An Efficient Non Linear Algorithm Predictive Model of a Robust Optimal Portfolio

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Abstract

Decision making under uncertainties is a real and challenging problem to portfolio managers in the banking industry.

In this paper, the optimal portfolio choice problem has been modelled by the non linear expectation method. The mathematical modelling process shows an optimal problem when the objective function is the expectation of the utility function of the terminal wealth, the state function is the differential equation of the total asset portfolio, and the equality constraints are the optimal proportions and the capacity. This problem has been numerically solved by using the discretization of the fourth order Runge Kutta method because of its strong numerical convergence and easiest implementation by python programming language.

The obtained optimal results show the dynamic of the total asset portfolio, the evolution of the optimal proportions in accordance with the capacity, the dynamic of loans and the market securities. The significant gain control are confirmed.

Keywords: Portfolio optimization, Non-linear mathematical modelling, Predictive Runge Kutta algorithm, Banking Industry model

1 Introduction

In banking industry system, there is always an issue of optimal portfolio choice. This problem also concerns the management process of the banking operations. Many Authors tried to propose some solutions and control methods for banking industries. In order to master the optimal control process, an efficient non linear Algorithm predictive model of a robust optimal portfolio is set up in this paper.
The optimal portfolio choice problem is modelled by the non linear expectation method. The Mathematical modelling process shows an optimal problem when the objective function is the expectation of the utility function of the terminal wealth, the state function is the differential equation of the total asset portfolio, and the equality constraints are the optimal proportions and the capacity. This problem is numerically solved by using the discretization of the fourth order Runge Kutta method because of its strong numerical convergence and is implemented by python programming language. The applied algorithm shows optimal numerics results which are discussed with state of arts and compared to standards optimal proportions trajectories.

2 Banking Industry model

We consider a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) on a time horizon, \(T\), with filtration: \(\{\mathcal{F}_t\}, \ t \geq 0\), generated by two independent standard Brownian motions \(\{W_S(t), W_L(t)\}, \ t \geq 0\) and \(\mathbb{P}\) is a probability measure on \(\Omega\). The bank’s balance sheet assets, capital and liabilities satisfy the relation: total asset = total liabilities + bank capital. In particular, at any time \(t\), the bank’s stylized balance sheet is represented as follows:

\[
R(t) + L(t) + S(t) \equiv B(t) + D(t) + C(t),
\]

where \(R(t)\), \(L(t)\), \(S(t)\), \(B(t)\), \(D(t)\) and \(C(t)\) are respectively reserves, loans, securities, deposits, borrowings and bank capital. Each of these variables is given as a function defined from \(\Omega \times T \rightarrow \mathbb{R}_+\). We further assume equal proportionality between the reserves and the borrowing plus deposit. Here, equation (1) becomes

\[
L(t) + S(t) \equiv C(t),
\]

Following the standard practice in the banking industry (see [1]), we put the bank securities into two categories; The first contains the riskless assets called treasuries issued by national treasuries in most countries as means of borrowing money to meet government expenditure not covered by tax revenues. The second group contains risky assets called market securities (e.g. loans, equities etc.). The dynamics of treasury is given as follows :

\[
\frac{dS_0(t)}{S_0(t)} = r(t)dt.
\]

On the other hand, the dynamic of market security price is given by:

\[
\frac{dS(t)}{S(t)} = (r(t) + \lambda_S)dt + \sigma_S dW_S(t)
\]
where \( \sigma_S \) is the security volatility and \( \lambda_S \) denotes the risk premium. Under the Capital Asset Pricing Model (CAPM), the risk premium can be quantified by the relation 

\[
\lambda_S = \beta [E(R_m) - R_f],
\]

with \( E(R_m) \) representing the market expected return, \( R_f \) the risk-free interest rate, and \( \beta \) the sensitivity of the expected excess asset returns to the expected excess market return. The dynamics of the loans is given as follows:

\[
\frac{dL(t)}{L(t)} = (r(t) + \lambda_L)dt + \sigma_L dW_L(t),
\]

