right],$$

(1)
Mergers and acquisitions terms under stochastic demand

\[
\max_{\psi} E \left[ (h(\psi) V^*(\psi) - A) e^{-rt} \right],
\]

(2)

so that, if the seller places the offer, then we have that \( g(\psi) = g(\psi_S) = \theta - \psi_S \)
and \( h(\psi) = h(\psi_S) = \psi_S - 1 \), otherwise we have that \( g(\psi) = g(\psi_B) = \psi_B - 1 \)
and \( h(\psi) = h(\psi_B) = \theta - \psi_B \).

We now make the following standing assumptions:

**Assumptions 2.1.** (i) The price of the company \( V_t \) is uncertain and its dynamics is described by the following stochastic differential equation

\[
\begin{aligned}
dV_t &= \eta V_t dt + \sigma V_t dW^{(1)}_t + V_t dD_t, \quad t > 0, \\
V(0) &= V_0,
\end{aligned}
\]

where \( \eta \) is an endogenous parameter tied to the made investments, \( D_t \) is the demand which we will assume as random, \( \sigma \) is the volatility parameter which is positive and \( W^{(1)}_t \) is a standard Brownian motion. The term \( V_t dD_t \) measure the impact of demand on the firm’s price.

(ii) Demand has a dynamics that follows a mean reverting process \( D_t \) satisfying the following stochastic differential equation

\[
\begin{aligned}
dD_t &= (a_0 + a_1 D_t) dt + b dW^{(2)}_t, \quad t > 0, \\
D(0) &= D_0,
\end{aligned}
\]

where \( a_0 \) and \( b \) are strictly positive constants, \( a_1 \) is negative constant and \( W^{(2)}_t \) is another standard Brownian motion correlated to \( W^{(1)}_t \) by parameter \( \rho \) measuring the market’s response to the company policies.

(iii) The agents are risk-neutral, and the term structure of interest rates is flat and fixed to a level \( r \), such that \( r \geq \eta + a_0 - (a_0 + \frac{1}{2} b^2 + \rho \sigma b) \mathbf{1}_{\{a_1 < 0\}} \).

**Remark 2.2.** If \( a_1 = 0 \), the demand process follows an arithmetic Brownian motion. In general, a demand modeled by a process with constant volatility is widely used since its closed form solution facilitates the process. However, the process can become negative but the probability that the process \( D_t \) becomes negative is very small.

**Theorem 2.3.** Under Assumptions 2.1 and conjecturing that the solution has the form

\[
F(V, D) = \begin{cases} 
KV^\beta e^{-\beta D}, & \text{if } a_1 < 0 \\
KV^B, & \text{if } a_1 = 0,
\end{cases}
\]

(5)
the solution of \((1)\) is given by

\[
F(V, D) = \begin{cases} 
(g(\psi)V^* - A) \left(\frac{V}{V^*}\right)^\beta e^{-\beta(D-D^*)} & \text{if } a_1 < 0 \\
(g(\psi)V^* - A) \left(\frac{V}{V^*}\right)^\beta & \text{if } a_1 = 0
\end{cases},
\]

where

\[
\beta = \begin{cases} 
-\left(\frac{1}{2} \sigma^2 - \frac{b}{2} - \rho \sigma b\right) + \sqrt{(\eta - \frac{1}{2} \sigma^2 - \frac{1}{2} b^2 - \rho \sigma b)^2 + 2 \sigma^2} & \text{if } a_1 < 0 \\
-\left(\frac{1}{2} \sigma^2 - \frac{b}{2} - \rho \sigma b\right) + \sqrt{(\eta + a_0 - \frac{1}{2} \sigma^2 - \frac{1}{2} b^2 - \rho \sigma b)^2 + 2r(\sigma^2 + b^2 + 2 \rho \sigma b)} & \text{if } a_1 = 0
\end{cases},
\]

\[
V^*(\psi) = \frac{\beta}{\beta - 1} \frac{A}{g(\psi)}, \quad \text{and} \quad D^* = D(t^*),
\]

with \(t^*\) is solution of maximization problem \((1)\).

**Proof.** As in [10], and following [7], it is an optimal stopping problem. First of all, from the assumption \((\text{ii})\), we have that the price of company is given by

\[
dV_t = (\eta + a_0 + a_1 D_t)V_t dt + \sigma V_t dW_t^{(1)} + b V_t dW_t^{(2)},
\]

where \(W_t\) is a Brownian motion independent of \(W_t^{(2)}\).