(5)

where, \( \lambda_L = \lambda, \sigma_L + \delta \), \( \sigma_L \) the loan volatility, \( \delta \) the default risk premium and \( \lambda \), the constant premium of interest rate risk. Then by equations (3)-(5), the dynamics of the total asset portfolio is represented by the following equation

\[
\frac{dx(t)}{x(t)} = (1 - \pi_L(t) - \pi_S(t)) \frac{dS_0(t)}{S_0(t)} + \pi_L(t) \frac{dL(t)}{L(t)} + \pi_S(t) \frac{dS(t)}{S(t)},
\]

\[
= (r(t) + \pi_L(t)\lambda_L + \pi_S(t)\lambda_S)dt + \pi_L(t)\sigma_L dW_L + \pi_S(t)\sigma_S dW_S \tag{6}
\]

where \((1 - \pi_L(t) - \pi_S(t)), \pi_L(t)\) and \(\pi_S(t)\) are the proportions, invested in treasuries, loans and market securities respectively. In most economic situations as well in Burundi financial markets, a portfolio manager can doubt the probability distribution of assets in the market. This implies the presence of more than one martingale measure in the market. To deal with such cases we let \( \mathbb{P} \) be a reference probability measure and \( \mathbb{Q} \) the set of probability measures \( \mathbb{Q} \) such that \( \mathbb{Q} \sim \mathbb{P} \). Now, for a bank with a strictly initial asset value \( X(0) \), and share-holders enjoying a power utility function \( U \), the bank manager faces an additional problem of maximizing share-holders utility from terminal wealth with model ambiguity. This leads to a search for robust optimal strategies of the following form

\[
\max_{\pi(t) \in \mathcal{A}} \min_{\mathbb{Q} \in \mathbb{Q}} \mathbb{E}_{\mathbb{Q}}^{Q(\pi)}[u(x_T)] \tag{7}
\]

where, \( \mathcal{A} \) is the set of control processes for an ambiguity-neutral regulator in a given market and \( E_{\mathbb{Q}}^{\mathbb{Q}}[\cdot] = E_{\mathbb{Q}}^{\mathbb{Q}}[-|F_t] \) represents the conditional expectation under the probability measure \( \mathbb{Q} \). From this, we can see that \( \min_{\mathbb{Q} \in \mathbb{Q}} \mathbb{E}_{\mathbb{Q}}^{Q(\pi)}[u(x_T)] = \min_{\mathbb{Q} \in \mathbb{Q}} \int_{\Omega} u(x_T)d\mathbb{Q} = \int_{\Omega} u(x_T) \min_{\mathbb{Q} \in \mathbb{Q}} d\mathbb{Q} = \int_{\Omega} u(x_T)d\{\min_{\mathbb{Q} \in \mathbb{Q}} \mathbb{Q}\} \). If the portfolio manager assumes that there exist a capacity \( c \) such that \( c(\cdot) = \min_{\mathbb{Q} \in \mathbb{Q}} \mathbb{Q}(\cdot) \), then we have \( \min_{\mathbb{Q} \in \mathbb{Q}} \mathbb{E}_{\mathbb{Q}}^{Q(\pi)}[u(x_T)] = \int_{\Omega} u(x_T)dc = \mathbb{E}_c[u(x_T)]. \)

In order to find the Choquet expectation, we describe the ambiguity aversion by a binomial tree as in figure 1 where we have the joint capacity on each asset.
As proven in [2], with the Choquet random walk in continuous time \( t \in [0, T] \),
\[ dW_L(t) = \mu dt + s dB_L(t), \quad dW_S(t) = \mu dt + s dB_S(t) \]
where \( \mu = 2c - 1 \), \( s^2 = 4c(1 - c) \). The attitude towards ambiguity with the Choquet Brownian motion modifies the drift and the volatility terms of the total asset as follows:

\[
\frac{dx(t)}{x(t)} = (r(t) + \pi_L \lambda_L + \pi_S \lambda_S) dt + \pi_L \sigma_L \left( \mu dt + s dB_L(t) \right) + \pi_S \sigma_S \left( \mu dt + s dB_S(t) \right) = \\
\left( r(t) + \pi_L \lambda_L + \pi_S \lambda_S + \pi_L \sigma_L \mu + \pi_S \sigma_S \mu \right) dt + \pi_L \sigma_L s dB_L(t) + \pi_S \sigma_S s dB_S(t)
\]

Then this implies that the objective function to maximize is now given by

\[
\max_{\pi(t) \in A} \mathbb{E}^c_t \left[ u(x_T) \right]
\]

\[
\frac{dx(t)}{x(t)} = \left( r(t) + \pi_L \lambda_L + \pi_S \lambda_S + \pi_L \sigma_L \mu + \pi_S \sigma_S \mu \right) dt + \pi_L \sigma_L s dB_L(t) + \pi_S \sigma_S s dB_S(t),
\]

\[
\sum_i \pi_i(t) = 1, \quad \sum_i c_i = 1.
\]

where, \( u \) is the utility function, \( \mathbb{E}^c_t \) is the Choquet expectation.
3 Optimization problem Modelling and solving algorithm

The problem as defined in the relation below is an optimal control problem with instantaneous constraints. To model the problem solving algorithm, we write the optimal problem as follows:

\[
\begin{aligned}
\max_{\pi(t) \in A} & \mathbb{E}_{i}^{t} \left[ u(x_T) \right] \\
\dot{x}(t) &= f(x(t), u(t)) \\
k_1(x, u, t) &= 0, k_2(x, u, t) = 0,
\end{aligned}
\]

where \( f(x(t), u(t)) = x(t) \left[ (r(t) + \pi_L \lambda_L + \pi_S \lambda_S + \pi_L \sigma_L \mu + \pi_S \sigma_S \mu) dt + \pi_L \sigma_L dB_L(t) + \pi_S \sigma_S dB_S(t) \right] \) is the state modelling function and \( k_1 = \sum_i \pi_i(t) - 11, k_2 = \sum_i c_i - 1 \) the equality constraints functions. The fourth order Runge-Kutta method is used to solve the differential system [5, 6, 7, 8]. This method is chosen because of its higher order while avoiding the disadvantages of Taylor methods requiring the evaluation of partial derivatives of \( f \).

Algorithm 1:

1. Let us subdivide the time interval \([t_0, t_f]\) as \( h = t_{n+1} - t_n = \frac{t_f - t_0}{N} \), where \( N \) is the number of samples in numerical schema.

2. For \( 0 \leq n \leq N \),

\[
\begin{align*}
&\max_{u \in U} J_{12}(x_n, u_n) \\
l_1 &= h f(t_n, x_n, u_n), l_2 = h f(t_n + \frac{h}{2}, x_n + \frac{l_1}{2}, u_n) \\
l_3 &= h f(t_n + \frac{h}{2}, x_n + \frac{l_2}{2}, u_n), l_4 = h f(t_n + h, x_n + l_3, u_n), t_{n+1} = t_n + h \\
x_{n+1} &= x_n + \frac{h}{6}(l_1 + 2l_2 + 2l_3 + l_4) \\
\mu_1 k_1(x(t_n), u(t_n), t_n) &= 0, \mu_2 k_2(x(t_n), u(t_n), t_n) = 0, \mu_1 \leq 0, \mu_2 \geq 0 \]

Write \( t_{n+1} = t_n + h, x_{n+1} \).

3. Stop.

4 Main Numerical Results

In this section, we provide a numerical simulation of the evolution of the optimal portfolio strategy derived in the previous section. We assume the investment horizon to be \( T = 2 \) years. The parameters of the simulation are as follows: \( c = 0.2, \lambda_L = 0.06, \lambda_S = 0.05, \sigma_L = 0.01, \sigma_S = 0.01, \gamma = 0.5 \) and \( r = 0.06 \).
Figure (2) Dynamic of the total Asset Portfolio and Optimal proportions strategy

The figure 2 shows the dynamic of the total Asset Portfolio and the optimal proportions strategy for the optimal proportions of loans and market securities, we can see that they increase with the capacity while the optimal proportions of treasuries decrease when the capacity increases. These models coincide with the classical optimal strategies as shown by Melton in [4].

The figure 3a shows the dynamic of market Securities when the volatility $\sigma = 0.01$ and the optimal investment is increasing from zero to 1.8 but figure 3b shows that if the volatility increases to 0.03, the optimal investment is exponentially decreasing from zero to -600.
Figure 4a shows that the optimal investment strategy starts at a value equal to zero and increase while oscillating up to a value of 4.5. These oscillations are due to the increase of volatility which shows the risk. Figure 4b shows the dynamic of loans for the volatility value when the optimal investment strategy is moving up and down for all the process.

Figures 5a and 5b shows the dynamic of loans for different values of the volatility when the optimal investment strategy is moving up and down to a value of 15.7. All the numerical results are obtained by applying the robust optimization method with a high convergence order and precision. If we do not apply the optimization as the standard banking system used to solve the optimal portfolio choice problem, this can cause the bankrupt.
5 Conclusion

In this paper, the optimal portfolio choice problem has been modelled by the non linear expectation method. The Mathematical modelling process shows an optimal problem when the objective function is the expectation of the utility function of the terminal wealth, the state function is the differential equation of the total asset portfolio, and the equality constraints are the optimal proportions and the capacity. This problem has been numerically solved by using the discretization of the fourth order Runge Kutta method because of its strong numerical convergence and easiest implementation by python programming language. The obtained optimal results show the dynamic of the total asset portfolio, the evolution of the optimal proportions in accordance with the capacity, the dynamic of loans and the market securities. The significant gain control are confirmed.

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