By Ito’s Lemma, \(F(V, D)\) must satisfy the following Bellman equation

\[
(\eta + a_0 + a_1 D)\frac{\partial F}{\partial V} + (a_0 + a_1 D)\frac{\partial F}{\partial D} + \frac{1}{2} V^2 \sigma^2 + b^2 + 2 \rho \sigma b \frac{\partial^2 F}{\partial V^2} + \frac{1}{2} b^2 \frac{\partial^2 F}{\partial D^2} + bV(\rho \sigma + b) \frac{\partial^2 F}{\partial V \partial D} = rF.
\]

By (5), it follows that (9) is satisfied if

\[
\left[\frac{1}{2} \sigma^2 + \frac{1}{2} (2 \rho \sigma b + b^2) \mathbf{1}_{\{a_1=0\}}\right] \beta^2 + (\eta + a_0 \mathbf{1}_{\{a_1=0\}} - \frac{1}{2} \sigma^2 - \frac{1}{2} b^2 - \rho \sigma b) \beta - r = 0.
\]

The previous 2-nd degree equation has a positive solution given by (7).

Since \(F(V, D)\) must satisfy the following boundary condition

\[
K(V^*)^\beta e^{-\beta D^*} = g(\psi)V^* - A,
\]

\[
K \beta (V^*)^\beta - 1 e^{-\beta D^*} = g(\psi),
\]

with \(D^*\) defined in (8); it follows that

\[
V^*(\psi) = \frac{\beta}{\beta - 1} \frac{A}{g(\psi)},
\]

and consequently (6).
On the other hand, the bidding company chooses $\psi$ such that it is a solution of the maximization problem (2). We have the following results.

**Proposition 2.4.** Under Assumptions 2.1, if the seller places the offer, then the optimal premium is

$$
\psi^*_S = \frac{\beta \theta + \beta - 1}{2 \beta - 1},
$$

(10)

otherwise, if the buyer places the offer, then the optimal premium is

$$
\psi^*_B = \frac{\beta \theta + \beta - \theta}{2 \beta - 1}.
$$

(11)

The optimal price is given by

$$
V^*(\psi_S) = V^*(\psi_D) = \frac{\beta}{(\beta - 1)^2} \frac{2 \beta - 1}{\theta - 1} A,\n$$

(12)

and is independent of who places the bid.

**Proof.** By Theorem 2.3, it follows that

$$
\max_{\psi} \mathbb{E} \left[ (h(\psi) V^*(\psi) - A) e^{-rt} \right] = \max_{\psi} \left[ (h(\psi) V^*(\psi) - A) \left( \frac{V_0}{V^*(\psi)} \right)^\beta e^{-\beta(D_0 - D^*)} \right];
$$

consequently, since the demand level doesn’t depend on $\psi$, we have that $\psi$ is the solution of the equation

$$
\frac{h'(\psi) g(\psi) - h(\psi) g'(\psi)}{g(\psi)(\beta - 1)} + \left( \frac{\beta}{\beta - 1} \frac{h(\psi)}{g(\psi)} - 1 \right) g'(\psi) = 0.
$$

If the bid is placed by the seller, that is $g(\psi) = g(\psi_S) = \theta - \psi_S$ and $h(\psi) = h(\psi_S) = \psi_S - 1$, it follows that (10); while if the buyer places the offer, $g(\psi) = g(\psi_B) = \psi_B - 1$ and $h(\psi) = h(\psi_B) = \theta - \psi_B$, it follows that (11). Moreover, in the both cases, the optimal price is given by (12). \qed

As in [10], we can determine the total surplus, generated by the acquisition: how is it shared out among the parties? In the next proposition, we show that the total surplus is independent of who places the offer, and we can establish that the offering party holds a greater fraction of the surplus.

**Proposition 2.5.** Under Assumptions 2.1, the total surplus is given by

$$
G(V_0, D_0) = \frac{3 \beta - 2}{(\beta - 1)^2} A \left( \frac{V_0}{V^*} \right)^\beta e^{-\beta(D_0 - D^*)}.
$$

(13)

Moreover, the offering and reacting parties hold a share $\alpha$ and $1 - \alpha$ of the surplus, respectively, with $\alpha = \frac{2 \beta - 1}{3 \beta - 2}.$
Proof. Denoting by $G_S(V_0, D_0)$ and $G_B(V_0, D_0)$, the surplus of each parties, respectively, the total surplus is given by their sum, i.e.

$$G(V_0, D_0) = G_S(V_0, D_0) + G_B(V_0, D_0). \tag{14}$$

If the seller places the offer, by (10) and (12), we have that the surplus obtained by him is given by

$$G_S(V_0, D_0) = \left[ (\psi^* S - \psi^* S - 1) \right] V^* - A \left( \frac{V_0}{V^*} \right)^{\beta} e^{-\beta(D_0 - D^*)} \left[ \frac{2\beta - 1}{(\beta - 1)^2} A \left( \frac{V_0}{V^*} \right)^{\beta} e^{-\beta(D_0 - D^*)} \right],$$

while, the buyer obtains a surplus given by

$$G_B(V_0, D_0) = \left[ (\theta - \psi^* S) V^* - A \right] \left( \frac{V_0}{V^*} \right)^{\beta} e^{-\beta(D_0 - D^*)} \left[ \frac{\beta - 1}{(\beta - 1)^2} A \left( \frac{V_0}{V^*} \right)^{\beta} e^{-\beta(D_0 - D^*)} \right].$$

Consequently, by (14), we get that the total surplus is given by (13), and the share $\alpha$ is given by

$$\alpha = \frac{G_S(V_0, D_0)}{G(V_0, D_0)} = \frac{2\beta - 1}{3\beta - 2}.$$  

Analogously, we can achieve the same results, when the buyer places the offer.

\[ \square \]

3 Sensitivities Analysis

This section is dedicated to the analysis of the optimal terms behavior, with respect to the model parameters. It is natural to expect that if the seller (or the buyer) places the offer, then a growth of drift parameters implies that the agent can choose a higher premium (or lower premium). Moreover, independently of who places the offer, the optimal price increases with respect to drift parameters. Finally, the share $\alpha$ of the surplus, held by the offering party, increases with respect to a drift parameters growth. We have the following results.

**Lemma 3.1.** Under Assumptions (2.1):

1. If $a_1 = 0$, $\beta$ decreases with respect drift’s parameters $\eta$ and $a_0$.
2. If $a_1 < 0$, $\beta$ decreases with respect drift’s parameter $\eta$. 


Proof. We only prove the Point 1., the proof of Point 2. is similar. First of all, we observe that the parameters \( \eta \) and \( a_0 \) have the same influence on parameter \( \beta \), hence

\[
\frac{\partial \beta}{\partial \eta} = \frac{\partial \beta}{\partial a_0},
\]

so that it is sufficient to determine the sign of only one of two:

\[
\frac{\partial \beta}{\partial \eta} = -1 + \frac{\eta + a_0 - \frac{1}{2} \sigma^2 - \frac{1}{2} b^2 - \rho \sigma b}{\sqrt{(\eta + a_0 - \frac{1}{2} \sigma^2 - \frac{1}{2} b^2 - \rho \sigma b)^2 + 2r(\sigma^2 + b^2 + 2 \rho \sigma b)}}.
\]

By (iii) in Assumptions 2.1, the previous term is negative. \( \square \)

As a consequence, we obtain the following proposition:

**Proposition 3.2.** Under Assumptions 2.1:

1. If the seller (or the buyer) places the offer, the optimal premium is an increasing function (or decreasing function) with respect to drift parameters \( \eta \) and \( a_0 \).

2. The optimal price \( V^* \) is an increasing function with respect to model parameters.

3. The share \( \alpha \) is an increasing function with respect to \( \eta \) and \( a_0 \).

Proof. Let \( x \) be a drift parameter. Applying the chain rule as a formula for computing the derivative of the composition of two functions, we have

\[
\frac{\partial \psi^*_S}{\partial x} = \frac{\partial \psi^*_S}{\partial \beta} \frac{\partial \beta}{\partial x}, \quad \frac{\partial \psi^*_B}{\partial x} = \frac{\partial \psi^*_B}{\partial \beta} \frac{\partial \beta}{\partial x}, \quad \frac{\partial V^*}{\partial x} = \frac{\partial V^*}{\partial \beta} \frac{\partial \beta}{\partial x}, \quad \text{and} \quad \frac{\partial \alpha}{\partial x} = \frac{\partial \alpha}{\partial \beta} \frac{\partial \beta}{\partial x}.
\]

By Proposition 2.4, taking into account that

\[
\frac{\partial \psi^*_S}{\partial \beta} = -\frac{\theta - 1}{(2\beta - 1)^2}, \quad \frac{\partial \psi^*_B}{\partial \beta} = \frac{\theta - 1}{(2\beta - 1)^2}, \quad \frac{\partial V^*}{\partial \beta} = -\frac{A}{\theta - 1} \frac{2\theta - 1}{(\beta - 1)^3}, \quad \text{and} \quad \frac{\partial \alpha}{\partial \beta} = -\frac{1}{(3\beta - 2)^2},
\]

and thanks to previous lemma, we have reached a conclusion. \( \square \)

How do the optimal terms behave, with respect to a variation of volatilities parameters? The impact of uncertainty parameters depends on their influence on the process price \( V_t \) (defined in (3)), and, in particular, it depends on the order of magnitude of the correlation between the processes and the parameters \( \sigma \) and \( b \), as we can see in the next results.

**Lemma 3.3.** Let \( M \) be the maximum value between \(-\frac{a_1}{b}\) and \(-\frac{b}{a_1}\). Under Assumptions 2.1:

1. If \( a_1 = 0 \), \( \beta \) decreases with respect to \( \rho \). Moreover, \( \beta \) is a decreasing parameter with respect to \( \sigma \) and \( b \), when \( M < \rho \leq 1 \); while, \( \beta \) increases in \( \sigma \) (or \( b \)) and decreases in \( b \) (or \( \sigma \)), when \(-1 \leq \rho < M \) and \( b > \sigma \) (or \( b < \sigma \)).
2. If \( a_1 < 0 \), \( \beta \) increases with respect to \( \rho \), and decreases with respect to \( \sigma \). Moreover, \( \beta \) increases (or decreases) with respect to \( b \), if \( -\frac{b}{\sigma} < \rho \leq 1 \) (or \( -1 \leq \rho < -\frac{b}{\sigma} \)) and \( b < \sigma \). Otherwise, if \( b \geq \sigma \), \( \beta \) increases with respect to \( b \).

**Proof.** As in Lemma 3.1, we prove Point 1., the proof of Point 2. is similar. We start computing the derivatives of \( \beta \) with respect to \( \sigma \) and \( b \):

\[
\frac{\partial \beta}{\partial \sigma} = -2 \left( \frac{\eta + a_0}{\sigma^2 + b^2 + 2\rho \sigma b} \right) \left[ 1 - \frac{\left( \frac{\eta + a_0}{\sigma^2 + b^2 + 2\rho \sigma b} - \frac{1}{2} \right) + \frac{r}{\eta + a_0}}{\sqrt{\left( \frac{\eta + a_0}{\sigma^2 + b^2 + 2\rho \sigma b} - \frac{1}{2} \right)^2 + \frac{2r}{\eta + a_0}}} \right],
\]

and

\[
\frac{\partial \beta}{\partial b} = 2 \left( \frac{\eta + a_0}{\sigma^2 + b^2 + 2\rho \sigma b} \right) \left[ 1 - \frac{\left( \frac{\eta + a_0}{\sigma^2 + b^2 + 2\rho \sigma b} - \frac{1}{2} \right) + \frac{r}{\eta + a_0}}{\sqrt{\left( \frac{\eta + a_0}{\sigma^2 + b^2 + 2\rho \sigma b} - \frac{1}{2} \right)^2 + \frac{2r}{\eta + a_0}}} \right].
\]

Consequently, when \( \rho > \max \left( -\frac{b}{\sigma}, -\frac{\sigma}{b} \right) \), we have that \( \frac{\partial \beta}{\partial \sigma} \) and \( \frac{\partial \beta}{\partial b} \) are both positive; on the other hand, when \( \rho < \max \left( -\frac{b}{\sigma}, -\frac{\sigma}{b} \right) \) and \( b > \sigma \), we have that

\[
\frac{\partial \beta}{\partial \sigma} > 0 \quad \text{and} \quad \frac{\partial \beta}{\partial b} < 0.
\]

When \( b < \sigma \), the derivatives have opposite signs. To end the proof, computing the derivatives of \( \beta \) with respect to \( \rho \):

\[
\frac{\partial \beta}{\partial \rho} = 2 \left( \frac{\eta + a_0}{\sigma^2 + b^2 + 2\rho \sigma b} \right) \left[ 1 - \frac{\left( \frac{\eta + a_0}{\sigma^2 + b^2 + 2\rho \sigma b} - \frac{1}{2} \right) + \frac{r}{\eta + a_0}}{\sqrt{\left( \frac{\eta + a_0}{\sigma^2 + b^2 + 2\rho \sigma b} - \frac{1}{2} \right)^2 + \frac{2r}{\eta + a_0}}} \right].
\]

Noting that it is always negative, we have reached a conclusion. \( \square \)

Finally, we can establish the behavior of optimal terms and the share \( \alpha \) of surplus, with respect to uncertainty parameters. The proof is omitted because it is similar to the proof of Proposition 3.2.

**Proposition 3.4.** Under Assumptions 2.1, setting \( M \) as in the previous lemma, we have that

**(i) if \( a_1 = 0 \):**

1. If the seller places the offer, the optimal premium \( \psi^*_S \) is an increasing function with respect to \( \rho \) and with respect to \( \sigma \) and \( b \) when \( M < \rho \leq 1 \); the function \( \psi^*_S \) decreases in \( \sigma \) (or \( b \)) and increases in \( b \) (or \( \sigma \)), when \( -1 \leq \rho < M \) and \( b > \sigma \) (or \( b < \sigma \)). If the buyer places the offer, the opposite case holds.
2. The optimal price $V^*$ is an increasing function with respect to $\rho$ and with respect to $\sigma$ and $b$ when $M < \rho \leq 1$; the function $V^*$ decreases in $\sigma$ (or $b$) and increases in $b$ (or $\sigma$), when $-1 \leq \rho < M$ and $b > \sigma$ (or $b < \sigma$).

3. The share $\alpha$ of the offering party’s surplus is an increasing function with respect to $\rho$ and with respect to $\sigma$ and $b$ when $M < \rho \leq 1$; the function $\alpha$ decreases in $\sigma$ (or $b$) and increases in $b$ (or $\sigma$), when $-1 \leq \rho < M$ and $b > \sigma$ (or $b < \sigma$). For the share $1 - \alpha$ of the reacting party’s surplus, the opposite case holds.

(ii) If $a_1 < 0$:

1. If the seller places the offer, the optimal premium $\psi^*_S$ is an increasing function (or decreasing function) with respect to $\sigma$ (or $\rho$) and with respect to $b$, when $\rho < -\frac{b}{\sigma}$ (or $b$, when $\rho > -\frac{b}{\sigma}$). If the buyer places the offer, the opposite case holds.

2. The optimal price $V^*$ is an increasing function (or decreasing function) with respect to $\sigma$ (or $\rho$) and with respect to $b$, when $\rho < -\frac{b}{\sigma}$ (or $b$, when $\rho > -\frac{b}{\sigma}$).

3. The share $\alpha$ of the offering party’s surplus is an increasing function (or decreasing function) with respect to $\sigma$ (or $\rho$) and with respect to $b$ when $\rho < -\frac{b}{\sigma}$ (or $\rho > -\frac{b}{\sigma}$). For the share $1 - \alpha$ of the reacting party’s surplus, the opposite case holds.

4 Conclusions

We determine the optimal terms in acquisition problems, using a real options approach. In our model, the company’s price depends on demand. We consider that the demand is described by arithmetic Brownian motion or by mean-reverting stochastic process.

The obtained formulas have the same spirit as the result in [10]. We can notice that the optimal premia and the optimal price don’t directly depend on the level of demand, but on its parameters. The demand parameters affect $\beta$-parameter, through the response of the market to the company policies. If the market best responds to the company policies, the parameter $\beta$ is higher; otherwise, it will assume lower value. The parameter $\beta$ depends on the parameter $a_0$ only if the demand process follows an Arithmetic Brownian motion. In our paper, the total surplus depends on demand and it is shared out among the parties in order to obtain a greater surplus for the offering party.

From our research, it is clear that the drift parameters growth implies the possibility to bargain a higher optimal price; while the behavior of the optimal
premium depends on who places the offer. As regards to the behavior of optimal terms with respect to variation of uncertainty’s parameters, we cannot state anything conclusive without knowing the order of magnitude of the correlation between the processes and the parameters $\sigma$ and $b$. In every studied case, if $\rho > 0$, the growth of $\sigma$ implies that the reacting party can choose a higher optimal price.

**References**


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